Research Article

Asymptotic Behaviors of a Delayed Nonautonomous Predator-Prey System Governed by Difference Equations

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Based on a predator-prey differential system with continuously distributed delays, we derive a corresponding difference version by using the method of a discretization technique. A good understanding of permanence of system and global attractivity of positive solutions of system is gained. An example and its numerical simulations are presented to substantiate our theoretical results.

1. Introduction

It is well known that the classical Lotka-Volterra differential equations for predator-prey interaction can be written as

$$x'(t) = x(t)\left[a - bx(t) - cy(t)\right],$$
$$y'(t) = y(t)\left[-d + px(t) - qy(t)\right],$$

(1.1)

where $x(0) > 0$, $y(0) > 0$, $x(t)$, and $y(t)$ are the population densities of prey and predator, respectively, at time t, and the coefficients a, b, c, d, p, q are positive constants. a is the intrinsic growth rate of prey species, and d represents the death rate of predator species; b, q and c, p, denote their self-inhibition and interaction rates, respectively. To account for time lag effects

in their interaction, Cushing [1] suggested visualizing system (1.1) as a special case of the following more general system

$$x'(t) = x(t)\left[a - \int_0^{+\infty} x(t-\theta)dl_1(\theta) - \int_0^{+\infty} y(t-\theta)dl_2(\theta)\right],$$

$$y'(t) = y(t)\left[-d + \int_0^{+\infty} x(t-\theta)dl_3(\theta) - \int_0^{+\infty} y(t-\theta)dl_4(\theta)\right],$$

(1.2)

where $l_i(\theta)$ are nondecreasing functions on $[0, +\infty)$ with $\int_0^{+\infty} dl_i(\theta) < +\infty$ for each $i = 1, \ldots, 4$, and with $\lim_{\theta \to 0^+}[l_i(\theta) - l_i(0)]$, respectively, equal to b, c, p, q for $i = 1, \ldots, 4$.

As Hale [2] pointed out, one expects that the lag effects tend to diminish gradually in ever-moderating pace as one moves backward in time after the initial instantaneous effect described by the assumption $\lim_{\theta \to 0^+}[l_i(\theta) - l_i(0)] > 0$. The effects are negligible after a certain length of time $\tau > 0$. Mathematically, for $\theta > 0$, it is reasonable that $dl_i(\theta)$ can be assumed to be of the form $dl_i(\theta) = k_i(\theta)d\theta$, $i = 1, \ldots, 4$, where

(S_1) $k_i(s) \geq 0$ and continuous for $s \in [0, \tau]$; $k_i(s) = 0$ for $s \in [\tau, +\infty]$;

(S_2) $k_i'(s) \leq 0$ and continuous for $s \in (0, \tau)$; $\lim_{s \to 0^+} k_i'(s)$ and $\lim_{s \to \tau^-} k_i'(s)$ both exist;

(S_3) $k_i''(s) \geq 0$ and continuous for $s \in (0, \tau)$.

The assumption (S_2) represents gradual diminishing effect as one moves backward in time, and the assumption (S_3) stands for the moderating pace of the decrease (see [2] or [3] for the assumptions (S_1) to (S_3)). Leung [4] revised system (1.2) and established the following predator-prey model with continuously distributed delays

$$x'(t) = x(t)\left[a - bx(t) - cy(t) - \int_0^{\tau} k_1(s)x(t-s)ds - \int_0^{\tau} k_2(s)y(t-s)ds\right],$$

$$y'(t) = y(t)\left[-d + px(t) - qy(t) + \int_0^{\tau} k_3(s)x(t-s)ds - \int_0^{\tau} k_4(s)y(t-s)ds\right].$$

(1.3)

The parameters of system (1.3) are assumed to be constant; however, in the real world the parameters are not fixed constants owing to the variation of environment. The effect of a varying environment is significant for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, it is realistic to assume that the parameters of system (1.3) are continuous functions with respect to t, and then we obtain the following form:

$$x'(t) = x(t)\left[a(t) - b(t)x(t) - c(t)y(t) - \int_0^{\tau} k_1(s)x(t-s)ds - \int_0^{\tau} k_2(s)y(t-s)ds\right],$$

$$y'(t) = y(t)\left[-d(t) + p(t)x(t) - q(t)y(t) + \int_0^{\tau} k_3(s)x(t-s)ds - \int_0^{\tau} k_4(s)y(t-s)ds\right],$$

(1.4)

where the coefficients $a(t), b(t), c(t), d(t), p(t), q(t)$ are positive continuous functions and $k_i(s)ds$ satisfy $\int_0^{\tau} k_i(s)ds = 1$ for $i = 1, \ldots, 4$.

Although much progress on the predator-prey models with discrete or distributed delays has been made, such models are not sufficiently researched in the sense that most results are continuous time versions related (see [5–13]). Many authors have argued that the discrete time models governed by difference equations are more appropriate than the continuous time ones when populations have a short life expectancy, nonoverlapping generations in the real world (see [14–18]). Meanwhile, discrete time models can provide efficient computational models of continuous time models for numerical simulations. To the best of our knowledge, no such work has been done for the corresponding discrete version of system (1.4).

