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Research Article

Positive Solutions to a Second-Order Discrete Boundary Value Problem

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We are concerned with second-order discrete boundary value problems and obtain some sufficient conditions for the existence of at least one positive solution by using the fixed point theorem due to Krasnosel'skii on a cone.

1. Introduction

Boundary value problems for difference equations have been studied extensively by many authors, for example, [1–10] to name a few. Many techniques arose in the studies of this kind of problem. For example, Agarwal et al. [1] employed the critical point theory to establish the existence of multiple solutions of some regular as well as singular discrete boundary value problems. Cai and Yu [2] applied the Linking Theorem and the Mountain Pass Lemma in the critical point theory to study second-order discrete boundary value problems and obtained some new results for the existence of solutions. Li and Sun [3, 4] were concerned with discrete system boundary value problems and gave some sufficient conditions for the existence of one or two positive solutions by using a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed point theorem in a cone. Pang et al. [5] provided sufficient conditions for the existence of at least three positive solutions for quasilinear boundary value problems for finite difference equations by using a generalization of the Leggett-Williams fixed point theorem due to Avery and Peterson. Du [6], Lin and Liu [7] discussed triple positive solutions of some second-order discrete boundary value problems by making use of the Leggett-Williams fixed-point theorem, respectively.

This paper deals with the following three-point boundary value problem for second-order difference equation of the form

$$\Delta^2 y(k-1) + h(k) f\big(y(k)\big) = 0, \quad k \in \{1, \ldots, T\},$$
$$y(0) - \alpha \Delta y(0) = 0, \qquad y(T+1) = \beta y(n), \tag{1.1}$$

where $\Delta y(k-1) = y(k) - y(k-1)$, $\Delta^2 y(k-1) = y(k+1) - 2y(k) + y(k-1), k \in \{1, \ldots, T\}$.

Throughout this paper, we will assume that the following conditions are satisfied:

(A1) $T \geq 3$ is a fixed positive integer, $n \in \{2, \ldots, T-1\}$, constant $\alpha, \beta > 0$ such that $H := T + 1 - \beta n + \alpha(1-\beta) > 0$ and $T + 1 - \beta n > 0$;

(A2) $f \in C([0, +\infty), [0, +\infty))$, f is either superlinear or sublinear, that is, either $f_0 = 0$, $f_\infty = \infty$ or $f_0 = \infty$, $f_\infty = 0$, where

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}; \tag{1.2}$$

(A3) $h(k)$ is nonnegative on $\{1, \ldots, T\}$ and $h(k) \equiv 0$ does not hold on $\{n, \ldots, T\}$.

In the paper, we show the existence of positive solutions of (1.1) under some assumptions. We also establish the associate Green's function. Readers may find that it is useful to define a cone on which a positive operator was defined, and a fixed point theorem due to Krasnosel'skii [11] will be applied to yield the existence of at least one positive solution.

2. Preliminary and Green's Function

Let $\mathbf{N}$ be the nonnegative integers; we let $\mathbf{N}_{i,j} = \{k \in \mathbf{N} : i \leq k \leq j\}$ and $\mathbf{N}_p = \mathbf{N}_{0,p}$.

By a *positive solution* y of problem (1.1), we mean $y : \mathbf{N}_{T+1} \to R$, y satisfies the first equation of (1.1) on $\mathbf{N}_{1,T}$, y fulfills $y(0) - \alpha \Delta y(0) = 0$, $y(T+1) = \beta y(n)$, and y is nonnegative on $\mathbf{N}_{T+1}$ and positive on $\mathbf{N}_{1,T}$.

We shall need the following fixed point theorem due to Krasnosel'skii [8, 11].

Theorem A. *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $A : K \bigcap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(1) $\|Au\| \leq \|u\|$, $u \in K \bigcap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$, $u \in K \bigcap \partial\Omega_2$ *or*

(2) $\|Au\| \geq \|u\|$, $u \in K \bigcap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$, $u \in K \bigcap \partial\Omega_2$.

