Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 659503, 12 pages
doi:10.1155/2011/659503

Research Article

Oscillation Criteria Based on a New Weighted Function for Linear Matrix Hamiltonian Systems

Yingxin Guo[1,2] and Junchang Wang[3]

[1] *College of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, China*
[2] *Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China*
[3] *Department of Mathematics, Shangqiu Normal University, Shangqiu, Henan 476000, China*

Correspondence should be addressed to Junchang Wang, jcw178@163.com

Received 26 February 2011; Accepted 3 April 2011

Academic Editor: Mingshu Peng

By employing a generalized Riccati technique and an integral averaging technique, some new oscillation criteria are established for the second-order matrix differential system $U' = A(x)U + B(t)V$, $V' = C(x)U - A^*(t)V$, where $A(t)$, $B(t)$, and $C(t)$ are $(n\times n)$-matrices, and B, C are Hermitian. These results are sharper than some previous results.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the linear matrix Hamiltonian system of the form

$$\begin{aligned} U' &= A(x)U + B(t)V, \\ V' &= C(x)U - A^*(t)V, \end{aligned} \qquad t \geq t_0, \tag{1.1}$$

where $A(t)$, $B(t)$, and $C(t)$ are $(n \times n)$-matrices and B, C are Hermitian, that is, $B^*(t) = B(t)$, $C^*(t) = C(t)$. For any matrix A, the transpose of A is denoted by A^*.

For any real symmetric matrixes P, Q, R, we write $P \geq Q$ meaning that $P - Q \geq 0$; that is, $P - Q$ is positive semidefinite and $P > Q$ meaning that $P - Q > 0$; that is, $P - Q$ is positive definite.

Definition 1.1. A solution $(U(t), V(t))$ of (1.1) is called nontrivial if $\det U(t) \neq 0$ for at least one $t \geq t_0$.

Definition 1.2. A nontrivial solution $(U(t), V(t))$ of (1.1) is called prepared if $U^*(t)V(t) - V^*(t)U(T) = 0$ for every $t \geq t_0$.

Definition 1.3. System (1.1) is called oscillatory on $[t_0, \infty)$ if there is a nontrivial prepared solution $(U(t), V(t))$ of (1.1) having the property that $\det U(t)$ vanishes on $[T, \infty)$ for every $T > t_0$. Otherwise, it is called nonoscillatory.

Note 1. It follows from [1, Theorem 8.1, page 303] that if the system (1.1) is oscillatory on $[t_0, \infty)$, then every nontrivial prepared solution $(U(t), V(t))$ of (1.1) has the property that $\det U(t)$ vanishes on $[T, \infty)$ for every $T > t_0$.

The oscillation problem for system (1.1) and its various particular cases such as the second-order matrix differential systems

$$[Y(t)]'' + Q(t)Y(t) = 0, \quad t \in [t_0, \infty), \tag{1.2}$$

$$[P(t)Y(t)]'' + Q(t)Y(t) = 0, \quad t \in [t_0, \infty), \tag{1.3}$$

has been studied extensively in recent years, for example, see [1–23]. Some of the most important conditions that guarantee that system (1.2) is oscillatory are as follows:

$\lim_{t\to\infty}\lambda_1\{\int_{t_0}^{t} Q(s)ds\} = \infty$ (see [4, 6]),

$\lim_{t\to\infty}\inf(1/t)\int_{t_0}^{t}\int_{t_0}^{s}\operatorname{tr} Q(\tau)d\tau\, ds > -\infty$ and

$\lim_{t\to\infty}\sup(1/t)\int_{t_0}^{t}\lambda_1[\int_{t_0}^{s} Q(\tau)d\tau]ds = \infty$ or

$\lim_{t\to\infty}\sup(1/t)\int_{t_0}^{t}\{\lambda_1[\int_{t_0}^{s} Q(\tau)d\tau]\}^2 ds = \infty$ (see [5]),

$\lim_{t\to\infty}\sup(1/t^{m-1})\lambda_1[\int_{t_0}^{t}(t-s)^{m-1}Q(s)ds]ds = \infty$, $m > 2$ is an integer (see [2]).

We particularly mention the other results of Erbe et al. [2] who proved the following theorem.

Erbe, Kong, and Ruan's Theorem

Let $H(t,s)$ and $h(t,s)$ be continuous on $D = \{f(t,s) : t \geq s \geq t_0\}$ such that $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $t > s \geq t_0$. We assume further that the partial derivative $(\partial/\partial s)H(t,s) = H_s(t,s)$ is nonpositive and continuous for $t \geq s \geq t_0$ and $h(t,s)$ is defined by

$$H_s(t,s) = -h(t,s)[H(t,s)]^{1/2}, \quad (t,s) \in D. \tag{1.4}$$

Finally, we assume that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\lambda_1\left[\int_{t_0}^{t}\left(H(t,s)Q(s) - \frac{1}{4}h^2(t,s)I\right)ds\right] = \infty, \tag{1.5}$$

where $\lambda_1[A] \geq \lambda_2[A] \geq \cdots \geq \lambda_n[A]$ denotes the usual ordering of the eigenvalues of the symmetric matrix A; I is the $n \times n$ identity matrix. Then system (1.2) is oscillatory.

