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Research Article

Generalized Projective Synchronization for Different Hyperchaotic Dynamical Systems

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Projective synchronization and generalized projective synchronization have recently been observed in the coupled hyperchaotic systems. In this paper a generalized projective synchronization technique is applied in the hyperchaotic Lorenz system and the hyperchaotic Lü. The sufficient conditions for achieving projective synchronization of two different hyperchaotic systems are derived. Numerical simulations are used to verify the effectiveness of the proposed synchronization techniques.

1. Introduction

Chaos is an interesting phenomenon in nonlinear dynamical systems research area. In the last three decades, chaos has been extensively studied within the scientific, engineering, and mathematical communities [1–6].

A chaotic system is a nonlinear deterministic system that displays complex, noisy-like and unpredictable behavior. These motions are trajectories in which infinite unstable periodic orbits (UPOs) are embedded. Chaos is generally undesirable in many fields. This irregular and complex phenomenon can lead systems to harmful or even catastrophic situations. In these troublesome cases chaos should be suppressed as much as possible or totally eliminated. Therefore controlling chaos has become one of the most considerable research area in the nonlinear problems ranging from biology, physics and chemistry to economics.

Since Pecora and Carroll [7, 8] showed that it is possible to synchronize two identical chaotic systems, chaos synchronization has been intensively and extensively studied due to its potential applications in secure communication, ecological systems, system identification, and so forth.

Among all kinds of chaos synchronizations, projective synchronization is one of the most noticeable ones. This kind of synchronization was first observed in continuous systems

[9] where a part of state variables possessed some partial linearity [10]. Its typical feature is that the state variables of the two-coupled system may synchronize up to a scaling factor but the Lyapunov exponents and fractal dimensions remain unchanged. Such synchronization has been relatively understood well [11–14].

In 1999, Mainieri and Rehacek [10] first reported the projective synchronization phenomenon and explained the mechanism of the formation of projective synchronization in three-dimensional systems and further attempted to predict the scaling factor by introducing a vector field. However, they only provided a guideline of predicting the scaling factor rather than a concrete theoretical solution.

Generalized synchronization [15–23] is another interesting chaos synchronization technique. It means that there exists a transformation which is able to map asymptotically the trajectories of the master attractor into those of the slave one. To understand such kind of synchronization needs much mathematics. Till now, there are relatively few publications for generalized synchronization.

A focused problem in the study of chaos synchronization is how to design a physically available and simple controller to guarantee the realization of high-quality synchronization in coupled chaotic systems. Linear feedback is of course a practical technique, but the shortcoming is that it needs to find the suitable feedback constant. Recently, Huang proposed a simple adaptive feedback control method, which dose not need to estimate or find feedback constant, to effectively synchronize two almost arbitrary identical hyperchaotic systems [24–26]. This technique has been adopted by some authors to realize the identical synchronization of almost all kinds of coupled identical neural networks with time—varying delay [27] and the complete synchronization in uncertain complex networks [28].

In this paper, we introduce a new synchronization technique, which is different from projective synchronization, but share the same typical feature of projective synchronization, that is, the Lyapunov exponents and fractal dimensions are also invariant during the synchronization process. To some extent, the synchronization presented here is very similar to the generalized synchronization.

The rest of the paper is organized as follows. In Section 2, a mathematical description of generalized projective synchronization is presented. In Section 3 system description is introduced. In Section 4, the projective synchronization problem of a hyperchaotic Lorenz system is investigated and numerical simulation results are demonstrated in Subsection 4.1. In Section 5, the generalized projective synchronization problem of hyperchaotic Lü system is presented, and numerical simulation results are given in Subsection 5.1. In Section 6, the generalized projective synchronization problem between hyperchaotic Lorenz system and hyperchaotic Lü system is presented, and numerical simulation results are given in Subsection 6.1. Finally, in Section 7 the conclusion of the paper is given.

2. Generalized Projective Synchronization of Chaotic Systems

First, both projective and generalized synchronization are introduced.

A partial linear system is often expressed as

$$\dot{u}(\mathbf{t}) = \mathbf{M}(\mathbf{z})\mathbf{u}, \qquad \dot{z}(\mathbf{t}) = \mathbf{f}(\mathbf{u}, \mathbf{z}), \tag{2.1}$$

in which the state vector u is linearly related to $\dot{u}$ with respect to t, while the matrix $\mathbf{M}(\mathbf{z})$ only depends upon the variable z which is nonlinearly related to the variable u. Projective synchronization often occurs when two identical system are coupled through the variable z in the form as

$$\dot{u}_d = \mathbf{M}(\mathbf{z})\mathbf{u}_d, \quad \dot{z}(\mathbf{t}) = \mathbf{f}(\mathbf{u}_d, \mathbf{z}), \quad \dot{u}_r = \mathbf{M}(\mathbf{z})\mathbf{u}_r. \tag{2.2}$$

The subscripts d and r stand for the driver (or master) and response (or slave) systems, respectively.

If there exists a constant $\alpha \in R$ $(\alpha \neq 0)$ such that $\lim_{t\to\infty}\|u_r - \alpha\, u_d\| = 0$, then the projective synchronization between the drive system and response system is achieved, and we call α as "scaling factor."

Consider the following coupled system:

$$\dot{x}_d = f(x_d), \quad \dot{y}_r = g(y_r, h_\mu(x_d)), \tag{2.3}$$

where $x_d \in R^n$, $y_r \in R^k$, $f : R^n \to R^n$, $h : R^n \to R^k$, and $g : R^{2k} \to R^k$. When $\mu = 0$, y_r evolves independently and has no relation to x_d, and we assume that both systems are chaotic. When $\mu \neq 0$, the chaotic trajectories of the two systems are said to be generalized synchronization if there exists a transformation $\varphi : x_d \to y_r$ which is able to map asymptotically the trajectories of the master attractor into those of the slave attractor $y_r(t) = \varphi(x_d(t))$, regardless of the initial conditions in the basin of the synchronization manifold $M = \{(x_d,\ y_r : y_r(t) = \varphi(x_d(t))\}$ [22, 23]. In general. φ is difficult to be determined.

In what follows, a new definition is introduced. Consider the following chaotic equations:

$$\dot{x}_d = f(x_d), \quad \dot{x}_r = g(x_r,\ u(x_d, x_r)), \tag{2.4}$$

where $x_d,\ x_r \in R^n$, $u : R^{2n} \to R^n$, $g : R^{2n} \to R^n$, and $u(0,0) = 0$, $g(x,\ u(0,0)) = f(x) : R^{2k} \to R^k$. If there exists a constant $\alpha \in R$ $(\alpha \neq 0)$ such that $\lim_{t\to\infty}\|x_r - \alpha\, x_d\| = 0$, then we call them "generalized projective synchronization."

Remark 2.1. (i) This definition is very similar to that of generalized synchronization, see (2.3) and (2.4). (ii) The master attractor synchronizes to the slave one up to a scaling factor a. Obviously, the Lyapnove exponents and fractal dimension remain invariant. (iii) From the last equation of (2.4), u can be regarded as a feedback controller (or "synchronizer"), that is, similar to [7, 10], and if and only if such feedback controller u is applied to the slave system, generalized projective synchronization may occur.