In the following, we employ the discretization technique to derive the discrete version of system (1.4). Throughout this paper, let $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{R}^2$ denote the sets of all integers, nonnegative integers, and two-dimensional vector space, respectively. We begin to approximate the continuous time system by replacing the integral term with discrete sums of the form

$$
\int_0^\tau k_i(s)x(t-s)ds \approx \sum_{[s/h]=0}^{M} k_i\left(\left[\frac{s}{h}\right]h\right)x\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)w(h) \tag{1.5}
$$

for $t \in [nh, (n+1)h)$, $s \in [rh, (r+1)h)$, $n, r \in \mathbb{Z}^+$, $i = 1, \ldots, 4$, where $[\cdot]$ denotes the greatest integer function. $M = [\tau/h]$, $w(h) = h + O(h)$ for $h > 0$, where the fixed number h denotes an uniform discretization step size. We approximate system (1.4) by differential equations with piecewise constant arguments of the form

$$
x'(t) = x(t)\left[a\left(\left[\frac{t}{h}\right]h\right) - b\left(\left[\frac{t}{h}\right]h\right)x\left(\left[\frac{t}{h}\right]h\right) - c\left(\left[\frac{t}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h\right)\right.
$$

$$
\left. - \sum_{[s/h]=0}^{M} k_1\left(\left[\frac{s}{h}\right]h\right)x\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)h - \sum_{[s/h]=0}^{M} k_2\left(\left[\frac{s}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)h\right],
$$

$$
y'(t) = y(t)\left[-d\left(\left[\frac{t}{h}\right]h\right) + p\left(\left[\frac{t}{h}\right]h\right)x\left(\left[\frac{t}{h}\right]h\right) - q\left(\left[\frac{t}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h\right)\right.
$$

$$
\left. + \sum_{[s/h]=0}^{M} k_3\left(\left[\frac{s}{h}\right]h\right)x\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)h - \sum_{[s/h]=0}^{M} k_4\left(\left[\frac{s}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)h\right]
$$

(1.6)

for $t \in [nh, (n+1)h)$, $s \in [rh, (r+1)h)$, $n, r \in \mathbb{Z}^+$. Noting that $[t/h] = n$, $[s/h] = r$, we integrate (1.6) over $[nh, t)$, where $t < (n+1)h$, then (1.6) can be reformulated as

$$
x'(t) = x(t)\left[a(nh) - b(nh)x(nh) - c(nh)y(nh) - \sum_{r=0}^{M} k_1(rh)x(nh-rh)h - \sum_{r=0}^{M} k_2(rh)y(nh-rh)h\right]
$$

$$
y'(t) = y(t)\left[-d(nh) + p(nh)x(nh) - q(nh)y(nh) + \sum_{r=0}^{M} k_3(rh)x(nh-rh)h - \sum_{r=0}^{M} k_4(rh)y(nh-rh)h\right].
$$

(1.7)

Denoting $x(n) = x(nh), y(n) = y(nh), a(n) = a(nh), b(n) = b(nh), c(n) = c(nh), d(n) = d(nh),$
$p(n) = p(nh), q(n) = q(nh), k_i(r) = k_i(rh)h, i = 1,\ldots,4$, then we have

$$x(t) = x(n) \exp\left\{\left[a(n) - b(n)x(n) - c(n)y(n) - \sum_{r=0}^{M}k_1(r)x(n-r) - \sum_{r=0}^{M}k_2(r)y(n-r)\right](t-nh)\right\},$$

$$y(t) = y(n) \exp\left\{\left[-d(n) + p(n)x(n) - q(n)y(n) + \sum_{r=0}^{M}k_3(r)x(n-r) - \sum_{r=0}^{M}k_4(r)y(n-r)\right](t-nh)\right\}.$$

$$(1.8)$$

Setting $t \to (n+1)h$ in (1.8) and simplifying, we get a discrete time analogue of continuous time system (1.4) with the form

$$x(n+1) = x(n) \exp\left\{a(n) - b(n)x(n) - c(n)y(n) - \sum_{r=0}^{M}k_1(r)x(n-r) - \sum_{r=0}^{M}k_2(r)y(n-r)\right\}$$

$$y(n+1) = y(n) \exp\left\{-d(n) + p(n)x(n) - q(n)y(n) + \sum_{r=0}^{M}k_3(r)x(n-r) - \sum_{r=0}^{M}k_4(r)y(n-r)\right\}.$$

$$(1.9)$$

Our main purpose of this paper is to derive a set of easily verifiable sufficient conditions concerning the permanence and global attractivity of system (1.9). For convenience, we introduce the following definitions and notations.