And, later, Meng et al. [3] gave the following oscillation criteria.

Meng, Wang, and Zheng's Theorem

Let $H(t,s)$ and $h(t,s)$ be continuous on $D = \{(t,s) : t \geq s \geq t_0\}$ such that $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $t > s \geq t_0$. We assume further that the partial derivative $(\partial/\partial s)H(t,s) = H_s(t,s)$ is nonpositive and continuous for $t \geq s \geq t_0$ and $h(t,s)$ is defined by

$$H_s(t,s) = -h(t,s)[H(t,s)]^{1/2}, \quad (t,s) \in D. \tag{1.6}$$

If there exists a function $f \in C^1[t_0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\lambda_1\left[\int_{t_0}^{t}\left(H(t,s)R(s) - \frac{1}{4}a(s)h^2(t,s)I\right)ds\right] = \infty, \tag{1.7}$$

where $a(t) = \exp\{-2\int t f(s)ds\}$, $R(t) = a(t)\{Q(t) + f^2(t)I - f'(t)I\}$. Then system (1.2) is oscillatory.

However, all these results are given in the form of $\lim_{t\to\infty}\sup \lambda_1[\cdot] = +\infty$. In this paper, using the generalized Riccati technique and the integral averaging technique, we establish some new oscillation criteria which are different from most known ones in the sense that they are based on a new weighted function $\hbar(t,s,l)$ and which are presented in the form of $\lim_{t\to\infty}\sup \lambda_1[\cdot] >$ const. Our results are presented in the form of a high degree of generality. Although the conditions in our main results (Theorem 2.1) seem to be more complicated compared to the known ones, with appropriate choices of the functions $\hbar$, f, we derive a number of oscillation criteria (see also (2.2)), which extend, improve, and unify a number of existing results and handle the cases not covered by known criteria. In particular, this can be seen by the examples given at the end of this paper.

2. Main Results

In the last literature, most oscillation results involve a function $H = H(t,s) \in C(D,R+)$, where $D = \{(t,s) : t_0 \leq s \leq t < \infty\}$, which satisfies $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$ and has partial derivative $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial s} = -h(t,s)[H(t,s)]^{1/2}, \tag{2.1}$$

where h is locally integrable with respect to s in D.

In this paper, let a function $\hbar = \hbar(t,s,l)$ be continuous on $D = \{(t,s,l) : t_0 \leq l \leq s \leq t < +\infty\}$, which satisfies $\hbar(t,t,l) = 0$, $\hbar(t,s,l) > 0$ for $l \leq s < t$ and has the partial derivative $\partial\hbar/\partial s$ on D such that $\partial\hbar/\partial s$ is locally integrable with respect to s in D, and we call the two positive numbers γ and δ admissible [22] if they satisfy the condition $\gamma\delta > 1$.

Theorem 2.1. *If there exist a function $f \in C^1[t_0, \infty)$ and two admissible numbers γ, δ such that*

$$\limsup_{t\to\infty} \frac{1}{\hbar^2(t,t_0,t_0)}\lambda_1\left[\int_{t_0}^{t}\left(\hbar^2(t,s,t_0)\Psi(s)+\gamma\delta P(t,s,t_0)\right)ds\right]=\infty, \tag{2.2}$$

where $\Psi(s) = b(s)[-C - f(A + A^*) + f^2B - f'I](s)$, *$I$ is the $n \times n$ identity matrix,* $b(s) = \exp(-2\int_{x_0}^{s} f(\varsigma)d\varsigma)$, *and*

$$\begin{aligned} P(t,s,t_0) &= b(s)\hbar^2(t,s,t_0)\left[f(A+A^*)-A^*B^{-1}A\right](s)\\ &\quad - b(s)\hbar(t,s,t_0)\left[\hbar_s'(t,s,t_0)-f(s)\hbar(t,s,t_0)\right]\times\left[A^*B^{-1}+B^{-1}A\right](s)\\ &\quad - b(s)\left[\left(\hbar_s'(t,s,t_0)-f(s)\hbar(t,s,t_0)\right)B^{-1/2}(s)-f(s)\hbar(t,s,t_0)B^{1/2}(s)\right]^2, \end{aligned} \tag{2.3}$$

then system (1.1) *is oscillatory.*

Proof. Suppose to the contrary that system (1.1) is nonoscillatory. Then there exists a nontrivial prepared solution $(U(t), V(t))$ of (1.1) such that $U(t)$ is nonsingular on $[T, \infty)$ for some $T > t_0$. Without loss of generality, we may assume that $\det U(t) \neq 0$ for $t \geq t_0$. Define