Remark 2.2. From the definition, one has $\lim_{t\to\infty}\|x_r - \alpha\, x_d\|$, the limit of $\alpha(t)$ as $t \to \infty$ is still written as α. So one gets $\lim_{t\to\infty}\log|\alpha(t)| = \lim_{t\to\infty}\log|x_r/x_d|$.

3. System Description

Very recently, based on Lorenz system [29, 30] and Lü system [31, 32], two hyperchaotic systems, we are constructed by introducing state feedback controller function, which were named as hyperchaotic Lorenz system and hyperchaotic Lü system, respectively.

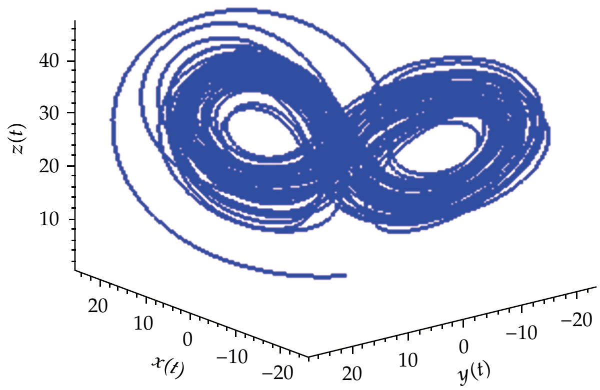

Figure 1: It shows the attractor of hyperchaotic Lorenz dynamical system at $a = 10, r = 28, b = 8/3$, and $d = 1.3$ in x, y, z subspace.

The hyperchaotic Lorenz system is described by

$$\begin{aligned} \dot{x} &= a(y - x) + w, \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= -bz + xy, \\ \dot{w} &= dw - xz. \end{aligned} \tag{3.1}$$

When parameters $a = 10, r = 28, b = 8/3$, and $0.85 < d \leq 1.3$, the system (3.1) shows hyperchaotic behavior, see Figure 1

The hyperchaotic Lü system is described by

$$\begin{aligned} \dot{x} &= a_1(y - x) + w \\ \dot{y} &= -xz + c_1 y \\ \dot{z} &= -b_1 z + xy \\ \dot{w} &= d_1 w + xz. \end{aligned} \tag{3.2}$$

When parameters $a_1 = 36, b_1 = 3, c_1 = 20$, and $-0.35 < d_1 \leq 1.3$, the system (3.2) has hyperchaotic attractor, see Figure 2

4. Generalized Projective Synchronization for Hyperchaotic Lorenz System

In order to observe generalized projective synchronization between two identical hyperchaotic Lorenz systems, we assume that the drive system with four state variables denoted by the subscript 1 and the response system having identical equations denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response

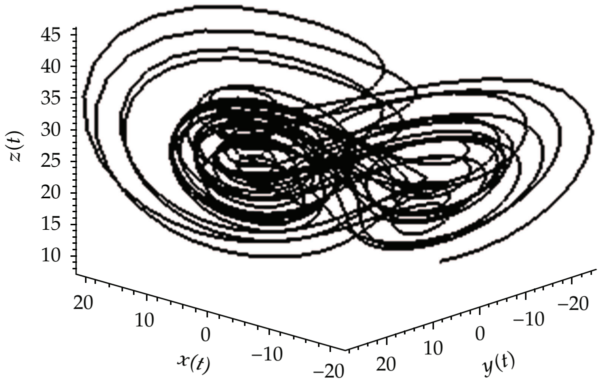

Figure 2: It shows the attractor of hyperchaotic Lü dynamical system at $a_1 = 36, b_1 = 3, c_1 = 20$, and $d_1 = 1.3$ in x, y, z subspace.

system. The drive and response systems are defined below, respectively,

$$\begin{aligned}
\dot{x}_1 &= a(y_1 - x_1) + w_1,\\
\dot{y}_1 &= -x_1 z_1 + r x_1 - y_1,\\
\dot{z}_1 &= -b z_1 + x_1 y_1,\\
\dot{w}_1 &= d w_1 - x_1 z_1,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\dot{x}_2 &= a(y_2 - x_2) + w_2 + u_1,\\
\dot{y}_2 &= -x_2 z_2 + r x_2 - y_2 + u_2,\\
\dot{z}_2 &= -b z_2 + x_2 y_2 + u_3,\\
\dot{w}_2 &= d w_2 - x_2 z_2 + u_4,
\end{aligned} \tag{4.2}$$

where $U = [u_1 \quad u_2 \quad u_3 \quad u_4]^T$ is the controller functions. The controller U is to be determined for the purpose of projective synchronizing the two identical hyperchaotic Lorenz systems.

In order to get generalized projective synchronization, we define the error system as the difference between the system (4.2) and (4.1). Set

$$e_x = x_2 - \alpha x_1, \qquad e_y = y_2 - \alpha y_1, \qquad e_z = z_2 - \alpha z_1, \qquad e_w = w_2 - \alpha w_1, \tag{4.3}$$

then one obtains the error dynamical system between (4.2) and(4.1)

$$\begin{aligned}
\dot{e}_x &= a(e_y - e_x) + e_w + u_1,\\
\dot{e}_y &= r e_x - e_y - x_2 z_2 + \alpha x_1 z_1 + u_2,\\
\dot{e}_z &= -b e_z + x_2 y_2 - \alpha x_1 y_1 + u_3,\\
\dot{e}_w &= d e_w - x_2 z_2 + \alpha x_1 z_1 + u_4.
\end{aligned} \tag{4.4}$$

Let

$$V_1 = u_1,$$
$$V_2 = \alpha x_1 z_1 - x_2 z_2 + u_2,$$
$$V_3 = x_2 y_2 - \alpha x_1 y_1 + u_3, \quad (4.5)$$
$$V_4 = \alpha x_1 z_1 - x_2 z_2 + u_4,$$

then the error dynamical system can be rewritten as

$$\dot{e}_x = a(e_y - e_x) + e_w + V_1,$$
$$\dot{e}_y = re_x - e_y + V_2,$$
$$\dot{e}_z = -be_z + V_3, \quad (4.6)$$
$$\dot{e}_w = de_w + V_4.$$

To get the projective synchronization to occur, the zero solutions of error system must be stable, that is to say, the error evolution of the drive system and response system tends to zero as $t \to \infty$. As we know, if all the eigenvalues of the Jacobian matrix of closed-loop system have negative real parts, the system is stable. Based on this theory, we desired the $(V_1, V_2, V_3, V_4)^T$ to guarantee that all the eigenvalues of closed-loop system (4.6) have negative real part. There are of course some other choices of $(V_1, V_2, V_3, V_4)^T$, but here the choice is very easy and convenient. For simplicity, choose $(V_1, V_2, V_3, V_4)^T$ as follows:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = M \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 0 & -a & 0 & -1 \\ -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2d \end{pmatrix}. \quad (4.7)$$

System (4.6) has four negative eigenvalues $-10, -1, -8/3$, and -1.3 when setting $a = 10, r = 28, b = 8/3$, and $d = 1.3$. That is to say, the error states e_x, e_x, e_z, and e_w converge to zero as $t \to \infty$. So the generalized projective synchronization is achieved.

