Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 791610, 10 pages
doi:10.1155/2010/791610

Research Article

Asymptotic Stability for a Class of Nonlinear Difference Equations

Chang-you Wang,[1, 2, 3] Shu Wang,[3] Zhi-wei Wang,[4] Fei Gong,[2, 5] and Rui-fang Wang[2, 5]

[1] *College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*
[2] *Key Laboratory of Network Control & Intelligent Instrument, Chongqing University of Posts and Telecommunications, Ministry of Education, Chongqing 400065, China*
[3] *College of Applied Sciences, Beijing University of Technology, Beijing 100124, China*
[4] *School of Communication and Information Engineering, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*
[5] *College of Automation, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*

Correspondence should be addressed to Chang-you Wang, wangcy@cqupt.edu.cn

Received 9 January 2010; Accepted 5 February 2010

Academic Editor: Guang Zhang

We study the global asymptotic stability of the equilibrium point for the fractional difference equation $x_{n+1} = (ax_{n-l}x_{n-k})/(\alpha + bx_{n-s} + cx_{n-t})$, $n = 0, 1, \ldots$, where the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_1, x_0$ are arbitrary positive real numbers of the interval $(0, \alpha/2a)$, l, k, s, t are nonnegative integers, $r = \max\{l, k, s, t\}$ and α, a, b, c are positive constants. Moreover, some numerical simulations are given to illustrate our results.

1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, and so forth [1]. The study of nonlinear difference equations is of paramount importance not only in their own field but in understanding the behavior of their differential counterparts. There has been a lot of work concerning the global asymptotic behavior of solutions of rational difference equations [2–6]. In particular, Ladas [7] put forward the idea of investigating the global asymptotic stability of the following difference equation:

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in (0, +\infty)$.

In [8], Nesemann utilized the strong negative feedback property of [2] to study the following difference equation:

$$x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_{n-1}x_n + x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in (0, +\infty)$.

By using semicycle analysis methods, the authors of [9] got a sufficient condition which guarantees the global asymptotic stability of the following difference equation:

$$x_{n+1} = \frac{x_{n-1}^b + x_n x_{n-2}^b + a}{x_{n-1}^b x_n + x_{n-2}^b + a}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where $a, b \in [0, +\infty)$ and the initial values $x_{-2}, x_{-1}, x_0 \in (0, +\infty)$.

Yang et al. [10] investigated the invariant intervals, the global attractivity of equilibrium points, and the asymptotic behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{a x_{n-1} + b x_{n-2}}{c + d x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots. \tag{1.4}$$

Berenhaut et al. [11] generalized the result reported in [12] to the following rational equation

$$x_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k} y_{n-m}}, \quad n = 0, 1, \ldots. \tag{1.5}$$

This work is motivated from [13–15]. For more similar work, one can refer to [12, 16–20] and references therein.

The purpose of this paper is to investigate the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the difference equation:

$$x_{n+1} = \frac{a x_{n-l} x_{n-k}}{\alpha + b x_{n-s} + c x_{n-t}}, \quad n = 0, 1, \ldots, \tag{1.6}$$

where the initial conditions $x_{-r}, x_{-r+1}, \ldots, x_1, x_0$ are arbitrary positive real numbers of the interval $(0, \alpha/2a), l, k, s, t$ is nonnegative integer, and $r = \max\{l, k, s, t\}$ and α, a, b, c are positive constants. Moreover, some numerical simulations to the special case of (1.6) are given to illustrate our results.

This paper is arranged as follows: in Section 2, we give some definitions and preliminary results. The main results and their proofs are given in Section 3. Finally, some numerical simulations are given to illustrate our theoretical analysis.

2. Some Preliminary Results

To prove the main results in this paper, we first give some definitions and preliminary results [21, 22] which are basically used throughout this paper.

Lemma 2.1. *Let I be some interval of real numbers and let*

$$f : I^{k+1} \longrightarrow I \tag{2.1}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{2.2}$$

has a unique solution $\{x_n\}_{n=-k}^{+\infty}$.