$$W(t) = b(t)\left[V(t)U^{-1}(t) + f(t)I\right], \quad t \geq t_0. \tag{2.4}$$

Then $W(t)$ is well defined, Hermitian, and it satisfies the Riccati equation

$$\left\{W' + WA + A^*W + \frac{1}{b}WBW - f[WB + BW - 2W] + \Psi\right\}(t) = 0 \tag{2.5}$$

on $[t_0, \infty)$. Multiplying (2.5), with t replaced by s, by $\hbar^2(t, s, t_0)$, integrating from t_0 to t, and picking two admissible numbers γ and δ, we obtain

$$\begin{aligned} \int_{t_0}^{t}\hbar^2(t,s,t_0)\Psi(s)ds &= -\int_{t_0}^{t}\hbar^2(t,s,t_0)W'(s)ds - \int_{t_0}^{t}\frac{\hbar^2(t,s,t_0)}{b(s)}[WBW](s)ds\\ &\quad - \int_{t_0}^{t}\hbar^2(t,s,t_0)\left[WA + A^*W - f(WB+BW-2W)\right](s)ds\\ &= \hbar^2(t,t_0,t_0)W(t_0) - \int_{t_0}^{t}\frac{\hbar^2(t,s,t_0)}{b(s)}[WBW](s)ds\\ &\quad - \int_{t_0}^{t}\hbar^2(t,s,t_0)\left[WA + A^*W - f(WB+BW)\right](s)ds \end{aligned}$$

$$
\begin{aligned}
&+2\int_{t_0}^{t} \hbar(t,s,t_0)\left[\hbar_s'(t,s,t_0)-f(s)\hbar(t,s,t_0)\right]W(s)ds \\
&= \hbar^2(t,t_0,t_0)W(t_0)-\frac{1}{\gamma}\int_{t_0}^{t}[(Q^*Q)(t,s,t_0)]ds-\gamma\delta\int_{t_0}^{t}[P(t,s,t_0)]ds \\
&-\frac{\gamma\delta-1}{\gamma\delta}\int_{t_0}^{t}\frac{\hbar^2(t,s,t_0)}{b(s)}\left[(RW)^*(RW)\right](s)ds,
\end{aligned}
\tag{2.6}
$$

where $R(t)=\sqrt{B(t)}$ and

$$
\begin{aligned}
Q(t,s,t_0) = {} & \frac{\hbar(t,s,t_0)}{\sqrt{\delta b(s)}}(RW)(s)-\gamma\left[\sqrt{\delta b(s)}\hbar(t,s,t_0)\right]\left(fR-R^{-1}A\right)(s) \\
&+\gamma\sqrt{\delta b(s)}\left[\hbar_s'(t,s,t_0)-f(s)\hbar(t,s,t_0)\right]R^{-1}(s).
\end{aligned}
\tag{2.7}
$$

Then

$$
\begin{aligned}
\int_{t_0}^{t}\left(\hbar^2(t,s,t_0)\Psi(s)+\gamma\delta P(t,s,t_0)\right)ds &= \hbar^2(t,t_0,t_0)W(t_0)-\frac{1}{\gamma}\int_{t_0}^{t}[(Q^*Q)(t,s,t_0)]ds \\
&-\frac{\gamma\delta-1}{\gamma\delta}\int_{t_0}^{t}\frac{\hbar^2(t,s,t_0)}{b(s)}\left[(RW)^*(RW)\right](s)ds \\
&\le \hbar^2(t,t_0,t_0)W(t_0).
\end{aligned}
\tag{2.8}
$$

This implies that

$$
\lambda_1\left[\int_{t_0}^{t}\left(\hbar^2(t,s,t_0)\Psi(s)+\gamma\delta P(t,s,t_0)\right)ds\right]\le \hbar^2(t,t_0,t_0)\lambda_1(W(t_0)),
\tag{2.9}
$$

and then

$$
\frac{1}{\hbar^2(t,t_0,t_0)}\lambda_1\left[\int_{t_0}^{t}\left(\hbar^2(t,s,t_0)\Psi(s)+\gamma\delta P(t,s,t_0)\right)ds\right]\le \lambda_1(W(t_0)).
\tag{2.10}
$$

Taking the upper limit in both sides of (2.10) as $t\to\infty$, the right-hand side is always bounded, which contradicts condition (2.2). This completes the proof of Theorem 2.1. □

By applying the matrix theory [8, 21], we have the following theorem from Theorem 2.1.