Definition 1.1. It is said that system (1.9) is permanent if there exist positive constants m_i, M_i, $i = 1, 2$ and any positive solution $\{(x(n), y(n))^\top\}$ of system (1.9) satisfies

$$m_1 \le \lim_{n\to+\infty}\inf x(n) \le \lim_{n\to+\infty}\sup x(n) \le M_1,$$

$$m_2 \le \lim_{n\to+\infty}\inf y(n) \le \lim_{n\to+\infty}\sup y(n) \le M_2.$$

$$(1.10)$$

Definition 1.2. A positive solution $\{(x(n), y(n))^\top\}$ of system (1.9) is global attractive if each other positive solution $\{(x^*(n), y^*(n))^\top\}$ of system (1.9) satisfies

$$\lim_{n\to+\infty}|x(n) - x^*(n)| = 0, \qquad \lim_{n\to+\infty}|y(n) - y^*(n)| = 0. \tag{1.11}$$

Let $f(n)$ be a bounded sequence; we denote

$$f^u = \sup_{n\in\mathbb{Z}^+}\{f(n)\}, \qquad f^l = \inf_{n\in\mathbb{Z}^+}\{f(n)\}. \tag{1.12}$$

Meanwhile, we make a convention that $\prod_{n=m}^{n=k}f(n) = 1$ for all $m > k$.

This paper proceeds as follows. System (1.9) is analyzed to study the permanence and global attractivity of system (1.9) in the next two sections, respectively. In the final section, we

give an example, and its numerical simulations are presented to substantiate our theoretical results.

2. Permanence of System (1.9)

In this section, we devote to investigating the permanence of system (1.9). To do so, we introduce the following lemmas.

Lemma 2.1 (see [19]). *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n + 1) \leq x(n) \exp\left[r(n)(1 - ax(n))\right] \tag{2.1}$$

for $n \in [n_1, +\infty)$, *where* a *is a positive constant and* $n_1 \in \mathbb{Z}^+$. *Then*

$$\lim_{n \to +\infty} \sup x(n) \leq \frac{1}{ar^u} \exp(r^u - 1). \tag{2.2}$$

Lemma 2.2 (see [19]). *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n + 1) \geq x(n) \exp[r(n)(1 - ax(n))] \tag{2.3}$$

for $n \in [n_2, +\infty)$, $\lim\sup_{n \to +\infty} x(n) \leq H$ *and* $x(n_2) > 0$, *where* a *is a positive constant such that* $aH > 1$ *and* $n_2 \in \mathbb{Z}^+$. *Then*

$$\lim_{n \to +\infty} \inf x(n) \geq \frac{1}{a} \exp[r^u(1 - aH)]. \tag{2.4}$$

For the simplicity of description, we denote

$$
\begin{aligned}
\Delta_1^\varepsilon &= -d^l + \left[p^u + (M + 1)k_3^u\right](M_1 + \varepsilon), \\
\Delta_2^\varepsilon &= a^l - c^u(M_2 + \varepsilon) - (M + 1)\left[k_1^u(M_1 + \varepsilon) + k_2^u(M_2 + \varepsilon)\right], \\
\Delta_3^\varepsilon &= -d^u + p^l(m_1 - \varepsilon) + (M + 1)\left[k_3^l(m_1 - \varepsilon) - k_4^u(M_2 + \varepsilon)\right], \\
\Delta_i &= \lim_{\varepsilon \to 0} \Delta_i^\varepsilon, \qquad i = 1, 2, 3,
\end{aligned}
\tag{2.5}
$$

where M_1, M_2, m_1 are, respectively, defined in (2.8) and (2.11)–(2.14) and ε is a sufficient small positive constant. Now, we begin to search the conditions for the permanence of (1.9).

Theorem 2.3. *If the conditions*

$$\min\{\Delta_i, \ i = 1, 2, 3\} > 0 \tag{2.6}$$

hold, then (1.9) *is permanent.*

Proof. It is easy to verify that $\Omega = \{(x(n), y(n))^\top \mid x(n) > 0, \ y(n) > 0\}$ is a positive invariant set of (1.9) with $x(0) > 0$, $y(0) > 0$. Suppose that $\{(x(n), y(n))^\top\}$ is any positive solution of (1.9), we prove Theorem 2.3 by the following two steps.

Step 1. We show that $\{(x(n), y(n))^\top\}$ is uniformly ultimately upper bounded.
From the first equation of (1.9), we have

$$x(n+1) \le x(n) \exp[a(n) - b(n)x(n)] \le x(n) \exp\left\{a(n)\left[1 - \frac{b^l}{a^u}x(n)\right]\right\}. \qquad (2.7)$$

Applying Lemma 2.1, we can obtain that

$$\lim_{n \to +\infty} \sup x(n) \le \frac{\exp(a^u - 1)}{b^l} \overset{\text{def}}{=} M_1. \qquad (2.8)$$