Definition 2.2. A point $\overline{x} \in I$ is called an equilibrium point of (2.2), if

$$\overline{x} = f(\overline{x}, \overline{x}, \dots, \overline{x}). \tag{2.3}$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (2.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2.3. Let p, q be two nonnegative integers such that $p + q = n$. Splitting $x = (x_1, x_2, \dots, x_n)$ into $x = ([x]_p, [x]_q)$, where $[x]_\sigma$ denotes a vector with σ-components of x, we say that the function $f(x_1, x_2, \dots, x_n)$ possesses a mixed monotone property in subsets I^n of R^n if $f([x]_p, [x]_q)$ is monotone nondecreasing in each component of $[x]_p$ and is monotone nonincreasing in each component of $[x]_q$ for $x \in I^n$. In particular, if $q = 0$, then it is said to be monotone nondecreasing in I^n.

Definition 2.4. Let $\overline{x}$ be an equilibrium point of (2.2).

(i) $\overline{x}$ is stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_{-k}, x_{-k+1}, \dots, x_0) \in I^{k+1}$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta$, $|x_n - \overline{x}| < \varepsilon$holds for $n = 1, 2, \dots$.

(ii) $\overline{x}$ is a local attractor if there exists $\gamma > 0$ such that $x_n \to \overline{x}$ holds for any initial conditions $(x_{-k}, x_{-k+1}, \dots, x_0) \in I^{k+1}$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma$.

(iii) $\overline{x}$ is locally asymptotically stable if it is stable and is a local attractor.

(iv) $\overline{x}$ is a global attractor if $x_n \to \overline{x}$ holds for any initial conditions $(x_{-k}, x_{-k+1}, \dots, x_0) \in I^{k+1}$.

(v) $\overline{x}$ is globally asymptotically stable if it is stable and is a global attractor.

(vi) $\overline{x}$ is unstable if it is not locally stable.

The linearized equation of (2.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=1}^{k} \frac{\partial f(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{2.4}$$

Now assume that the characteristic equation associated with (2.4) is

$$P(\lambda) = P_0\lambda^k + P_1\lambda^{k-1} + \cdots + P_{k-1}\lambda + P_k = 0, \tag{2.5}$$

where $P_i = \partial f(\overline{x}, \overline{x}, \dots, \overline{x}) / \partial x_{n-i}$.

Lemma 2.5. *Assume that* $P_1, P_2, \ldots, P_k \in R$ *and* $k \in \{0, 1, 2, \ldots\}$. *Then*

$$|P_1| + |P_2| + \cdots + |P_k| < 1 \tag{2.6}$$

is a sufficient condition for the local asymptotically stability of the difference equation

$$x_{n+k} + P_1 x_{n+k-1} + \cdots + P_k x_n = 0, \quad n = 0, 1, \ldots. \tag{2.7}$$

Lemma 2.6. *Assume that* f *is a* C^1 *function and let* $\overline{x}$ *be an equilibrium of* (2.2). *Then the following statements are true.*

(a) *If all roots of the polynomial equation* (2.5) *lie in the open unite* disk $|\lambda| < 1$, *then the equilibrium point* $\overline{x}$ *of* (2.2) *is locally asymptotically stable.*

(b) *If at least one root of* (2.2) *has absolute value greater than one, then the equilibrium point* $\overline{x}$ *of* (2.2) *is unstable.*

Remark 2.7. The condition (2.6) implies that all the roots of the polynomial equation (2.5) lie in the open unite disk $|\lambda| < 1$.

3. The Main Results and Their Proofs

In this section, we investigate the global asymptotic stability of the equilibrium point of (1.6).