4.1. Numerical Results

By using MAPLE 12, the systems of differential equations (4.1) and (4.2) are solved numerically. The parameters are chosen as $a = 10, r = 28, b = 8/3$, and $d = 1.3$ in all simulations so that the hyperchaotic Lorenz system exhibits a chaotic behavior if no control is applied (see Figure 1). The initial states of the drive system are $x_1(0) = 0.1$, $y_1(0) = 0.1$, $z_1(0) = 0.1$, and $w_1(0) = 0.1$, and initial states of the response system are $x_2(0) = 1$, $y_2(0) = -1$, $z_2(0) = 1$, and $w_2(0) = 1$.

Choosing $\alpha = -2$ then the error system (4.4) has the initial values $e_x(0) = 1.2, e_y(0) = -0.8, e_z(0) = 1.2$, and $e_w(0) = 1.2$. Figure 3 shows that the trajectories of $e_x(t), e_y(t), e_z(t)$, and $e_w(t)$ tended to zero after $t \geq 5$. Figure 4 shows the evaluation of the ratios $\log|x_r/x_d| = \log|x_2/x_1|, \log|y_r/y_d| = \log|y_2/y_1|, \log|z_r/z_d| = \log|z_2/z_1|$, and $\log|w_r/w_d| = \log|w_2/w_1|$ whose limits are equal to $\log 2 = 0.693$.

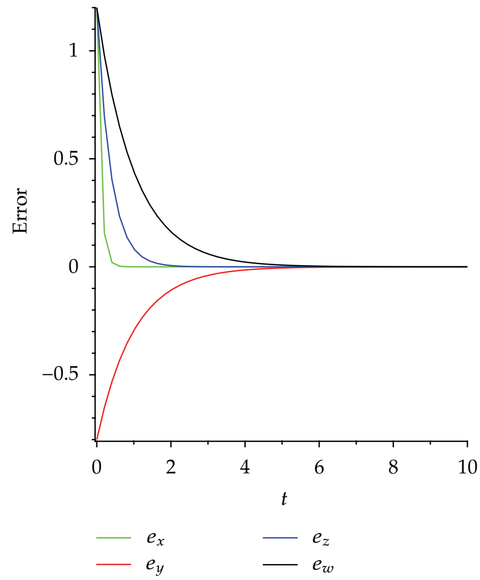

Figure 3: It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lorenz system error system tends to zero as t tends to 5 when the scaling factor $\alpha = -2$.

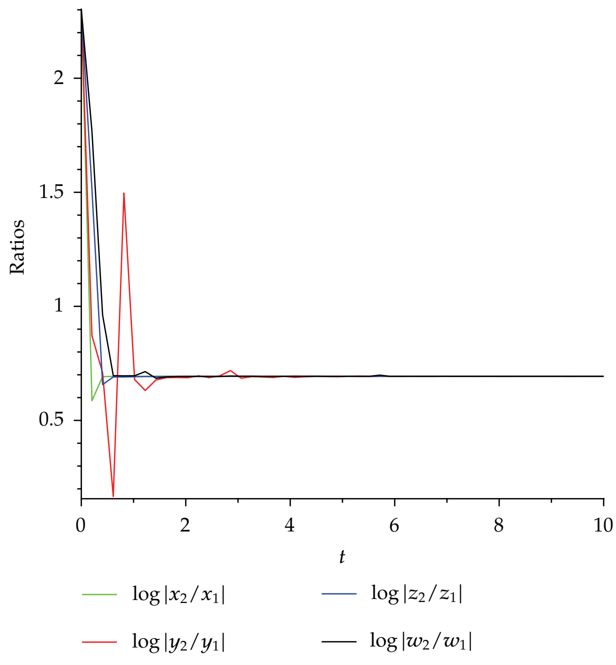

Figure 4: It shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 2 = 0.693$.

Choosing $\alpha = 5$ then the error system (4.4) has the initial values $e_x(0) = 0.5$, $e_y(0) = -1.5$, $e_z(0) = 0.5$, and $e_w(0) = 0.5$. Figure 5 shows that the trajectories of $e_x(t)$, $e_y(t)$, $e_z(t)$ and $e_w(t)$ tended to zero after $t \geq 5$. Figure 6 shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 5 = 1.609$.

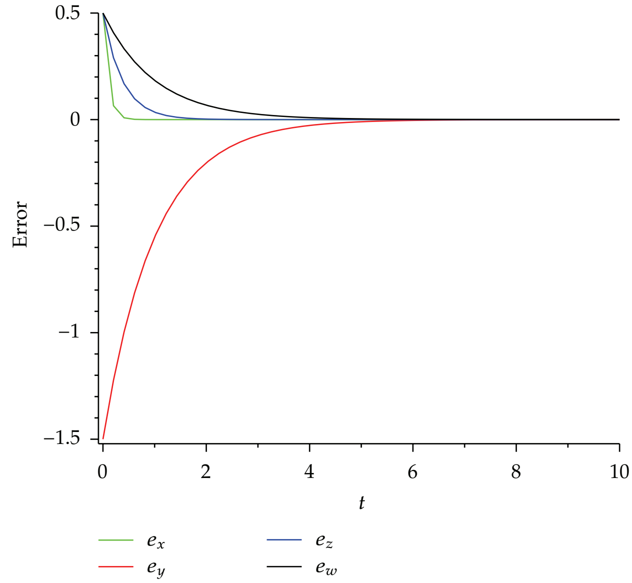

Figure 5: It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lorenz system error system tends to zero as t tends to 5 when the scaling factor $\alpha = 5$.

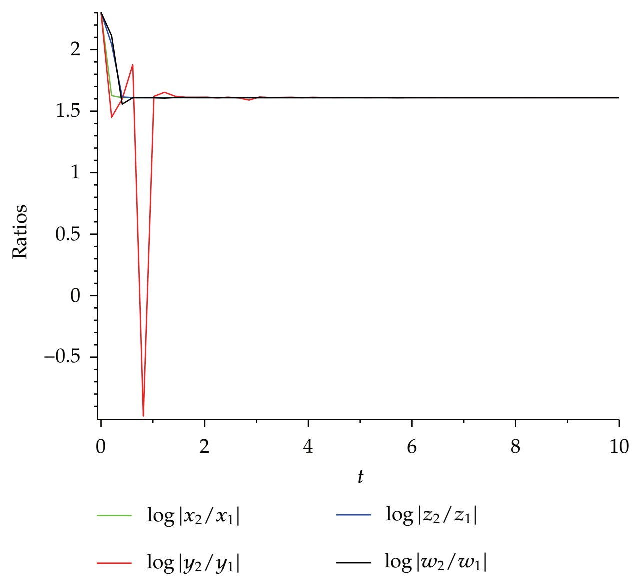

Figure 6: It shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to to $\log 5 = 1.609$.