From (2.8), there exists a sufficient large $n_1 \in \mathbb{Z}^+$ and any constant $\varepsilon > 0$ such that

$$x(n) \le M_1 + \varepsilon, \qquad n \ge n_1. \qquad (2.9)$$

It follows from the second equation of (1.9) that

$$y(n+1) \le y(n) \exp\left\{-d^l + [p^u + (M+1)k_3^u](M_1 + \varepsilon) - q^l y(n)\right\}$$

$$= y(n) \exp\left\{\Delta_1^\varepsilon\left[1 - \frac{q^l}{\Delta_1^\varepsilon}y(n)\right]\right\}. \qquad (2.10)$$

So by Lemma 2.1 and letting $\varepsilon \to 0$ in (2.10), we have

$$\lim_{n \to +\infty} \sup y(n) \le \frac{\exp(\Delta_1 - 1)}{q^l} \overset{\text{def}}{=} M_2. \qquad (2.11)$$

Step 2. We prove that $\{(x(n), y(n))^\top\}$ is uniformly ultimately lower bounded.
For any sufficient small $\varepsilon > 0$, it follows from (2.6) that $\Delta_2^\varepsilon > 0$. According to (2.11), there exists an $n_0 > n_1$ such that $y(n) \le M_2 + \varepsilon$ for $n \ge n_0$ and the above constant ε. By the first equation of (1.9), it gives that

$$x(n+1) \ge x(n) \exp\left\{a^l - c^u(M_2 + \varepsilon) - (M+1)\left[k_1^u(M_1 + \varepsilon) + k_2^u(M_2 + \varepsilon)\right] - b^u x(n)\right\}$$

$$= x(n) \exp\left\{\Delta_2^\varepsilon\left[1 - \frac{b^u}{\Delta_2^\varepsilon}x(n)\right]\right\}. \qquad (2.12)$$

Note that

$$\frac{b^u}{\Delta_2^\varepsilon}M_1 = \frac{b^u}{\Delta_2^\varepsilon}\frac{\exp(a^u - 1)}{b^l} \ge \frac{b^u}{a^l}\frac{\exp(a^u - 1)}{b^l} > \frac{\exp(a^u - 1)}{a^u} > 1, \qquad (2.13)$$

where we employ the inequality result that $\exp(x - 1) > x$ for $x > 0$. Therefore, applying Lemma 2.2 and setting $\varepsilon \to 0$, one has

$$\lim_{n \to +\infty} \inf x(n) \geq \frac{\Delta_2}{b^u} \exp\left[\Delta_2\left(1 - \frac{b^u}{\Delta_2}M_1\right)\right] \stackrel{\text{def}}{=} m_1. \tag{2.14}$$

For the above constant $\varepsilon > 0$, it follows from (2.14) that there exists a sufficient large $n_1^* \in \mathbb{Z}^+$ such that

$$x(n) \geq m_1 - \varepsilon, \quad n \geq n_1^*. \tag{2.15}$$

From the second equation of (1.9), we have

$$y(n + 1) \geq y(n) \exp\left\{-d^u + p^l(m_1 - \varepsilon) + (M + 1)\left[k_3^l(m_1 - \varepsilon) - k_4^u(M_2 + \varepsilon)\right] - q^u y(n)\right\}$$

$$= y(n) \exp\left\{\Delta_3^\varepsilon\left[1 - \frac{q^u}{\Delta_3^\varepsilon}y(n)\right]\right\}. \tag{2.16}$$

Clearly,

$$\frac{q^u}{\Delta_3^\varepsilon}M_2 = \frac{q^u}{\Delta_3^\varepsilon}\frac{\exp(\Delta_1 - 1)}{q^l} \geq \frac{q^l \exp\left\{-d^u + \left[p^l + (M + 1)k_3^l\right]m_1 - 1\right\}}{\left[-d^u + \left(p^l + (M + 1)k_3^l\right)m_1\right]q^l} > 1. \tag{2.17}$$

Thus, applying Lemma 2.2 and letting $\varepsilon \to 0$, one has

$$\lim_{n \to +\infty} \inf y(n) \geq \frac{\Delta_3}{q^u} \exp\left[\Delta_3\left(1 - \frac{q^u}{\Delta_3}M_2\right)\right] \stackrel{\text{def}}{=} m_2. \tag{2.18}$$

Based on the above Step 1 with Step 2, we can see that system (1.9) is permanent. This completes the proof of Theorem 2.3. □

3. Global Attractivity of Positive Solutions of System (1.9)

In this section, we investigate the global attractivity of positive solutions of system (1.9).

Theorem 3.1. *Assume that there exists a positive constant η such that*

$$\min\left\{b^l, \frac{2}{M_1} - b^u\right\} - (M + 1)\left(k_1^u + k_3^u\right) - p^u \geq \eta,$$

$$\min\left\{q^l, \frac{2}{M_2} - q^u\right\} - (M + 1)\left(k_2^u + k_4^u\right) - c^u \geq \eta \tag{3.1}$$

hold. Then any positive solution $\{(x(n), y(n))^\top\}$ of system (1.9) is global attractive.