Let $f : (0, \infty)^4 \to (0, \infty)$ be a function defined by

$$f(u, v, w, s) = \frac{auv}{\alpha + bw + cs}, \tag{3.1}$$

then it follows that

$$\begin{aligned} f_u(u, v, w, s) &= \frac{av}{\alpha + bw + cs}, & f_v(u, v, w, s) &= \frac{au}{\alpha + bw + cs}, \\ f_w(u, v, w, s) &= -\frac{abuv}{(\alpha + bw + cs)^2}, & f_s(u, v, w, s) &= -\frac{acuv}{(\alpha + bw + cs)^2}. \end{aligned} \tag{3.2}$$

Let $\overline{x}$, $\overline{\overline{x}}$ be the equilibrium points of (1.6), then we have

$$\overline{x} = 0, \qquad \overline{\overline{x}} = \frac{\alpha}{a - (b + c)}, \tag{3.3}$$

where $a \neq b + c$. If $a = b + c$, then $\overline{x} = 0$ is a unique equilibrium point.

Moreover,

$$\begin{gathered} f_u(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = f_v(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = f_w(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = f_s(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0, \\ f_u\left(\overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}\right) = f_v\left(\overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}\right) = 1, \qquad f_w\left(\overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}\right) = -\frac{b}{a}, \qquad f_s\left(\overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}, \overline{\overline{x}}\right) = -\frac{c}{a}. \end{gathered} \tag{3.4}$$

Thus, the linearized equations of (1.6) about equilibrium points $\overline{x}$ and $\overline{\overline{x}}$ are, respectively,

$$z_{n+1} = 0, \tag{3.5}$$

$$z_{n+1} = z_{n-k} + z_{n-l} - \frac{b}{a} z_{n-s} - \frac{c}{a} z_{n-t}, \tag{3.6}$$

where l, k, s, t are nonnegative different integers.

The characteristic equation associated with (3.6) is

$$P(\lambda) = \lambda^{r-k} + \lambda^{r-l} - \frac{b}{a} \lambda^{r-s} - \frac{c}{a} \lambda^{r-t} = 0, \tag{3.7}$$

where $r = \max\{l, k, s, t\}$.

By Lemmas 2.5 and 2.6, we have the following result.

Theorem 3.1. *The equilibrium point* $\overline{x} = 0$ *of* (1.6) *is locally asymptotically stable. Moreover, we have the following.*

(a) *If all roots of the characteristic equation* (3.7) *lie in the open unite* disk $|\lambda| < 1$, *then the equilibrium point* $\overline{\overline{x}}$ *of* (1.6) *is locally asymptotically stable.*

(b) *If at least one root of* (3.7) *has absolute value greater than one, then the equilibrium point* $\overline{\overline{x}}$ *of* (1.6) *is unstable.*

Theorem 3.2. *Let* $[\gamma, \delta]$ *be an interval of real numbers and assume that* $f : [\gamma, \delta]^{k+1} \to R$ *is a continuous function satisfying the mixed monotone property. If there exists*

$$m_0 \le \min\{x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0\} \le \max\{x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0\} \le M_0 \tag{3.8}$$

such that

$$m_0 \le f\left([m_0]_p, [M_0]_q\right) \le f\left([M_0]_p, [m_0]_q\right) \le M_0, \tag{3.9}$$

then there exist $(m, M) \in [m_0, M_0]^2$ *satisfying*

$$M = f\left([M]_p, [m]_q\right), \qquad m = f\left([m]_p, [M]_q\right). \tag{3.10}$$

Moreover, if $m = M$, *then* (2.2) *has a unique equilibrium point* $\overline{x} \in [m_0, M_0]$ *and every solution of* (2.2) *converges to* $\overline{x}$.

Proof. Using m_0 and M_0 as a couple of initial iteration, we construct two sequences $\{m_i\}$ and $\{M_i\}$ $(i = 1, 2, \ldots)$ from the equation

$$m_i = f\left([m_{i-1}]_p, [M_{i-1}]_q\right), \qquad M_i = f\left([M_{i-1}]_p, [m_{i-1}]_q\right). \tag{3.11}$$