5. Generalized Projective Synchronization for Hyperchaotic Lü System

In order to observe generalized projective synchronization between two identical hyperchaotic Lü systems, we assume that the drive system with four state variables denoted subscript 1 and the response system having identical equations denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response system. The drive and response systems are defined below, respectively,

$$\begin{aligned}
\dot{x}_1 &= a_1(y_1 - x_1) + w_1,\\
\dot{y}_1 &= -x_1 z_1 + c_1 y_1,\\
\dot{z}_1 &= -b_1 z_1 + x_1 y_1,\\
\dot{w}_1 &= d_1 w_1 + x_1 z_1,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
\dot{x}_2 &= a_1(y_2 - x_2) + w_2 + u_1,\\
\dot{y}_2 &= -x_2 z_2 + c_1 y_2 + u_2,\\
\dot{z}_2 &= -b_1 z_2 + x_2 y_2 + u_3,\\
\dot{w}_2 &= d_1 w_2 + x_2 z_2 + u_4,
\end{aligned} \tag{5.2}$$

where $U = [u_1 \quad u_2 \quad u_3 \quad u_4]^T$ is the controller functions. The controller U is to be determined for the purpose of projective synchronizing the two identical hyperchaotic Lü systems.

In order to get generalized projective synchronization, we define the error system as the difference between the systems (5.2) and (5.1). Set

$$e_x = x_2 - \alpha x_1, \qquad e_y = y_2 - \alpha y_1, \qquad e_z = z_2 - \alpha z_1, \qquad e_w = w_2 - \alpha w_1, \tag{5.3}$$

then one obtains the error dynamical system between (5.2) and(5.1)

$$\begin{aligned}
\dot{e}_x &= a_1(e_y - e_x) + e_w + u_1,\\
\dot{e}_y &= c_1 e_y - x_2 z_2 + \alpha x_1 z_1 + u_2,\\
\dot{e}_z &= -b_1 e_z + x_2 y_2 - \alpha x_1 y_1 + u_3,\\
\dot{e}_w &= d_1 e_w + x_2 z_2 - \alpha x_1 z_1 + u_4.
\end{aligned} \tag{5.4}$$

Let

$$\begin{aligned}
V_1 &= u_1,\\
V_2 &= \alpha x_1 z_1 - x_2 z_2 + u_2,\\
V_3 &= x_2 y_2 - \alpha x_1 y_1 + u_3,\\
V_4 &= x_2 z_2 - \alpha x_1 z_1 + u_4,
\end{aligned} \tag{5.5}$$

then the error dynamical system can be rewritten as

$$\begin{aligned}
\dot{e}_x &= a_1(e_y - e_x) + e_w + V_1,\\
\dot{e}_y &= c_1 e_y + V_2,\\
\dot{e}_z &= -b_1 e_z + V_3,\\
\dot{e}_w &= d_1 e_w + V_4.
\end{aligned} \tag{5.6}$$

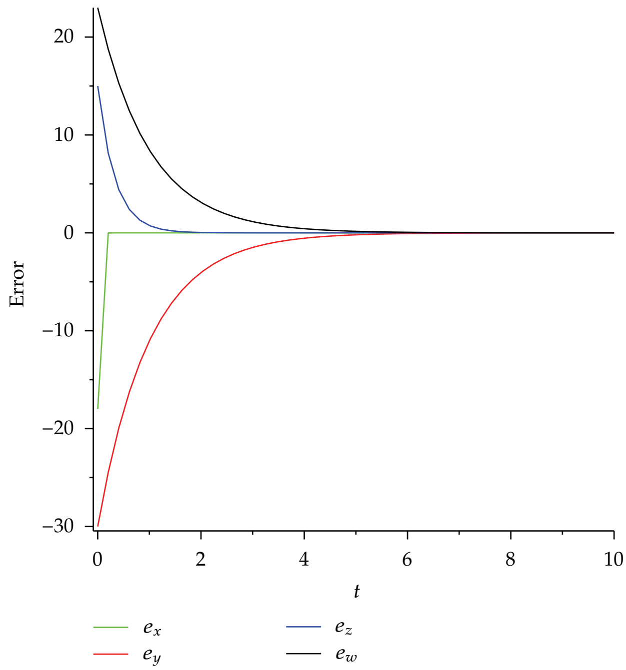

Figure 7: It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lü system error system tends to zero as t tends to 5 when the scaling factor $\alpha = -2$.

To get the projective synchronization, the zero solutions of error system must be stable, that is to say, the error evolution of the drive system and response system tends to zero as $t \to \infty$. As we know, if all the eigenvalues of the Jacobian matrix of closed-loop system have negative real parts, the system is stable. Based on this theory, we desired the $(V_1, V_2, V_3, V_4)^T$ to guarantee that all the eigenvalues of closed-loop system (5.6) have negative real part. There are of course some other choices of $(V_1, V_2, V_3, V_4)^T$, but here the choice is very easy and convenient. For simplicity, choose $(V_1, V_2, V_3, V_4)^T$ as follows:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = M \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 0 & -a_1 & 0 & -1 \\ 0 & -2c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2d \end{pmatrix}. \tag{5.7}$$

System (5.6) has four negative eigenvalues $-36, -20, -3$, and -1.3 when setting $a_1 = 36, b_1 = 3, c_1 = 20$, and $d_1 = 1.3$. That is to say, the error states e_x, e_x, e_z, and e_w converge to zero as $t \to \infty$. So the generalized projective synchronization is achieved.

5.1. Numerical Results

By using MAPLE 12, the systems of differential equations (5.1) and (5.2) are solved numerically. The parameters are chosen as $a_1 = 36, b_1 = 3, c_1 = 20$, and $d_1 = 1.3$ in all simulations so that the hyperchaotic Lü system exhibits a chaotic behavior if no control is applied (see Figure 2). The initial states of the drive system are $x_1(0) = -7$, $y_1(0) = -12, z_1(0) = 7$, and $w_1(0) = 11$ and initial states of the response system are $x_2(0) = -4, y_2(0) = -6, z_2(0) = 1$, and $w_2(0) = 1$.

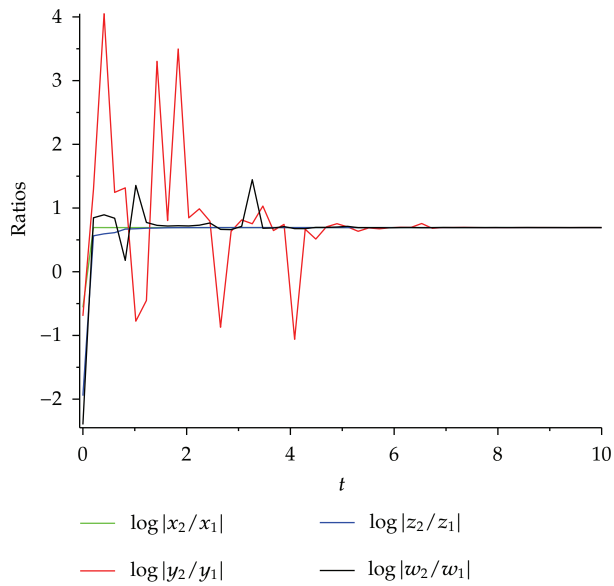

Figure 8: It shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 2 = 0.693$.