Proof. Denote $\{(x^*(n), y^*(n))^\top\}$ be any other positive solution of system (1.9). Let $V_{11}(n) = |\ln x(n) - \ln x^*(n)|$, then it follows from the first equation of (1.9) that

$$
\begin{aligned}
V_{11}(n+1) &= |\ln x(n+1) - \ln x^*(n+1)| \\
&\leq |\ln x(n) - \ln x^*(n) - b(n)(x(n) - x^*(n))| \\
&\quad + c(n)|y(n) - y^*(n)| \\
&\quad + \sum_{r=0}^{M} k_1(r)|x(n-r) - x^*(n-r)| \\
&\quad + \sum_{r=0}^{M} k_2(r)|y(n-r) - y^*(n-r)|.
\end{aligned}
\tag{3.2}
$$

By the Mean Value theorem, we get

$$
\ln x(n) - \ln x^*(n) = \frac{1}{\theta_1(n)}(x(n) - x^*(n)),
\tag{3.3}
$$

where $\theta_1(n)$ lies between $x(n)$ and $x^*(n)$. Then we have

$$
\begin{aligned}
&|\ln x(n) - \ln x^*(n) - b(n)(x(n) - x^*(n))| \\
&\quad = |\ln x(n) - \ln x^*(n)| - |\ln x(n) - \ln x^*(n)| + |\ln x(n) - \ln x^*(n) - b(n)(x(n) - x^*(n))| \\
&\quad = |\ln x(n) - \ln x^*(n)| - \frac{1}{\theta_1(n)}|x(n) - x^*(n)| + \left| \frac{1}{\theta_1(n)}(x(n) - x^*(n)) - b(n)(x(n) - x^*(n)) \right| \\
&\quad = |\ln x(n) - \ln x^*(n)| - \left(\frac{1}{\theta_1(n)} - \left| \frac{1}{\theta_1(n)} - b(n) \right| \right)|x(n) - x^*(n)|.
\end{aligned}
\tag{3.4}
$$

Combining (3.2) with (3.4), we have

$$
\begin{aligned}
\Delta V_{11}(n) &= V_{11}(n+1) - V_{11}(n) \\
&\leq -\left(\frac{1}{\theta_1(n)} - \left| \frac{1}{\theta_1(n)} - b(n) \right| \right)|x(n) - x^*(n)| + c(n)|y(n) - y^*(n)| \\
&\quad + \sum_{r=0}^{M} k_1(r)|x(n-r) - x^*(n-r)| + \sum_{r=0}^{M} k_2(r)|y(n-r) - y^*(n-r)|.
\end{aligned}
\tag{3.5}
$$

Next, we let

$$
V_{12}(n) = \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_1(r)|x(s) - x^*(s)| + \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_2(r)|y(s) - y^*(s)|.
\tag{3.6}
$$

Then we have

$$\Delta V_{12}(n) = V_{12}(n+1) - V_{12}(n)$$

$$= \sum_{r=0}^{M} \sum_{s=n+1-r}^{n} k_1(r)|x(s) - x^*(s)| - \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_1(r)|x(s) - x^*(s)|$$

$$+ \sum_{r=0}^{M} \sum_{s=n+1-r}^{n} k_2(r)|y(s) - y^*(s)| - \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_2(r)|y(s) - y^*(s)|$$

$$= \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_1(r)|x(s) - x^*(s)| + \sum_{r=0}^{M} k_1(r)|x(n) - x^*(n)|$$

$$- \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_1(r)|x(s) - x^*(s)| - \sum_{r=0}^{M} k_1(r)|x(n-r) - x^*(n-r)| \qquad (3.7)$$

$$+ \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_2(r)|y(s) - y^*(s)| + \sum_{r=0}^{M} k_2(r)|y(n) - y^*(n)|$$

$$- \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_2(r)|y(s) - y^*(s)| - \sum_{r=0}^{M} k_2(r)|y(n-r) - y^*(n-r)|$$

$$= \sum_{r=0}^{M} k_1(r)|x(n) - x^*(n)| - \sum_{r=0}^{M} k_1(r)|x(n-r) - x^*(n-r)|$$

$$+ \sum_{r=0}^{M} k_2(r)|y(n) - y^*(n)| - \sum_{r=0}^{M} k_2(r)|y(n-r) - y^*(n-r)|.$$

We set

$$V_1(n) = V_{11}(n) + V_{12}(n), \qquad (3.8)$$

then it follows from (3.5) and (3.7) that

$$\Delta V_1(n) = \Delta V_{11}(n) + \Delta V_{12}(n)$$

$$\leq -\left(\frac{1}{\theta_1(n)} - \left| \frac{1}{\theta_1(n)} - b(n) \right| - \sum_{r=0}^{M} k_1(r) \right) |x(n) - x^*(n)| \qquad (3.9)$$

$$+ \left(c(n) + \sum_{r=0}^{M} k_2(r) \right) |y(n) - y^*(n)|.$$

Similarly, we define

$$V_2(n) = V_{21}(n) + V_{22}(n), \qquad (3.10)$$

where

$$V_{21}(n) = \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_3(r)|x(s) - x^*(s)|$$

$$+ \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_4(r)|y(s) - y^*(s)|,$$

$$V_{22}(n) = |\ln y(n) - \ln y^*(n)|.$$

(3.11)

Then we have

$$\Delta V_{21}(n) = V_{21}(n+1) - V_{21}(n)$$

$$= \sum_{r=0}^{M} \sum_{s=n+1-r}^{n} k_3(r)|x(s) - x^*(s)| - \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_3(r)|x(s) - x^*(s)|$$

$$+ \sum_{r=0}^{M} \sum_{s=n+1-r}^{n} k_4(r)|y(s) - y^*(s)| - \sum_{r=0}^{M} \sum_{s=n-r}^{n-1} k_4(r)|y(s) - y^*(s)|$$

$$= \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_3(r)|x(s) - x^*(s)| + \sum_{r=0}^{M} k_3(r)|x(n) - x^*(n)|$$

$$- \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_3(r)|x(s) - x^*(s)| - \sum_{r=0}^{M} k_3(r)|x(n-r) - x^*(n-r)|$$

$$+ \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_4(r)|y(s) - y^*(s)| + \sum_{r=0}^{M} k_4(r)|y(n) - y^*(n)|$$

$$- \sum_{r=0}^{M} \sum_{s=n+1-r}^{n-1} k_4(r)|y(s) - y^*(s)| - \sum_{r=0}^{M} k_4(r)|y(n-r) - y^*(n-r)|$$

$$= \sum_{r=0}^{M} k_3(r)|x(n) - x^*(n)| - \sum_{r=0}^{M} k_3(r)|x(n-r) - x^*(n-r)|$$

$$+ \sum_{r=0}^{M} k_4(r)|y(n) - y^*(n)| - \sum_{r=0}^{M} k_4(r)|y(n-r) - y^*(n-r)|,$$

$$\Delta V_{22}(n) = V_{22}(n+1) - V_{22}(n)$$

$$\leq -\left(\frac{1}{\theta_2(n)} - \left| \frac{1}{\theta_2(n)} - q(n) \right| \right) |y(n) - y^*(n)| + p(n)|x(n) - x^*(n)|$$

$$+ \sum_{r=0}^{M} k_3(r)|x(n-r) - x^*(n-r)| + \sum_{r=0}^{M} k_4(r)|y(n-r) - y^*(n-r)|,$$

(3.12)

where $\theta_2(n)$ is between $y(n)$ and $y^*(n)$. Consequently, one has

$$
\begin{aligned}
\Delta V_2(n) &= \Delta V_{21}(n) + \Delta V_{22}(n) \\
&\leq -\left(\frac{1}{\theta_2(n)} - \left| \frac{1}{\theta_2(n)} - q(n) \right| - \sum_{r=0}^{M} k_4(r) \right) |y(n) - y^*(n)| \\
&\quad + \left(p(n) + \sum_{r=0}^{M} k_3(r) \right) |x(n) - x^*(n)|.
\end{aligned}
\tag{3.13}
$$

Now, we consider a Lyapunov-like discrete function $V(n)$ defined by

$$
V(n) = V_1(n) + V_2(n).
\tag{3.14}
$$

According to (2.8) and (2.11), there exists an $n^* \in \mathbb{Z}^+$ such that $x(n) \leq M_1 + \varepsilon$, $y(n) \leq M_2 + \varepsilon$ for $n \geq n^*$ and any positive constant ε. Obviously, $V(n) \geq 0$ for all $n \in \mathbb{Z}^+$ and $V(n^* + M) < +\infty$. In view of (3.1), we choose a sufficient small $\varepsilon > 0$ such that
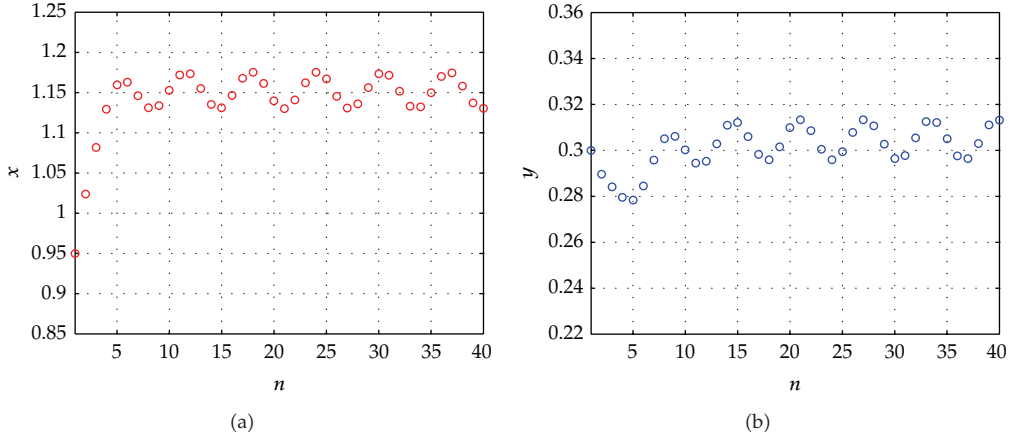
$$
\begin{aligned}
\min\left\{ b^l, \frac{2}{M_1 + \varepsilon} - b^u \right\} - (M+1)(k_1^u + k_3^u) - p^u \geq \eta, \\
\min\left\{ q^l, \frac{2}{M_2 + \varepsilon} - q^u \right\} - (M+1)(k_2^u + k_4^u) - c^u \geq \eta.
\end{aligned}
\tag{3.15}
$$