Theorem 2.2. *If there exist a function* $f \in C^1[t_0, \infty)$ *and two admissible numbers* γ, δ *such that*

$$\limsup_{t\to\infty} \frac{1}{\hbar^2(t,t_0,t_0)} \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\operatorname{tr}\Psi(s) + \gamma\delta \operatorname{tr} P(t,s,t_0)\right) ds\right] = \infty, \tag{2.11}$$

where $\Psi(s)$, $b(s)$, *and* $P(t,s,t_0)$ *are as in Theorem 2.1, then system* (1.1) *is oscillatory.*

By [8], the trace $\operatorname{tr} : S \to R$ is a positive linear functional on S, where the space S is the linear space of all real symmetric $n \times n$ matrices. And noting that two admissible numbers γ, δ satisfying $\gamma\delta > 1$, then we have the following corollary from Theorem 2.2.

Corollary 2.3. *If there exist a function* $f \in C^1[t_0, \infty)$ *and two admissible numbers* γ, δ *such that*

$$\limsup_{t\to\infty} \frac{1}{\hbar^2(t,t_0,t_0)} \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\operatorname{tr}\Psi(s) + \operatorname{tr} P(t,s,t_0)\right) ds\right] = \infty, \tag{2.12}$$

where $\Psi(s)$, $b(s)$, *and* $P(t,s,t_0)$ *are as in Theorem 2.1, then system* (1.1) *is oscillatory.*

Proof. By virtue of a simple property of limits

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{\hbar^2(t,t_0,t_0)} \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\operatorname{tr}\Psi(s) + \gamma\delta \operatorname{tr} P(t,s,t_0)\right) ds\right] \\ &\quad > \limsup_{t\to\infty} \frac{1}{\hbar^2(t,t_0,t_0)} \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\operatorname{tr}\Psi(s) + \operatorname{tr} P(t,s,t_0)\right) ds\right] \end{aligned} \tag{2.13}$$

and (2.12), the conclusion follows from Theorem 2.2. □

If we choose $\hbar(t,s,t_0) = \sqrt{H(t,s)/H(t,t_0)}$ in Theorem 2.1, then

$$\hbar(t,t_0,t_0) = \sqrt{\frac{H(t,t_0)}{H(t,t_0)}} = 1, \tag{2.14}$$

we have the following.

Corollary 2.4. *If there exist a function* $f \in C^1[t_0, \infty)$ *and two admissible numbers* γ, δ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \lambda_1 \left[\int_{t_0}^{t} \left(H(t,s)\Psi_1(s) + \gamma\delta P_1(t,s)\right) ds\right] = \infty, \tag{2.15}$$

where $H(t,s)$ *are as in Erbe, Kong, and Ruan's Theorem,* $\Psi_1(s) = b(s)[-C - \gamma\delta A^*B^{-1}A + f^2B - f'I + \gamma\delta f(A^*B^{-1} + B^{-1}A)](s)$*, I is the* $n \times n$ *identity matrix,* $b(s) = \exp(-2\int_{x_0}^{s} f(\varsigma)d\varsigma)$*, and*

$$\begin{aligned} P_1(t,s) = {} & \frac{b(s)h(t,s)\sqrt{H(t,s)}}{2}\left[A^*B^{-1} + B^{-1}A\right](s) \\ & - b(s)\left[\left(\frac{h(t,s)}{2} + f(s)\sqrt{H(t,s)}\right)B^{-1/2}(s) + f(s)\sqrt{H(t,s)}B^{1/2}(s)\right]^2, \end{aligned} \tag{2.16}$$

then system (1.1) *is oscillatory.*

Remark 2.5. In the last literature [1–4, 12, 15, 23], most oscillation results were given in the form of $\lim_{t\to\infty}\sup(1/(H(t,t_0)))\lambda_1[\cdot] = +\infty$. Obviously, Theorem 2.1 extends and improves a number of existing results and handles the cases not covered by known criteria, which can be seen from Corollary 2.4.

If we choose $f(t) = 0$ and let $\hbar(t,s,r) = \sqrt{(t-s)^{\alpha}/(t-r)^{\beta}}$ for $\alpha, \beta > 1/2$ in Theorem 2.1, then we have the following.

Corollary 2.6. *If there exist two real numbers* $\alpha, \beta > 1/2$ *and two admissible numbers* γ*,* δ *such that*

$$\begin{aligned} \lim_{t\to\infty}\sup\frac{1}{t^{\alpha}}\lambda_1\Bigg\{\int_{t_0}^{t}\frac{(t-s)^{\alpha}}{\gamma\delta}\Bigg[&\frac{\alpha(t-s)^{\alpha-1}}{2}\left(A^*B^{-1} + B^{-1}A\right) - A^*B^{-1}A \\ & -\frac{\alpha(t-s)^{2(\alpha-1)}}{4}B^{-1} - \gamma\delta C\Bigg]ds\Bigg\} = \infty, \end{aligned} \tag{2.17}$$

then system (1.1) *is oscillatory.*

If we choose appropriate f in Theorem 2.1 such that $b(t)B^{-1}(t) \le I$ for $t \ge t_0$ and let $\hbar(t,s,r) = \sqrt{(t-s)^{\alpha}/(t-r)}$ for $\alpha > 2$, then we have the following