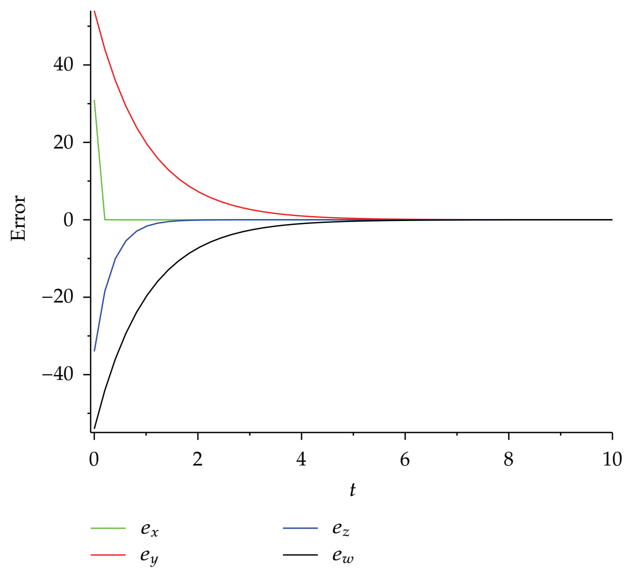

Figure 9: It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lü system error system tends to zero as t tends to 5 when the scaling factor $\alpha = 5$.

Choosing $\alpha = -2$ then the error system (5.4) has the initial values $e_x(0) = -18$, $e_y(0) = -30$, $e_z(0) = -13$, and $e_w(0) = 21$. Figure 7 shows that the trajectories of $e_x(t)$, $e_y(t)$, $e_z(t)$, and $e_w(t)$ tended to zero after $t \geq 5$. Figure 8 shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 2 = 0.693$.

Choosing $\alpha = 5$ then the error system (5.4) has the initial values $e_x(0) = 31$, $e_y(0) = 54$, $e_z(0) = -34$, and $e_w(0) = -54$. Figure 9 shows that the trajectories of $e_x(t), e_y(t), e_z(t)$,

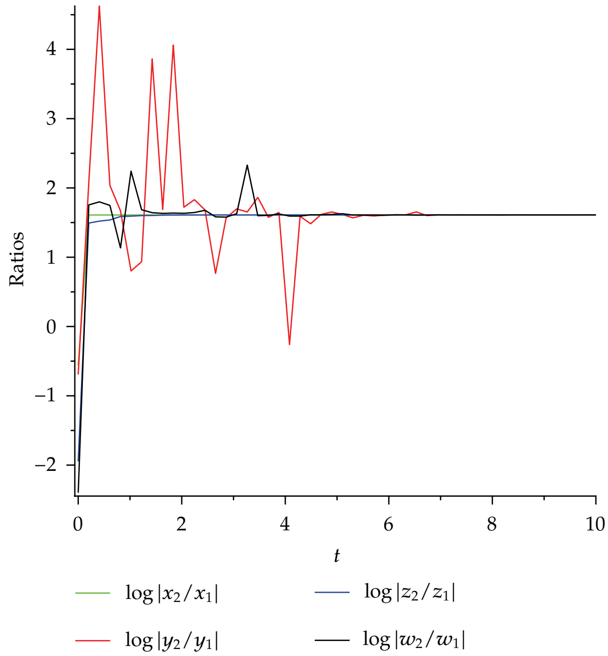

Figure 10: It shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to to $\log 5 = 1.609$.

and $e_w(t)$ tended to zero after $t \geq 5$. Figure 10 shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 5 = 1.609$.

6. Generalized Projective Synchronization between Hyperchaotic Lorenz System and Hyperchaotic Lü System

In order to observe generalized projective synchronization between hyperchaotic Lorenz system and hyperchaotic Lü system, we assume that hyperchaotic Lorenz system is the drive system and hyperchaotic Lü system is the response system. The drive system with four state variables denoted by the subscript 1 and the response system with four state variables denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response system. The drive and response systems are defined below, respectively,

$$\begin{aligned}
\dot{x}_1 &= a(y_1 - x_1) + w_1, \\
\dot{y}_1 &= -x_1 z_1 + r x_1 - y_1, \\
\dot{z}_1 &= -b z_1 + x_1 y_1, \\
\dot{w}_1 &= d w_1 - x_1 z_1,
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
\dot{x}_2 &= a_1(y_2 - x_2) + w_2 + u_1, \\
\dot{y}_2 &= -x_2 z_2 + c_1 y_2 + u_2, \\
\dot{z}_2 &= -b_1 z_2 + x_2 y_2 + u_3, \\
\dot{w}_2 &= d_1 w_2 + x_2 z_2 + u_4,
\end{aligned} \tag{6.2}$$

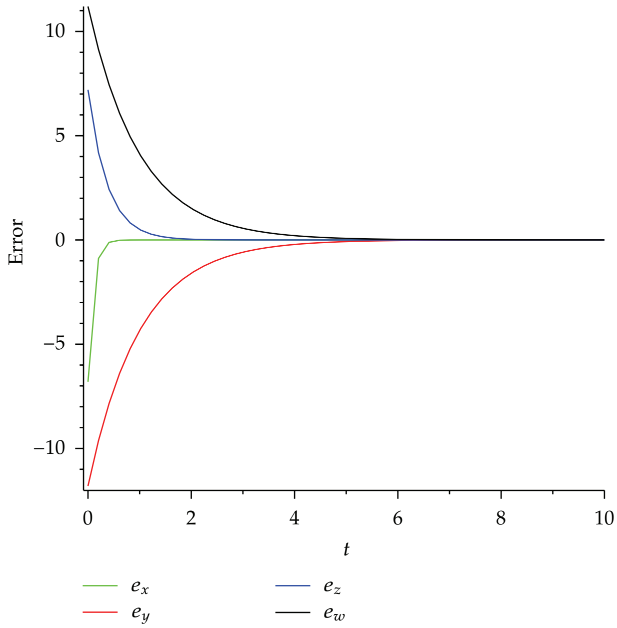

Figure 11: It shows the behaviour of the trajectories e_x, e_y, e_z, and e_w of the error system between hyperchaotic Lorenz system and hyperchaotic Lü system tends to zero as t tends to 5 when the scaling factor $\alpha = -2$.

where $U = [u_1 \quad u_2 \quad u_3 \quad u_4]^T$ is the controller functions. The controller U is to be determined for the purpose of projective synchronizing between hyperchaotic Lorenz system and hyperchaotic Lü system.

In order to get generalized projective synchronization, we define the error system as the difference between the systems (6.2) and (6.1). Set

$$e_x = x_2 - \alpha x_1, \qquad e_y = y_2 - \alpha y_1, \qquad e_z = z_2 - \alpha z_1, \qquad e_w = w_2 - \alpha w_1, \tag{6.3}$$

then one obtains the error dynamical system between (6.2) and(6.1)

$$\begin{aligned}
\dot{e}_x &= a_1(y_2 - x_2) + w_2 - a\alpha(y_1 - x_1) - \alpha w_1 + u_1, \\
\dot{e}_y &= -x_2 z_2 + c_1 y_2 + \alpha x_1 z_1 - \alpha r x_1 + \alpha y_1 + u_2, \\
\dot{e}_z &= -b_1 z_2 + x_2 y_2 + \alpha b z_1 - \alpha x_1 y_1 + u_3, \\
\dot{e}_w &= d_1 w_2 + x_2 z_2 - \alpha d w_1 + \alpha x_1 z_1 + u_4.
\end{aligned} \tag{6.4}$$

Let

$$\begin{aligned}
V_1 &= (a - a_1)(x_2 - y_2) + u_1, \\
V_2 &= \alpha x_1 z_1 - x_2 z_2 + (c_1 + 1) y_2 - r x_2 + u_2, \\
V_3 &= x_2 y_2 - \alpha x_1 y_1 - (b_1 - b) z_2 + u_3, \\
V_4 &= (d_1 + d) w_2 + x_2 z_2 + u_4 + \alpha x_1 z_1,
\end{aligned} \tag{6.5}$$