Then A has a fixed point in $K \bigcap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 2.1 (see [7]). *The function*

$$G(k,l)=\frac{1}{H}\begin{cases}(l+\alpha)\left[T+1-k-\beta(n-k)\right], & l\in\mathbf{N}_{1,k-1}\cap\mathbf{N}_{1,n-1},\\(l+\alpha)(T+1-k)+\beta(n+\alpha)(k-l), & l\in\mathbf{N}_{n,k-1},\\(k+\alpha)\left[T+1-l-\beta(n-l)\right], & l\in\mathbf{N}_{k,n-1},\\(k+\alpha)(T+1-l), & l\in\mathbf{N}_{k,T}\cap\mathbf{N}_{n,T},\end{cases}\tag{2.1}$$

is the Green's function of the problem

$$\begin{gathered}-\Delta^2 y(k-1)=0,\quad k\in\mathbf{N}_{1,T},\\y(0)-\alpha\Delta y(0)=0,\qquad y(T+1)=\beta y(n).\end{gathered}\tag{2.2}$$

Remark 2.2. We observe that the condition $H>0$ and $T+1-\beta n>0$ implies $G(k,l)$ is positive on $\mathbf{N}_{T+1}\times\mathbf{N}_{1,T}$, which means that the finite set

$$\left\{\frac{G(k,l)}{G(k,k)}:k\in\mathbf{N}_{T+1},\ \ l\in\mathbf{N}_{1,T}\right\}\tag{2.3}$$

takes positive values. Then we let

$$M_1=\min\left\{\frac{G(k,l)}{G(k,k)}:k\in\mathbf{N}_{T+1},\ l\in\mathbf{N}_{1,T}\right\},\tag{2.4}$$

$$M_2=\max\left\{\frac{G(k,l)}{G(k,k)}:k\in\mathbf{N}_{T+1},\ l\in\mathbf{N}_{1,T}\right\}.\tag{2.5}$$

3. Main Results

Theorem 3.1. *Assume that* $(A1)$–$(A3)$ *hold, then problem* (1.1) *has at least one positive solution.*

Proof. In the following, we denote

$$m=\min_{k\in\mathbf{N}_{n,T}}G(k,k),\qquad M=\max_{k\in\mathbf{N}_{T+1}}G(k,k).\tag{3.1}$$

Then $0<m<M$.

Let E be the Banach space defined by $E=\{y:\mathbf{N}_{T+1}\to R\}$. Define

$$K=\left\{y\in E:y(k)\geq 0,\ \text{ for } k\in\mathbf{N}_{T+1}\text{ and }\min_{k\in\mathbf{N}_{n,T}}y(k)\geq\sigma\|y\|\right\},\tag{3.2}$$

where $\sigma=M_1m/M_2M\in(0,1)$, $\|y\|=\max_{k\in\mathbf{N}_{T+1}}|y(k)|$. It is clear that K is a cone in E.

We define the operator $S : K \to E$ by

$$(Sy)(k) = \sum_{l=1}^{T} G(k,l)h(l)f(y(l)), \quad k \in \mathbf{N}_{T+1}. \tag{3.3}$$

It is clear that problem (1.1) has a solution y if and only if $y \in E$ is a solution of the operator equation $y(k) = (Sy)(k)$. We shall now show that the operator S maps K into itself. For this, let $y \in K$; from (A2), (A3), we find

$$(Sy)(k) = \sum_{l=1}^{T} G(k,l)h(l)f(y(l)) \geq 0, \quad \text{for } k \in \mathbf{N}_{T+1}. \tag{3.4}$$

From (2.5), we obtain

$$\begin{aligned}(Sy)(k) &= \sum_{l=1}^{T} G(k,l)h(l)f(y(l)) \leq M_2 \sum_{l=1}^{T} G(k,k)h(l)f(y(l)) \\ &\leq M_2 M \sum_{l=1}^{T} h(l)f(y(l)), \quad \text{for } k \in \mathbf{N}_{T+1}.\end{aligned} \tag{3.5}$$

Therefore

$$\|Sy\| \leq M_2 M \sum_{l=1}^{T} h(l)f(y(l)). \tag{3.6}$$

Now from (A2), (A3), (2.4), and (3.6), for $k \in \mathbf{N}_{n,T}$, we have

$$\begin{aligned}(Sy)(k) &\geq M_1 \sum_{l=1}^{T} G(k,k)h(l)f(y(l)) \geq M_1 m \sum_{l=1}^{T} h(l)f(y(l)) \\ &\geq \frac{M_1 m}{M_2 M}\|Sy\| = \sigma\|y\|.\end{aligned} \tag{3.7}$$

Then

$$\min_{k \in \mathbf{N}_{n,T}} (Sy)(k) \geq \sigma\|Sy\|. \tag{3.8}$$

From (3.4) and (3.6), we obtain $Sy \in K$. Hence $S(K) \subseteq K$. Also standard arguments yield that $S : K \to K$ is completely continuous. □

Case 1. Suppose f is superlinear. Now since $f_0 = 0$, we may choose $C_1 > 0$ such that $f(u) \leq \delta_1 u$, for $0 < u \leq C_1$, where δ_1 satisfies

$$\delta_1 M_2 M \sum_{l=1}^{T} h(l) \leq 1. \tag{3.9}$$

Let $y \in K$ be such that $\|y\| = C_1$; by using (2.5) and (3.9), we have

$$(Sy)(k) \le M_2 \sum_{l=1}^{T} G(k,k)h(l)f(y(l)) \le \delta_1 M_2 M \sum_{l=1}^{T} h(l)y(l) \\ \le \delta_1 M_2 M \sum_{l=1}^{T} h(l)\|y\| \le \|y\|. \tag{3.10}$$

Now if we let

$$\Omega_1 = \{y \in E : \|y\| < C_1\}, \tag{3.11}$$

then

$$\|Sy\| \le \|y\|, \quad \text{for } y \in K \cap \partial\Omega_1. \tag{3.12}$$

Next since $f_\infty = \infty$, there exists $\overline{C_2} > 0$, such that $f(u) \ge \delta_2 u$, for $u \ge \overline{C_2}$, where $\delta_2 > 0$ satisfying

$$\delta_2 M_1 \sigma \sum_{l=n}^{T} G(n,n)h(l) \ge 1. \tag{3.13}$$

Let $C_2 = \max\{2C_1, \overline{C_2}/\sigma\}$ and $\Omega_2 = \{y \in E : \|y\| < C_2\}$, and let $y \in K$ and $\|y\| = C_2$, then

$$\min_{k \in \mathbf{N}_{n,T}} y(k) \ge \sigma\|y\| \ge \overline{C_2}. \tag{3.14}$$

Applying (2.4) and (3.13), one has

$$(Sy)(n) = M_1 \sum_{l=1}^{T} G(n,l)h(l)f(y(l)) \ge M_1 \sum_{l=n}^{T} G(n,n)h(l)f(y(l)) \\ \ge \delta_2 M_1 \sum_{l=n}^{T} G(n,n)h(l)y(l) \ge \delta_2 M_1 \sigma \sum_{l=n}^{T} G(n,n)h(l)\|y\| \\ \ge \|y\|. \tag{3.15}$$

Thus

$$\|Sy\| \ge \|y\|, \quad y \in K \cap \partial\Omega_2. \tag{3.16}$$

In view of (3.12) and (3.16), it follows from Theorem A that S has a fixed point $y \in K \bigcap (\overline{\Omega_2} \setminus \Omega_1)$ such that $C_1 \le \|y\| \le C_2$.

Case 2. Suppose f is sublinear case. Since $f_0 = \infty$, we may choose $C_3 > 0$ such that $f(u) \geq \delta_3 u$ for $0 < u \leq C_3$, where $\delta_3 > 0$ satisfying

$$\delta_3 M_1 \sigma \sum_{l=n}^{T} G(n,n) h(l) \geq 1, \tag{3.17}$$

$\Omega_3 = \{y \in E : \|y\| < C_3\}$; let $y \in K$ and $\|y\| = C_3$. Using (2.4) and (3.17), one has

$$\begin{aligned}(Sy)(n) &\geq M_1 \sum_{l=n}^{T} G(n,n) h(l) f(y(l)) \geq \delta_3 M_1 \sum_{l=n}^{T} G(n,n) h(l) y(l) \\ &\geq \delta_3 M_1 \sigma \sum_{l=n}^{T} G(n,n) h(l) \|y\| \geq \|y\|.\end{aligned} \tag{3.18}$$

Then $\|Sy\| \geq \|y\|$, $y \in K \bigcap \partial\Omega_3$.

In view of $f_\infty = 0$, there exists $\overline{C_4} > 0$ such that $f(u) \leq \delta_4 u$ for $u \geq \overline{C_4}$, where $\delta_4 > 0$ satisfying

$$\delta_4 M_2 M \sum_{l=n}^{T} h(l) \leq 1. \tag{3.19}$$

There are two subcases to consider, that is, f is bounded and f is unbounded.

Subcase 2.1. Suppose f is bounded, that is, $f(y) \leq L$ for all $y \in [0, \infty)$ for some $L > 0$. Let

$$C_4 = \max\left\{ 2C_3, L M_2 M \sum_{l=1}^{T} h(l) \right\}. \tag{3.20}$$

Then, for $y \in K$ and $\|y\| = C_4$, one has

$$\begin{aligned}(Sy)(k) &\leq M_2 \sum_{l=1}^{T} G(k,k) h(l) f(y(l)) \leq L M_2 M \sum_{l=1}^{T} h(l) \\ &\leq C_4 = \|y\|.\end{aligned} \tag{3.21}$$

Hence $\|Sy\| \leq \|y\|$.

Subcase 2.2. Suppose f is unbounded, that is, there exists $C_4 > \max\{2C_3, \overline{C_4}/\sigma\}$ such that $f(u) \le f(C_4)$ for all $0 < u \le C_4$. Then, for $y \in K$ with $\|y\| = C_4$, from (2.5) and (3.19), we have

$$\begin{aligned}(Sy)(k) &\le M_2 \sum_{l=1}^{T} G(k,k)h(l)f(y(l)) \le M_2 M \sum_{l=1}^{T} h(l) f(C_4)\\ &\le \delta_4 M_2 M \sum_{l=1}^{T} h(l) C_4 \le C_4 = \|y\|.\end{aligned} \tag{3.22}$$

Thus in both Subcases 2.1 and 2.2, we may put $\Omega_4 = \{y \in E : \|y\| < C_4\}$. Then

$$\|Sy\| \le \|y\|, \quad y \in K \cap \partial\Omega_4. \tag{3.23}$$

By using the fixed point Theorem A, it follows that problem (1.1) has at least one positive solution, such that $C_3 \le \|y\| \le C_4$. The proof is finished.

Finally, we give an example to demonstrate our main result.

Example 3.2. Consider the following three-point boundary value problem:

$$\begin{gathered}\Delta^2 y(k-1) + \frac{2}{(-k^2+10k+33)^{1.5}}(y+20)^{1.5} = 0, \quad k \in \mathbf{N}_{1,8},\\ y(0) - \frac{13}{9}\Delta y(0) = 0, \qquad y(9) = \frac{22}{37} y(4),\end{gathered} \tag{3.24}$$

where $T = 8, n = 4, \alpha = 13/9, \beta = 22/37, T+1-\beta n+\alpha(1-\beta) = 800/111 > 0, T+1-\beta n = 245/37 > 0, h(k) = 2/(-k^2+10k+33)^{1.5}, k \in \mathbf{N}_{1,8}, f(y) = (y+20)^{1.5}$, then f is superlinear. Conditions of Theorem 3.1 are all satisfied. Then problem (3.24) has at least one positive solution y. Indeed $y = -k^2 + 10k + 13$ is one such positive solution.

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