Corollary 2.7. *If there exist a function* $f \in C^1[t_0,\infty)$ *and two admissible numbers* γ*,* δ *such that for some* $\alpha > 1/2$ *and for every* $r \ge t_0$*,*

$$\lim_{t\to\infty}\sup\frac{1}{t^{2\alpha+1}}\lambda_1\left[\int_r^t (t-s)^2(s-r)^{2\alpha}\left(\Psi(s) + \gamma\delta P_1(t,s,r)\right)ds\right] > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.18}$$

where $\Psi(s)$, $b(s)$ *are as in Theorem 2.1 and*

$$\begin{aligned}P_1(t,s,r) &= b(s)\Big[f(A+A^*)-A^*B^{-1}A\Big](s)\\&\quad-\frac{b(s)}{(t-s)(s-r)}[r+\alpha t-(\alpha+1)s]\Big[A^*B^{-1}+B^{-1}A\Big](s)\\&\quad+b(s)f(s)\Big[A^*B^{-1}+B^{-1}A-f\Big(B+2I+B^{-1}\Big)\Big](s)\\&\quad+\frac{2b(s)f(s)}{(t-s)(s-r)}[r+\alpha t-(\alpha+1)s]\Big(I+B^{-1}\Big)\\&\quad-\frac{b(s)}{(t-s)^2(s-r)^2}B^{-1},\end{aligned}\tag{2.19}$$

then system (1.1) *is oscillatory.*

Proof. Assume to the contrary that (1.1) is nonoscillatory. Then $U(t)$ is nonsingular for all sufficiently large t, say $t \geq T \geq t_0$. Similar to the proof of Theorem 2.1, for $t \geq T \geq t_0$, we have

$$\begin{aligned}\int_T^t\Big(\hbar^2(t,s,T)\Psi(s)+\gamma\delta P_1(t,s,T)\Big)ds &\leq \gamma\delta\int_T^t b(s)B^{-1}(s)\left[\hbar_s'(t,s,T)\right]^2ds\\&\leq \gamma\delta\int_T^t (s-T)^{2(\alpha-1)}[T+\alpha t-(\alpha+1)s]^2 I.\end{aligned}\tag{2.20}$$

This implies that

$$\begin{aligned}\lambda_1\left[\int_T^t\Big(\hbar^2(t,s,T)\Psi(s)+\gamma\delta P_1(t,s,T)\Big)ds\right] &\leq \int_T^t (s-T)^{2(\alpha-1)}[T+\alpha t-(\alpha+1)s]^2ds\\&=\frac{\alpha}{(2\alpha-1)(2\alpha+1)}(t-T)^{2(\alpha+1)}.\end{aligned}\tag{2.21}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\lambda_1\left[\int_T^t\Big(\hbar^2(t,s,T)\Psi(s)+\gamma\delta P_1(t,s,T)\Big)ds\right] \leq \frac{\alpha}{(2\alpha-1)(2\alpha+1)},\tag{2.22}$$

which contradicts assumption (2.18). This completes the proof of Corollary 2.7. □

When $A(t) \equiv 0$, $B^{-1}(t) = P(t)$ and $-C(t) = Q(t)$ for $t \geq t_0$, then system (1.1) reduces to system (1.3).

As an immediate result of Theorem 2.1, we have the following theorem.

Theorem 2.8. *If there exist a function $f \in C^1[t_0, \infty)$ and two admissible numbers γ, δ such that*

$$\lim_{t\to\infty} \sup \frac{1}{\hbar^2(t,t_0,t_0)} \lambda_1 \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\Psi_1(s) + \gamma\delta P_2(t,s,t_0) \right) ds \right] = \infty, \tag{2.23}$$

where $\Psi_1(s) = b(s)[Q(t) + f^2 P^{-1}(t) - f' I](s)$, I is the $n \times n$ identity matrix, $b(s) = \exp(-2\int_{x_0}^{s} f(\varsigma)d\varsigma)$, and

$$P_2(t,s,t_0) = -b(s)\left[\left(\hbar_s'(t,s,t_0) - f(s)\hbar(t,s,t_0) \right) P^{1/2}(s) - f(s)\hbar(t,s,t_0) P^{-1/2}(s) \right]^2, \tag{2.24}$$

then system (1.3) *is oscillatory.*

By applying the matrix theory [8, 21], we have the following theorem from Theorem 2.8.

Theorem 2.9. *If there exist a function $f \in C^1[t_0, \infty)$ and two admissible numbers γ, δ such that*

$$\lim_{t\to\infty} \sup \lambda_1 \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\, \mathrm{tr}\, \Psi_1(s) + \gamma\delta\, \mathrm{tr}\, P_2(t,s,t_0) \right) ds \right] = \infty, \tag{2.25}$$

where $\Psi_1(s)$, $b(s)$, and $P_2(t,s,t_0)$ are as in Theorem 2.8, then system (1.3) *is oscillatory.*

By [8], the trace $\mathrm{tr} : S \to R$ is a positive linear functional on S, where the space S is the linear space of all real symmetric $n \times n$ matrices. And noting that two admissible numbers γ, δ satisfying $\gamma\delta > 1$, then we have the following corollary from Theorem 2.9.