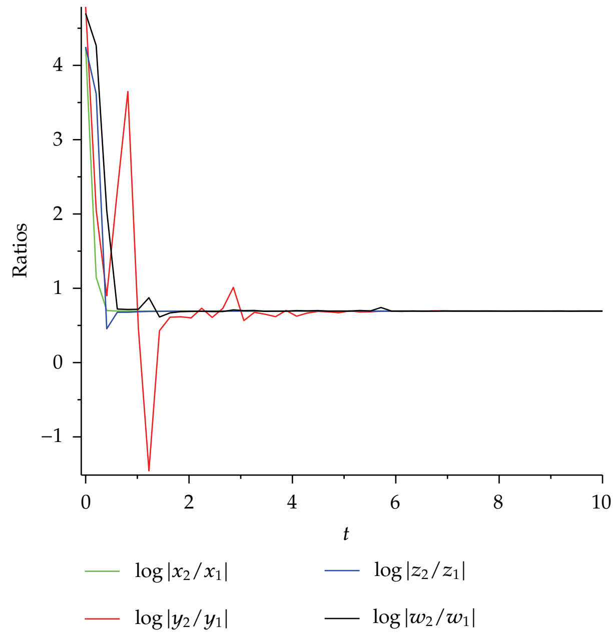

Figure 12: It shows that the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 2 = 0.693$.

then the error dynamical system can be rewritten as

$$\begin{aligned}
\dot{e}_x &= a(e_y - e_x) + e_w + V_1 \\
\dot{e}_y &= re_x - e_y + V_2 \\
\dot{e}_z &= -be_z + V_3 \\
\dot{e}_w &= -de_w + V_4.
\end{aligned} \tag{6.6}$$

To get the projective synchronization, the zero solutions of error system must be stable, that is to say, the error evolution of the drive system and response system tends to zero as $t \to \infty$. As we know, if all the eigenvalues of the Jacobian matrix of closed loop system have negative real parts, the system is stable. Based on this theory, we desired the $(V_1, V_2, V_3, V_4)^T$ to guarantee that all the eigenvalues of closed-loop system (6.7) have negative real part. There are of course some other choices of $(V_1, V_2, V_3, V_4)^T$, but here the choice is very easy and convenient. For simplicity, choose $(V_1, V_2, V_3, V_4)^T$ as follows:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = M \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 0 & -a & 0 & -1 \\ -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.7}$$

System (6.7) has four negative eigenvalues $-10, -1, -8/3$, and -1.3 when setting $a = 10, r = 28,\ b = 8/3$, and $d = 1.3$. That is to say, the error states e_x, e_x, e_z, and e_w converge to zero as $t \to \infty$. So the generalized projective synchronization is achieved.

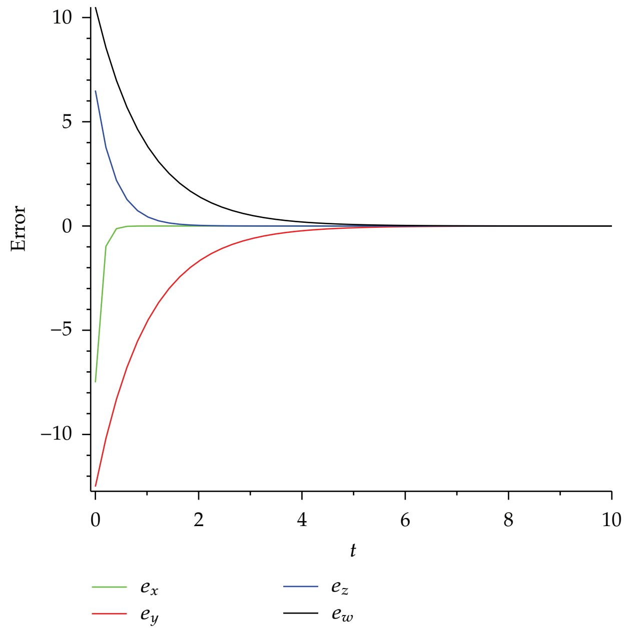

Figure 13: It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the error system between hyperchaotic Lorenz system and hyperchaotic Lü system tends to zero as t tends to 5 when the scaling factor $\alpha = 5$.

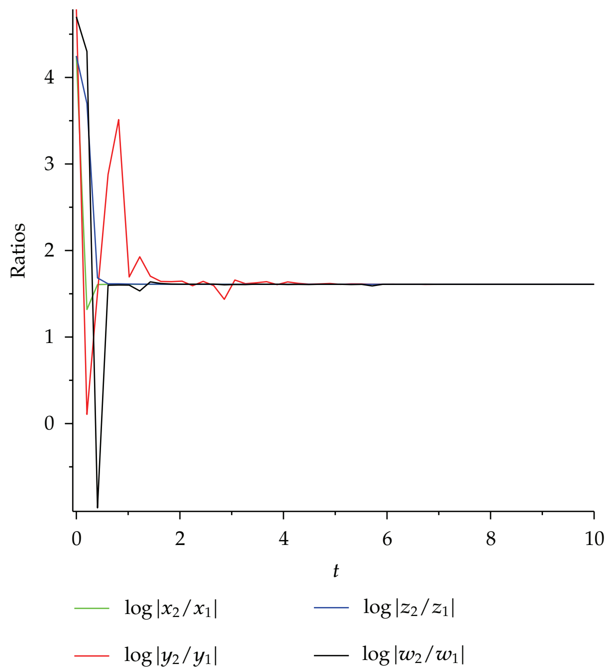

Figure 14: It shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 5 = 1.609$.

6.1. Numerical Results

By using MAPLE 12, the systems of differential equations (6.1) and (6.2) are solved numerically. The initial states of the drive system are $x_1(0) = 0.1$, $y_1(0) = 0.1$, $z_1(0) = 0.1$, and