Therefore, combining (3.9) with (3.13) for all $n \geq n^* + M$, we obtain

$$
\begin{aligned}
\Delta V(n) &= \Delta V_1(n) + \Delta V_2(n) \\
&\leq -\left(\frac{1}{\theta_1(n)} - \left| \frac{1}{\theta_1(n)} - b(n) \right| - \sum_{r=0}^{M} k_1(r) - \sum_{r=0}^{M} k_3(r) - p(n) \right) |x(n) - x^*(n)| \\
&\quad - \left(\frac{1}{\theta_2(n)} - \left| \frac{1}{\theta_2(n)} - q(n) \right| - \sum_{r=0}^{M} k_4(r) - \sum_{r=0}^{M} k_2(r) - c(n) \right) |y(n) - y^*(n)| \\
&\leq -\left(\min\left\{ b^l, \frac{2}{M_1 + \varepsilon} - b^u \right\} - (M+1)(k_1^u + k_3^u) - p^u \right) |x(n) - x^*(n)| \\
&\quad - \left(\min\left\{ q^l, \frac{2}{M_2 + \varepsilon} - q^u \right\} - (M+1)(k_2^u + k_4^u) - c^u \right) |y(n) - y^*(n)| \\
&\leq -\eta \left(|x(n) - x^*(n)| + |y(n) - y^*(n)| \right).
\end{aligned}
\tag{3.16}
$$

Summing both sides of (3.16) from $n^* + M$ to n, it derives that

$$
\sum_{s=n^*+M}^{n} \Delta V(s) = \sum_{s=n^*+M}^{n} (V(s+1) - V(s)) \leq -\eta \sum_{s=n^*+M}^{n} \left(|x(s) - x^*(s)| + |y(s) - y^*(s)| \right),
\tag{3.17}
$$

(a)

(b)

Figure 1: Permanence of (1.9) with $\{(x(-1), y(-1))^\top\} = \{(1.0, 0.29)^\top\}$ and $\{(x(0), y(0))^\top\} = \{(0.95, 0.3)^\top\}$. (a) Time series of x for $n \in [0, 40]$. (b) Time series of y for $n \in [0, 40]$.

which implies that

$$
V(n+1) + \eta \sum_{s=n^*+M}^{n} \left(|x(s) - x^*(s)| + |y(s) - y^*(s)| \right) \leq V(n^* + M), \quad \text{for any } n \geq n^* + M,
$$

$$
\sum_{n=n^*+M}^{+\infty} \left(|x(n) - x^*(n)| + |y(n) - y^*(n)| \right) \leq \frac{V(n^* + M)}{\eta} < +\infty,
$$

(3.18)

from which we conclude that

$$
\lim_{n \to +\infty} \left(|x(n) - x^*(n)| + |y(n) - y^*(n)| \right) = 0,
$$

(3.19)

that is,

$$
\lim_{n \to +\infty} |x(n) - x^*(n)| = 0, \qquad \lim_{n \to +\infty} |y(n) - y^*(n)| = 0.
$$

(3.20)

According to Definition 1.2, this result implies that $\{(x(n), y(n))^\top\}$ is global attractive. The proof is complete. □

4. An Example and Its Numerical Simulations

In this paper, we have investigated the asymptotic behaviors of a delayed nonautonomous predator-prey difference system. Sufficient conditions which guarantee the permanence of system and global attractivity of positive solutions of system are obtained, respectively. The theoretical results are substantiated by the following example and numerical results.
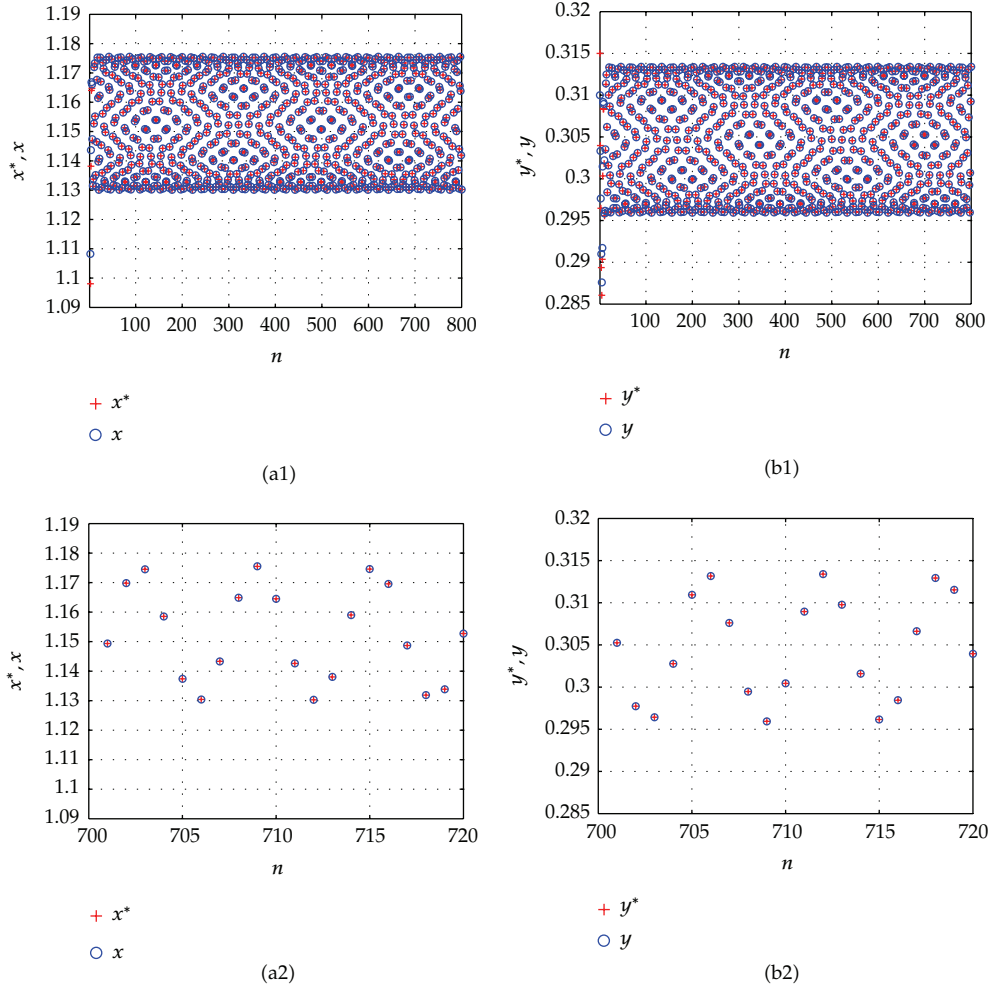
Figure 2: Dynamic behaviors of (1.9) with different initial values. (a1) and (a2) show x^*, x with $x^*(-1) = 1.06$, $x^*(0) = 1.0$, $x(-1) = 1.04$, $x(0) = 1.03$ for $k \in [0, 800]$ and $k \in [700, 720]$, respectively. (b1) and (b2) show y^*, y with $y^*(-1) = 0.32$, $y^*(0) = 0.315$, $y(-1) = 0.315$, $y(0) = 0.31$ for $k \in [0, 800]$ and $k \in [700, 720]$, respectively.

Example 4.1. Consider the following system:

$$
\begin{aligned}
x(n+1) &= x(n) \exp\{0.5 + 0.01 \sin n - (0.38 + 0.02 \sin n)x(n) - (0.02 + 0.01 \sin n)y(n) \\
&\qquad -0.024[x(n) + x(n-1)] - 0.001[y(n) + y(n-1)]\}, \\
y(n+1) &= y(n) \exp\{-0.015 - 0.005 \sin n + (0.26 + 0.03 \sin n)x(n) - (0.97 + 0.02 \sin n)y(n) \\
&\qquad +0.01[x(n) + x(n-1)] - 0.02[y(n) + y(n-1)]\}.
\end{aligned}
\tag{4.1}
$$

Obviously,

$$
\Delta_1 \approx 0.5175 > 0, \qquad \Delta_2 \approx 0.3875 > 0, \qquad \Delta_3 \approx 0.1347 > 0.
\tag{4.2}
$$

It is easy to verify that the assumptions of Theorem 2.3 are satisfied, so system (4.1) is permanent (see Figure 1). Furthermore, a calculation shows that

$$
\min\left\{b^l, \frac{2}{M_1} - b^u\right\} - (M+1)\left(k_1^u + k_3^u\right) - p^u \approx 0.0020 > 0.001,
$$

$$
\min\left\{q^l, \frac{2}{M_2} - q^u\right\} - (M+1)\left(k_2^u + k_4^u\right) - c^u \approx 0.8780 > 0.001.
$$

(4.3)

According to Theorem 3.1, any positive solution $\{(x(n), y(n))^\top\}$ of system (4.1) is global attractive (see Figure 2). From Figures 2(a1) and 2(a2), we see that x with $x(0) = 1.03$ tends to x^* with $x^*(0) = 1.0$. Similarly, from Figures 2(b1) and 2(b2), we see that y with $y(0) = 0.31$ tends to y^* with $y^*(0) = 0.315$.

Acknowledgments

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