Corollary 2.10. *If there exist a function $f \in C^1[t_0, \infty)$ and two admissible numbers γ, δ such that*

$$\lim_{t\to\infty} \sup \lambda_1 \left[\int_{t_0}^{t} \left(\hbar^2(t,s,t_0)\, \mathrm{tr}\, \Psi_1(s) + \mathrm{tr}\, P_2(t,s,t_0) \right) ds \right] = \infty, \tag{2.26}$$

where $\Psi_1(s)$, $b(s)$, and $P_2(t,s,t_0)$ are as in Theorem 2.8, then system (1.3) *is oscillatory.*

By Corollary 2.7 and (1.3), we easily get the following theorem:

Theorem 2.11. *If there exist a function $f \in C^1[t_0, \infty)$ and two admissible numbers γ, δ such that for some $\alpha > 1/2$ and for every $r \geq t_0$,*

$$\lim_{t\to\infty} \sup \frac{1}{t^{2\alpha+1}} \lambda_1 \left[\int_{r}^{t} (t-s)^2 (s-r)^{2\alpha} \left(\Psi_1(s) + \gamma\delta P_3(t,s,r) \right) ds \right] > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.27}$$

where $\Psi_1(s)$, $b(s)$ *are as in Theorem 2.8 and*

$$P_3(t,s,r) = -b(s)f^2(s)\left(P+2I+P^{-1}\right)(s) - \frac{b(s)}{(t-s)^2(s-r)^2}P(s) + \frac{2b(s)f(s)}{(t-s)(s-r)}[r+\alpha t-(\alpha+1)s](I+P)(s), \tag{2.28}$$

then system (1.3) *is oscillatory.*

3. Examples

Example 3.1. Consider the Euler differential system

$$Y'' + \operatorname{diag}\left(\frac{n}{t^2},\frac{m}{t^2}\right)Y = 0, \quad t \geq 1,\ m \geq n > 0. \tag{3.1}$$

If we choose $f(t)=0$, then $a(t)=1$, $\Psi_1(t)=\operatorname{diag}(n/t^2, m/t^2)$ and $P_3(t,s,r) = (1/((t-s)^2(s-r)^2))I$. Note that for each $r \geq 1$,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^{2\alpha+1}}\left[\int_r^t (t-s)^2(s-r)^{2\alpha}\left(\frac{m}{t^2}-\frac{\gamma\delta}{(t-s)^2(s-r)^2}\right)ds\right] \\
&= \lim_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_r^t \frac{m(t-s)^2(s-r)^{2\alpha}}{t^2}ds - \lim_{t\to\infty}\frac{\gamma\delta}{t^{2\alpha+1}}\int_r^t (s-r)^{2\alpha-2}ds \\
&= \frac{m}{\alpha(2\alpha-1)(2\alpha+1)}.
\end{aligned} \tag{3.2}$$

Obviously, for any $m > 1/4$, there exists $\alpha > 1/2$ such that

$$\frac{m}{\alpha(2\alpha-1)(2\alpha+1)} > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}. \tag{3.3}$$

This means that (2.25) holds. By Theorem 2.11, we find that system (3.1) is oscillatory for $m > 1/4$.

Remark 3.2. As pointed out in [3], the above-mentioned criteria (1.5) of Erbe, Kong, and Ruan cannot be applied to the Euler differential system (3.1), for

$$\lim_{t\to\infty}\sup\frac{1}{H(t,1)}\lambda_1\left[\int_1^t\left(H(t,s)Q(s)-\frac{1}{4}h^2(t,s)I\right)ds\right] \leq \lim_{t\to\infty}\int_1^t \frac{m}{s^2}ds = m < \infty. \tag{3.4}$$

Though the above-mentioned criteria (1.7) of Meng, Wang, and Zheng's Theorem can be applied to the Euler differential system, our results are sharper than theirs, which can be seen from Example 3.1.

Remark 3.3. It is interesting for the fact that If we choose $f(t) = 0$, $\hbar(t,s,T) = \sqrt{H(t,s)/H(t,T)}$, then for differential system (1.2), we have

$$\begin{aligned}\lim_{t\to\infty}\sup\lambda_1\left[\int_1^t\left(\hbar^2(t,s,1)\Psi_1(s)+P_2(t,s,1)\right)ds\right]\\ =\lim_{t\to\infty}\sup\frac{1}{H(t,1)}\lambda_1\left[\int_1^t\left(H(t,s)Q(s)-\frac{1}{4}h^2(t,s)I\right)ds\right],\end{aligned}\tag{3.5}$$

where $H(t,s)$ are as in Erbe, Kong, and Ruan's Theorem. Obviously, Theorem 2.8 extends and improves a number of existing results and handles the cases not covered by known criteria.