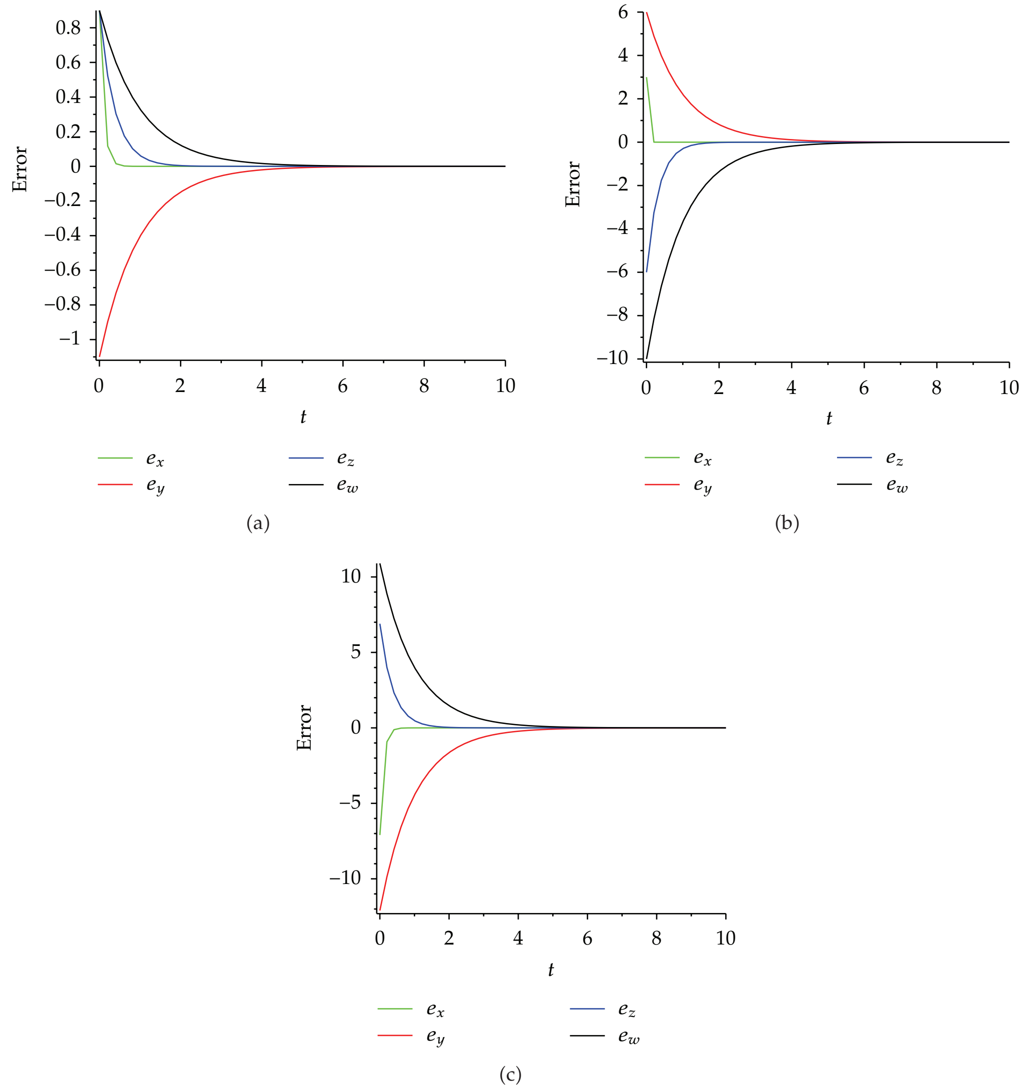

Figure 15: (a) It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lorenz system error system tends to zero as t tends to 5 when the scaling factor $\alpha = 1$. (b) It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lü system error system tends to zero as t tends to 5 when the scaling factor $\alpha = 1$. (c) It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the error system between hyperchaotic Lorenz system and hyperchaotic Lü system tends to zero as t tends to 5 when the scaling factor $\alpha = 1$.

$w_1(0) = 0.1$, and initial states of the response system are $x_2(0) = -7, y_2(0) = -12, z_2(0) = 7$, and $w_2(0) = 11$.

Choosing $\alpha = -2$, then the error system has the initial values $e_x(0) = -6.8, e_y(0) = -11.8, e_z(0) = 7.2$, and $e_w(0) = 11.2$. Figure 11 shows that the trajectories of $e_x(t), e_y(t), e_z(t)$, and $e_w(t)$ tended to zero after $t \geq 5$. Figure 12 shows the evaluation of the ratios $\log|x_2/x_1|, \log|y_2/y_1|, \log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 2 = 0.693$.

Choosing $\alpha = 5$, then the error system has the initial values $e_x(0) = -7.5$, $e_y(0) = -12.5$, $e_z(0) = 6.5$, and $e_w(0) = 10.5$. Figure 13 shows that the trajectories of $e_x(t), e_y(t), e_z(t)$,

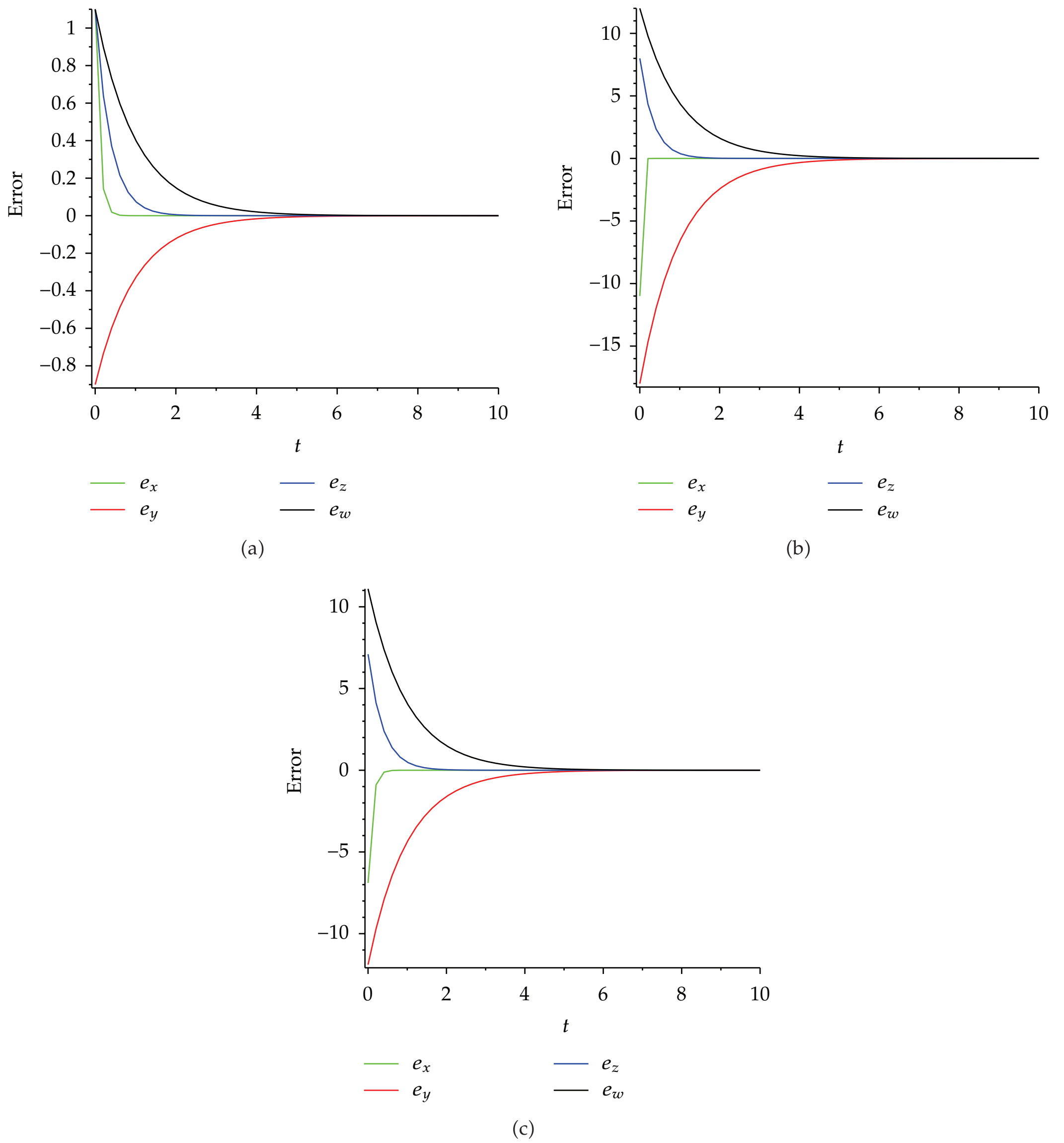

Figure 16: (a) Its shows the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lorenz system error system tends to zero as t tends to 5 when the scaling factor $\alpha = -1$. (b) It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the hyperchaotic Lü system error system tends to zero as t tends to 5 when the scaling factor $\alpha = -1$. (c) It shows that the behaviour of the trajectories e_x, e_y, e_z, and e_w of the error system between hyperchaotic Lorenz system and hyperchaotic Lü system tends to zero as t tends to 5 when the scaling factor $\alpha = -1$.

and $e_w(t)$ tended to zero after $t \geq 5$. Figure 14 shows the evaluation of the ratios $\log|x_2/x_1|$, $\log|y_2/y_1|$, $\log|z_2/z_1|$, and $\log|w_2/w_1|$ whose limits are equal to $\log 5 = 1.609$.