It is obvious from the mixed monotone property of f that the sequences $\{m_i\}$ and $\{M_i\}$ possess the following monotone property

$$m_0 \le m_1 \le \cdots \le m_i \le \cdots \le M_i \le \cdots \le M_1 \le M_0, \tag{3.12}$$

where $i = 0, 1, 2, \ldots$, and

$$m_i \le x_l \le M_i \quad \text{for } l \ge (k+1)i + 1. \tag{3.13}$$

Set

$$m = \lim_{i\to\infty} m_i, \qquad M = \lim_{i\to\infty} M_i, \tag{3.14}$$

then

$$m \le \lim_{i\to\infty} \inf x_i \le \lim_{i\to\infty} \sup x_i \le M. \tag{3.15}$$

By the continuity of f, we have

$$M = f\Big([M]_p, [m]_q\Big), \qquad m = f\Big([m]_p, [M]_q\Big). \tag{3.16}$$

Moreover, if $m = M$, then $m = M = \lim_{i\to\infty} x_i = \overline{x}$, and then the proof is complete. □

Theorem 3.3. *The equilibrium point* $\overline{x} = 0$ *of* (1.6) *is a global attractor for any initial conditions*

$$(x_{-r}, x_{-r+1}, \ldots, x_1, x_0) \in \left(0, \frac{\alpha}{2a}\right)^{r+1}. \tag{3.17}$$

Proof. Let $f : (0, \infty)^4 \to (0, \infty)$ be a function defined by

$$f(u, v, w, s) = \frac{auv}{\alpha + bw + cs}. \tag{3.18}$$

We can easily see that the function $f(u, v, w, s)$ is increasing in u, v and decreasing in w, s.

Let

$$M_0 = \max\{x_{-r}, x_{-r+1}, \ldots, x_{-1}, x_0\}, \qquad \frac{aM_0 - \alpha}{b + c} < m_0 < 0; \tag{3.19}$$

we have

$$m_0 \le \frac{am_0^2}{\alpha + bM_0 + cM_0} \le \frac{aM_0^2}{\alpha + bm_0 + cm_0} \le M_0. \tag{3.20}$$

Then from (1.6) and Theorem 3.2, there exist $m, M \in [m_0, M_0]$ satisfying

$$m = \frac{am^2}{\alpha + bM + cM}, \qquad M = \frac{aM^2}{\alpha + bm + cm}, \tag{3.21}$$

thus

$$[\alpha - a(m+M)](m-M) = 0. \tag{3.22}$$

In view of $2aM_0 < \alpha$, we have

$$\alpha - a(m+M) > 0. \tag{3.23}$$

Then

$$M = m. \tag{3.24}$$

It follows by Theorem 3.2 that the equilibrium point $\overline{x} = 0$ of (1.6) is a global attractor. The proof is therefore complete. □

Theorem 3.4. *The equilibrium point $\overline{x} = 0$ of* (1.6) *is a global asymptotic stability for any initial conditions*

$$(x_{-r}, x_{-r+1}, \ldots, x_1, x_0) \in \left(0, \frac{\alpha}{2a}\right)^{r+1}. \tag{3.25}$$

Proof. The result follows from Theorems 3.1 and 3.3. □

4. Numerical Simulations

In this section, we give numerical simulations to support our theoretical analysis via the software package Matlab7.0. As an example, we consider the following difference equations

$$x_{n+1} = \frac{x_n x_{n-1}}{5 + 2x_n + x_{n-1}}, \quad n = 0, 1, \ldots, \tag{4.1}$$

$$x_{n+1} = \frac{x_n x_{n-1}}{5 + 2x_{n-2} + x_{n-3}}, \quad n = 0, 1, \ldots, \tag{4.2}$$

where the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, 2.5)$. Let $m_0 = -0.5$, $M_0 = 2.5$; it is obvious that (4.1) and (4.2) satisfy the conditions of Theorems 3.2 and 3.3.