Example 3.4. Consider the 4-dimensional system (1.1) where

$$A(t)\equiv 0,\qquad B(t)=(t+1)^2I_2,\qquad C(t)=-\begin{bmatrix}\frac{\rho}{t^2} & 0\\ 0 & \frac{\sigma}{2t^2}\end{bmatrix},\tag{3.6}$$

and where $\rho\ge\sigma>0$ and $t\ge 1$. If we let $f(t)=0$, then $b(t)=1$ and $b(t)B^{-1}(t)\le I_2$ for $t\ge 1$. Thus, we have

$$\Psi(s)=\begin{bmatrix}\frac{\rho}{t^2} & 0\\ 0 & \frac{\sigma}{2t^2}\end{bmatrix},\qquad P_1(t,s,r)=-\frac{1}{(t-s)^2(s-r)^2(s+1)^2}I_2.\tag{3.7}$$

Thus, if we choose two admissible numbers γ, δ such that $\gamma\delta=3/2$, then for some $\alpha>1/2$ and for every $r\ge t_0$,

$$\lim_{t\to\infty}\frac{1}{t^{2\alpha+1}}\left[\int_r^t(t-s)^2(s-r)^{2\alpha}\left(\frac{\rho}{t^2}-\frac{3}{2(t-s)^2(s-r)^2(s+1)^2}\right)ds\right]=\frac{\rho}{\alpha(2\alpha-1)(2\alpha+1)}.\tag{3.8}$$

By Corollary 2.6, we find that system (3.1) is oscillatory for $\rho>1/4$.

Acknowledgment

This wrok was supported financially by the NNSF of China (10801088).

References

[1] W. T. Reid, *Sturmian Theory for Ordinary Differential Equations*, vol. 31 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1980.
[2] L. H. Erbe, Q. Kong, and S. G. Ruan, "Kamenev type theorems for second-order matrix differential systems," *Proceedings of the American Mathematical Society*, vol. 117, no. 4, pp. 957–962, 1993.

[3] F. Meng, J. Wang, and Z. Zheng, "A note on Kamenev type theorems for second order matrix differential systems," *Proceedings of the American Mathematical Society*, vol. 126, no. 2, pp. 391–395, 1998.

[4] R. Byers, B. J. Harris, and M. K. Kwong, "Weighted means and oscillation conditions for second order matrix differential equations," *Journal of Differential Equations*, vol. 61, no. 2, pp. 164–177, 1986.

[5] G. J. Butler, L. H. Erbe, and A. B. Mingarelli, "Riccati techniques and variational principles in oscillation theory for linear systems," *Transactions of the American Mathematical Society*, vol. 303, no. 1, pp. 263–282, 1987.

[6] D. B. Hinton and R. T. Lewis, "Oscillation theory for generalized second-order differential equations," *The Rocky Mountain Journal of Mathematics*, vol. 10, no. 4, pp. 751–766, 1980.

[7] Ch. G. Philos, "Oscillation theorems for linear differential equations of second order," *Archiv der Mathematik*, vol. 53, no. 5, pp. 482–492, 1989.

[8] G. J. Etgen and J. F. Pawlowski, "Oscillation criteria for second order self adjoint differential systems," *Pacific Journal of Mathematics*, vol. 66, no. 1, pp. 99–110, 1976.

[9] J. R. Yan, "Oscillation theorems for second order linear differential equations with damping," *Proceedings of the American Mathematical Society*, vol. 98, no. 2, pp. 276–282, 1986.

[10] P. Hartman, "On non-oscillatory linear differential equations of second order," *American Journal of Mathematics*, vol. 74, pp. 389–400, 1952.

[11] I. V. Kamenev, "An integral test for conjugacy for second order linear differential equations," *Matematicheskie Zametki*, vol. 23, no. 2, pp. 249–251, 1978.

[12] I. S. Kumari and S. Umamaheswaram, "Oscillation criteria for linear matrix Hamiltonian systems," *Journal of Differential Equations*, vol. 165, no. 1, pp. 174–198, 2000.

[13] H. J. Li, "Oscillation criteria for second order linear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 1, pp. 217–234, 1995.

[14] A. B. Mingarelli, "On a conjecture for oscillation of second-order ordinary differential systems," *Proceedings of the American Mathematical Society*, vol. 82, no. 4, pp. 593–598, 1981.

[15] T. Walters, "A characterization of positive linear functionals and oscillation criteria for matrix differential equations," *Proceedings of the American Mathematical Society*, vol. 78, no. 2, pp. 198–202, 1980.

[16] E. C. Tomastik, "Oscillation of systems of second order differential equations," *Journal of Differential Equations*, vol. 9, pp. 436–442, 1971.