7. Conclusion

This paper shows that the generalized projective synchronizations for the hyperchaotic Lorenz system and the hyperchaotic Lü system can be easily achieved by using the fundamental feedback techniques, where the scaling factors can be arbitrarily manipulated (amplified or reduced). We note that the driver and response systems achieve complete synchronization when α is equal to 1 (see Figure 15). Further, if α is equal to -1, then the two systems are said to be in anti-synchronization (see Figure 16). Numerical simulations are used to verify the effectiveness of the proposed generalized projective synchronization techniques.

References

[1] E. Ott, C. Grebogi, and J. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.

[2] A. W. Hubler, "Adaptive control of chaotic system," *Helvetica Physica Acta*, vol. 62, pp. 343–346, 1989.

[3] K. Pyragas, "Continuous control of chaos by self-controlling feedback," *Physics Letters A*, vol. 170, no. 6, pp. 421–428, 1992.

[4] G. Chen and X. Dong, "On feedback control of chaotic nonlinear dynamic systems," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 2, no. 2, pp. 407–411, 1992.

[5] G. Chen, "On some controllability conditions for chaotic dynamics control," *Chaos, Solitons and Fractals*, vol. 8, no. 9, pp. 1461–1470, 1997.

[6] K. Pyragas, "Control of chaos via extended delay feedback," *Physics Letters. A*, vol. 206, no. 5-6, pp. 323–330, 1995.

[7] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[8] T. L. Carroll and L. M. Pecora, "Synchronizing a chaotic systems," *IEEE Transactions on Circuits and Systems*, vol. 38, no. 4, pp. 453–456, 1991.

[9] J. M. Gonzăalez-Miranda, "Chaotic systems with a null conditional Lyapunov exponent under nonlinear driving," *Physical Review E*, vol. 53, no. 1, pp. R5–R8, 1996.

[10] R. Mainieri and J. Rehacek, "Projective synchronization in three-dimensional chaotic systems," *Physical Review Letters*, vol. 82, no. 15, pp. 3042–3045, 1999.

[11] L. Zhigang and X. Daolin, "Stability criterion for projective synchronization in three-dimensional chaotic systems," *Physics Letters A*, vol. 282, no. 3, pp. 175–179, 2001.

[12] X. Daolin and L. Zhigang, "Controlled projective synchronization in nonpartially-linear chaotic systems," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 12, no. 6, pp. 1395–1402, 2002.

[13] X. Daolin, Ong Wee-leng, and L. Zhigang, "Criteria for the occurrence of projective synchronization in chaotic systems of arbitrary dimension," *Physics Letters A*, vol. 305, no. 3-4, pp. 167–172, 2002.

[14] J.-p. Yan and C.-p. Li, "Generalized projective synchronization for the chaotic Lorenz system and the chaotic Chen system," *Journal of Shanghai University*, vol. 10, no. 4, pp. 299–304, 2006.

[15] Z.-M. Ge and C.-H. Yang, "The generalized synchronization of a quantum—CNN chaotic oscillator with different order systems," *Chaos, Solitons & Fractals*, vol. 35, pp. 980–990, 2008.

[16] J. Liu, S. H. Chen, and J. A. Lu, "Projective synchronization in a unified chaotic system and its control," *Acta Physica Sinica*, vol. 52, pp. 1595–1599, 2003.

[17] G.-H. Li, "Generalized projective synchronization of two chaotic systems by using active control," *Chaos, Solitons and Fractals*, vol. 30, no. 1, pp. 77–82, 2006.

[18] J. H. Park, "Adaptive control for modified projective synchronization of a four—dimensional chaotic system with uncertain parameters," *Journal of Computational and Applied Mathematics*, vol. 213, pp. 288–293, 2008.

[19] Z.-M. Ge and G.-H. Lin, "The complete, lag and anticipated synchronization of a BLDCM chaotic system," *Chaos, Solitons and Fractals*, vol. 34, no. 3, pp. 740–764, 2007.

[20] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. Abarbanel, "Generalized synchronization of chaos in directionally coupled chaotic systems," *Physical Review E*, vol. 51, no. 2, pp. 980–994, 1995.

[21] Y. L. Zou and J. Zhu, "Controlling projective synchronizationin coupled chaotic systems," *Chinese Physics*, vol. 15, no. 9, pp. 1965–1970, 2006.

[22] L. Kocarev and U. Parlitz, "Synchrionizing spatiotemporal chaos in coupled nonlinear oscillators," *Physical Review Letters*, vol. 77, pp. 2206–2209, 1996.

[23] H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, "Generalized synchronization of chaos: the auxiliary system approach," *Physical Review E*, vol. 53, pp. 4528–4535, 1996.

[24] D. Huang, "Simple adaptive feedback controller for identical chaos synchronization," *Physical Review E*, vol. 71, pp. 37–203, 2005.

[25] D. Huang, "Adaptive-feedback control algorithm," *Physical Review E*, vol. 73, no. 6, pp. 66–204, 2006.

[26] M. Hu and Z. Xu, "Adaptive feedback controller for projective synchronization," *Nonlinear Analysis*, vol. 9, no. 3, pp. 1253–1260, 2008.

[27] J. Cao and J. Lu, "Adaptive synchronization of neural networks with or without time-varying delay," *Chaos*, vol. 16, no. 1, pp. 13–101, 2006.

[28] M. Chen and D. Zhou, "Synchronization in uncertain complex networks," *Chaos*, vol. 16, no. 1, pp. 13–101, 2006.

[29] Q. Jia, "Hyperchaos generated from the Lorenz chaotic system and its control," *Physics Letters A*, vol. 366, no. 3, pp. 217–222, 2007.

[30] X. Wang and M. Wan, "A hyper chaos generated from Lorenz system," *Physica A*, vol. 387, pp. 3751–3758, 2008.

[31] Y. Li, W. K. S. Tang, and G. Chen, "Hyperchaos evolved from the generalized Lorenz equation," *International Journal of Circuit Theory and Applications*, vol. 33, no. 4, pp. 235–251, 2005.

[32] A. Chen, J. Lu, J. Lü, and S. Yu, "Generating hyperchaotic Lü attractor via state feedback control," *Physics A*, vol. 364, pp. 103–110, 2006.