By employing the software package MATLAB7.0, we can solve the numerical solutions of (4.1) and (4.2) which are shown, respectively, in Figures 1 and 2. More precisely, Figure 1 shows the numerical solution of (4.1) with $x_{-1} = 1.2$, $x_0 = 1.8$, and the relations that $m_i \le x_l \le M_i$ when $l \ge (k+1)i + 1$, $i = 0, 1, 2, \ldots$, and Figure 2 shows the numerical solutions of (4.2) with $x_{-3} = 1.5$, $x_{-2} = 1.8$, $x_{-1} = 1.3$, $x_0 = 1.4$, and the relations that $m_i \le x_l \le M_i$ when $l \ge (k+1)i + 1$, $i = 0, 1, 2, \ldots$.

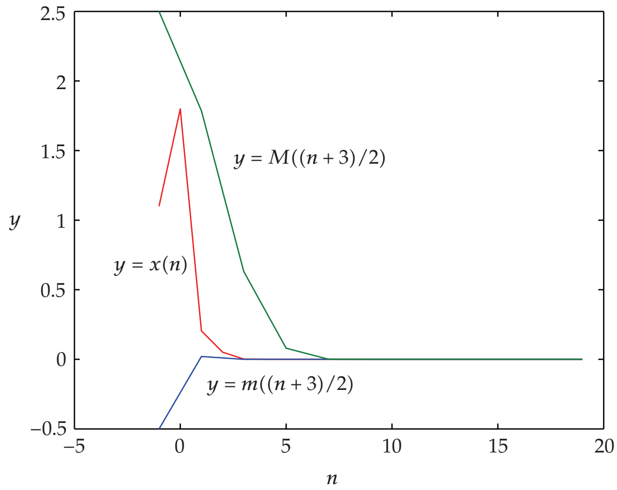

Figure 1: Chart of (4.1) with $x(-1) = 1.2,\ x(0) = 1.8$.

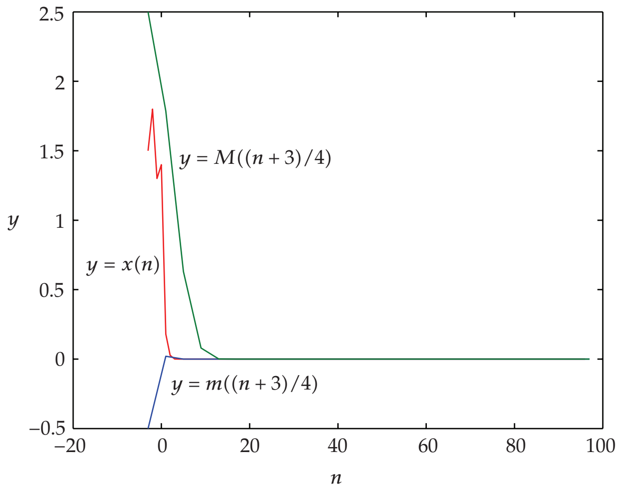

Figure 2: Chart of (4.2) with $x(-3) = 1.5,\ x(-2) = 1.8,\ x(-1) = 1.3,\ x(0) = 1.4$.

5. Conclusions

This paper presents the use of a variational iteration method for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. The numerical simulations show that this method is an effective and convenient one. The variational iteration method provides an efficient method to handle the nonlinear structure. Computations are performed using the software package MATLAB7.0.

We have dealt with the problem of global asymptotic stability analysis for a class of nonlinear difference equations. The general sufficient conditions have been obtained to ensure the existence, uniqueness, and global asymptotic stability of the equilibrium point for the nonlinear difference equation. These criteria generalize and improve some known results. In particular, an illustrate example is given to show the effectiveness of the obtained results. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equations.

Acknowledgments

The authors are grateful to the referee for her/his comments. This work is supported by Science and Technology Study Project of Chongqing Municipal Education Commission (Grant no. KJ 080511) of China, Natural Science Foundation Project of CQ CSTC (Grant no. 2008BB7415) of China, Foundation Project of Doctor Graduate Student Innovation of Beijing University of Technology of China, the NSFC (Grant no. 10471009), and BSFC (Grant no. 1052001) of China.

References

[1] W. Li and H. Sun, "Global attractivity in a rational recursive sequence," *Dynamic Systems and Applications*, vol. 11, pp. 339–346, 2002.
[2] A. M. Amleh, N. Kruse, and G. Ladas, "On a class of difference equations with strong negative feedback," *Journal of Difference Equations and Applications*, vol. 5, no. 6, pp. 497–515, 1999.
[3] K. Cunningham, M. R. S. Kulenovic, G. Ladas, and S. V. Valicenti, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_n)/(Bx_n + Cx_{n-1})$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, pp. 4603–4614, 2001.
[4] L. Li, "Stability properties of nonlinear difference equations and conditions for boundedness," *Computers & Mathematics with Applications*, vol. 38, pp. 29–35, 1999.
[5] M. R. S. Kulenovic, G. Ladas, and N. R. Prokup, "A rational difference equation," *Computers & Mathematics with Applications*, vol. 41, pp. 671–678, 2001.
[6] X. Yan, W. Li, and H. Sun, "Global attractivity in a higher order nonlinear difference equation," *Applied Mathematics E-Notes*, vol. 2, pp. 51–58, 2002.
[7] G. Ladas, "Open problem and conjectures," *Journal of Difference Equations and Applications*, vol. 3, pp. 317–321, 1997.
[8] T. Nesemann, "Positive nonlinear difference equations: some results and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 7, pp. 4707–4717, 2001.
[9] X. Li and D. Zhu, "Global asymptotic stability of a nonlinear recursive sequence," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 833–838, 2004.
[10] X. Yang, W. Su, B. Chen, G. M. Megson, and D. J. Evans, "On the recursive sequence $x_{n+1} = (ax_{n-1} + bx_{n-2})/(c + dx_{n-1}x_{n-2})$," *Applied Mathematics and Computation*, vol. 162, pp. 1485–1497, 2005.
[11] K. S. Berenhaut, J. D. Foley, and S. Stevic, "The global attractivity of the rational difference equation $y_n = (y_{n-k} + y_{n-m})/(1 + y_{n-k}y_{n-m})$," *Applied Mathematics Letters*, vol. 20, pp. 54–58, 2007.
[12] K. S. Berenhaut and S. Stevic, "The difference equation $x_{n+1} = (a + x_{n-k})/\sum_{i=0}^{k-1} c_i x_{n-i}$ has solutions converging to zero," *Journal of Mathematical Analysis and Applications*, vol. 326, pp. 1466–1471, 2007.
[13] C. Wang, "Existence and stability of periodic solutions for parabolic systems with time delays," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 2, pp. 1354–1361, 2008.
[14] C. Wang and S. Wang, "Oscillation of partial population model with diffusion and delay," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1793–1797, 2009.
[15] C. Wang, S. Wang, and X. Yan, "Global asymptotic stability of 3-species mutualism models with diffusion and delay effects," *Discrete Dynamics in Natural and Science*, vol. 2009, Article ID 317298, 20 pages, 2009.
[16] C. F. Li and Y. Zhou, "Existence of bounded and unbounded non-oscillatory solutions for partial difference equations," *Mathematical and Computer Modelling*, vol. 45, pp. 825–833, 2007.
[17] C. Wang, F. Gong, S. Wang, L. Li, and Q. Shi, "Asymptotic behavior of equilibrium point for a class of nonlinear difference equation," *Advances in Difference Equations*, vol. 2009, Article ID 214309, 8 pages, 2009.
[18] S. M. Chen and C. S. Li, "Nonoscillatory solutions of second order nonlinear difference equations," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 478–481, 2008.
[19] T. Sun and H. Xi, "Global attractivity for a family of nonlinear difference equations," *Applied Mathematics Letters*, vol. 20, no. 7, pp. 741–745, 2007.

[20] Z. Q. Liu, Y. G. Xu, and S. M. Kang, "Global solvability for a second order nonlinear neutral delay difference equation," *Computers and Mathematics with Applications*, vol. 57, no. 4, pp. 587–595, 2009.

[21] M. R. S. Kulenovic and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2001.

[22] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.