[17] Y. G. Sun, "New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping," *Journal of Mathematical Analysis and Applications*, vol. 291, no. 1, pp. 341–351, 2004.

[18] A. Wintner, "A criterion of oscillatory stability," *Quarterly of Applied Mathematics*, vol. 7, pp. 115–117, 1949.

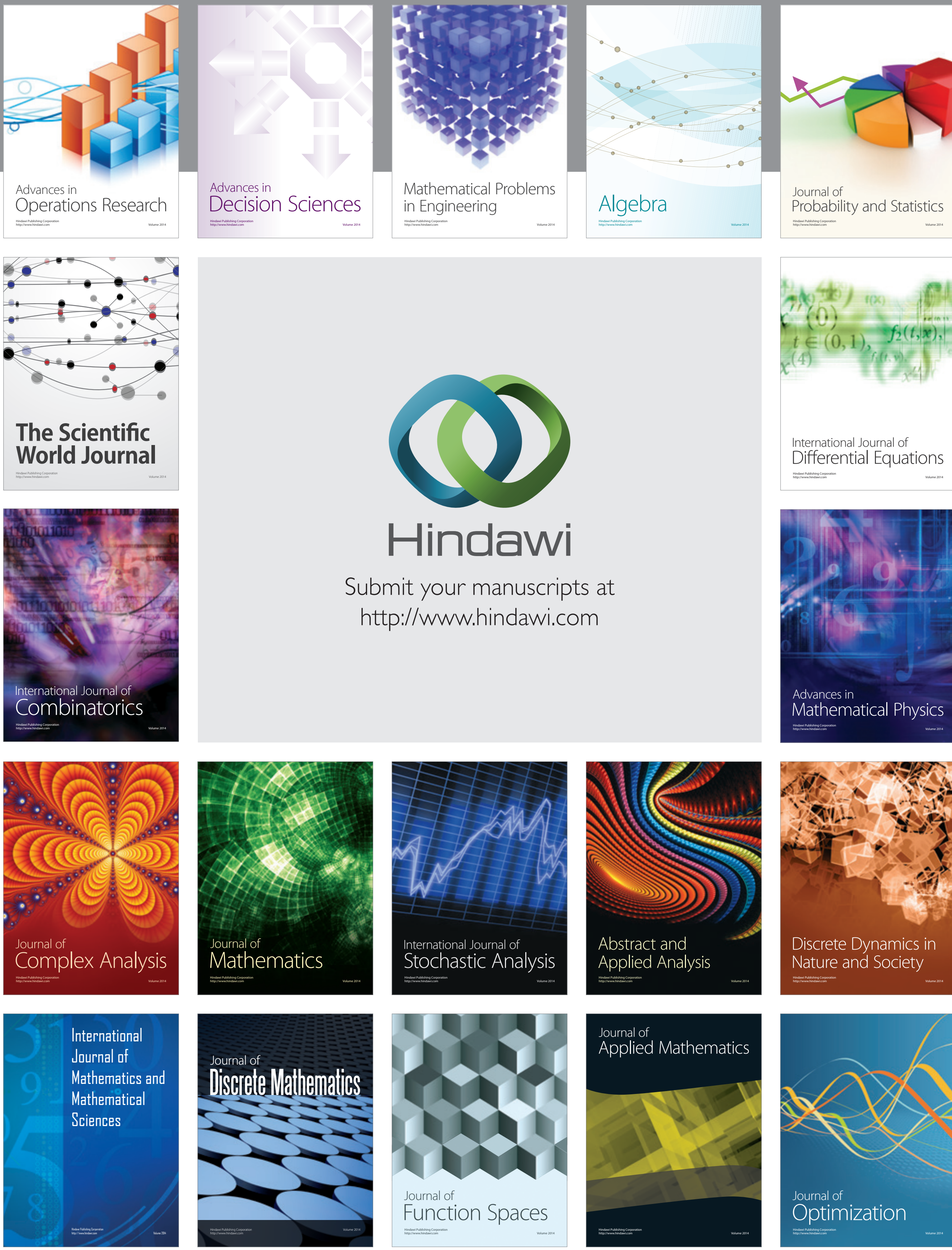
[19] Q. Kong, "Interval criteria for oscillation of second-order linear ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 229, no. 1, pp. 258–270, 1999.

[20] G. J. Butler and L. H. Erbe, "Oscillation results for second order differential systems," *SIAM Journal on Mathematical Analysis*, vol. 17, no. 1, pp. 19–29, 1986.

[21] P. Hartman, "Oscillation criteria for selfadjoint second-order differential systems and "principal sectional curvatures"," *Journal of Differential Equations*, vol. 34, no. 2, pp. 326–338, 1979.

[22] Y. V. Rogovchenko and F. Tuncay, "Yan's oscillation theorem revisited," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1740–1744, 2009.

[23] Q.-R. Wang, "Oscillation criteria related to integral averaging technique for linear matrix Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 40–54, 2004.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization