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Research Article

On the Recursive Sequence $x_n = A + x_{n-k}^p / x_{n-1}^r$

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This paper studies the dynamic behavior of the positive solutions to the difference equation $x_n = A + x_{n-k}^p / x_{n-1}^r$, $n = 1, 2, \ldots$, where A, p, and r are positive real numbers, and the initial conditions are arbitrary positive numbers. We establish some results regarding the stability and oscillation character of this equation for $p \in (0, 1)$.

1. Introduction

In recent years, there has been intense interest in the dynamic behavior of the positive solutions to a class of difference equations of the form

$$x_n = A + \frac{x_{n-k}^p}{x_{n-1}^p}, \quad n \in \mathbb{N}, \tag{1.1}$$

where A and p are positive real numbers. Now, let us make a brief review on the advances in this class of difference equations.

In 1999, Amleh et al. [1] studied the second-order rational difference equation

$$x_n = A + \frac{x_{n-2}}{x_{n-1}}, \quad n \in \mathbb{N}. \tag{1.2}$$

Later, Berenhaut and Stević [2], Stević [3], and El-Owaidy et al. [4] extended this work to the following more general second-order difference equation:

$$x_n = A + \frac{x_{n-2}^p}{x_{n-1}^p}, \quad n \in \mathbb{N}. \tag{1.3}$$

On the other hand, DeVault et al. [5] investigated the following higher-order version of (1.2):

$$x_n = A + \frac{x_{n-k}}{x_{n-1}}, \quad n \in \mathbb{N}. \tag{1.4}$$

By combining (1.3) and (1.4), Berenhaut and Stević [6] examined a larger class of difference equations, which are of the form

$$x_n = A + \frac{x_{n-k}^p}{x_{n-1}^p}, \quad n \in \mathbb{N}. \tag{1.5}$$

Very recently, Berenhaut et al. [7] studied the following generalization of (1.5):

$$x_n = A + \frac{x_{n-k}^p}{x_{n-m}^p}, \quad n \in \mathbb{N}. \tag{1.6}$$

For some related work, the interested reader is referred to [1, 3, 8–19].

Inspired by the previous work and by the work owing to Stević [15], this paper studies the behavior of the recursive equation

$$x_n = A + \frac{x_{n-k}^p}{x_{n-1}^r}, \quad n \in \mathbb{N}. \tag{1.7}$$

We establish some interesting results regarding the stability and oscillation character of this equation for $p \in (0, 1)$.

2. Stability Character

In this section we investigate the stability character of the positive solutions to (1.7).

A point $\overline{x} \in \mathbb{R}$ is an equilibrium point of (1.7) if and only if it is a root for the function

$$g(x) = x - x^{p-r} - A, \tag{2.1}$$

that is,

$$\overline{x} = \overline{x}^{p-r} + A. \tag{2.2}$$

Lemma 2.1. *Let* $0 < p < r + 1$, *then* (1.7) *has a unique equilibrium point* $\overline{x} > 1$.

Proof

Case 1. $p = r$. Then $\overline{x} = A + 1 > 1$.

Case 2. $r < p < r+1$. Then g defined by (2.1) is decreasing on $[0, (p-r)^{1/(r-p+1)}]$ and increasing on $[(p-r)^{1/(r-p+1)}, \infty)$. Since $g(1) = -A$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique zero $\overline{x} > 1$.
Case 3. $0 < p < r$. Since g is increasing on $[0,\infty)$, $g(1) = -A$ and $\lim_{x\to\infty} g(x) = \infty$, then g has a unique zero $\overline{x} > 1$.

Lemma 2.2. *Let* $0 < p < r+1$. *Assume that* $\overline{x}$ *is the equilibrium point of* (1.7). *If* $(p+r)^{(p-r)/(r+1-p)}(p+r-1) < A$, *then* $\overline{x}$ *is locally asymptotically stable.*

Proof. By the Linearized Stability Theorem [11], $\overline{x}$ is locally asymptotically stable if and only if $\overline{x}^{r+1-p} > p+r$. A simple calculations shows that

$$g\left((p+r)^{1/(r+1-p)}\right) = (p+r)^{(p-r)/(r+1-p)}(p+r-1) - A < 0, \tag{2.3}$$

where g is defined by (2.1). Then since $\lim_{x\to\infty} g(x) = \infty$, we have $\overline{x} > (p+r)^{1/(r+1-p)}$ and $\overline{x}^{r+1-p} > p+r$. The proof is complete. □

Lemma 2.3. *If* $p \in (0,1)$, *then every positive solution to* (1.7) *is bounded.*

Proof. Note that each $n \in \mathbb{N}$ can be written in the form $lk+i$ for some $l \in \mathbb{N}_0$ and $i \in \{0,1,\ldots,k-1\}$. From (1.7) and since $x_n > A$ for every $n \geq 0$, we have that

$$x_{lk+i} = A + \frac{x^p_{(l-1)k+i}}{x^r_{lk+i-1}} < A + \frac{x^p_{(l-1)k+i}}{A^r}, \tag{2.4}$$

for every $l \in \mathbb{N}_0$ and $i \in \{0,1,\ldots,k-1\}$. Let $(u^{(i)}_l)_{l\in\mathbb{N}_0}$ be the solution to the difference equation

$$u^{(i)}_l = A + \frac{\left(u^{(i)}_{l-1}\right)^p}{A^r}, \qquad u^{(i)}_0 = x_{-k+i}. \tag{2.5}$$

From (2.4) and by induction we see that $x_{(l-1)k+i} \leq u^{(i)}_l$, $l \in \mathbb{N}_0$. Hence it is enough to prove that the sequences $(u^{(i)}_l)_{l\geq 0}$, $i \in \{0,1,\ldots,k-1\}$ are bounded.

Since the function $f(x) = A + x^p/A^r$, $x \in (0,\infty)$ is increasing and concave for $p \in (0,1)$, it follows that there is a unique fixed point $\overline{x}$ of the equation $f(x) = x$ and that the function f satisfies

$$\left(f(x) - x\right)(x - \overline{x}) < 0, \quad x \in (0,\infty). \tag{2.6}$$

Using this fact it is easy to see that if $u^{(i)}_l \in (0,\overline{x}]$, the sequence is nondecreasing and bounded from above by $\overline{x}$, and if $u^{(i)}_l \geq \overline{x}$, it is nonincreasing and bounded from below by $\overline{x}$. Hence for every $u^{(i)}_0 \in (0,\infty)$, each of the sequences $u^{(i)}_l$, $i \in \{0,1,\ldots,k-1\}$ is bounded. The claimed result follows. □

Lemma 2.4 (see [18]). *Let s, t be distinct nonnegative integers. Consider the difference equation*

$$x_n = f(x_{n-s}, x_{n-t}), \quad n = 1, 2, 3, \dots,$$
$$x_{1-\max(s,t)}, x_{2-\max(s,t)}, \dots, x_0 \in [a, b]. \tag{2.7}$$

Suppose f satisfies the following conditions.

(H_1) *$f : [a,b]^2 \to [a,b]$ is a continuous function that is nondecreasing in the first argument and is nonincreasing in the second argument.*

(H_2) *The system*

$$x = f(x, y),$$
$$y = f(y, x) \tag{2.8}$$

has a unique solution $(\overline{x}, \overline{x}) \in [a,b] \times [a,b]$.

Then $\overline{x}$ is the global attractor of all solutions to (2.7).

Theorem 2.5. *Let $p + r \le 1$, then the unique equilibrium $\overline{x}$ to* (1.7) *is globally asymptotically stable.*

Proof. By Lemma 2.3, there must exist positive constants P and Q such that $P \le x_n \le Q$. Let $f(u,v) = A + u^p/v^r$, $u, v \in [P, Q]$, it is easy to verify that (H_1) holds. In addition, if

$$x = A + \frac{x^p}{y^r},$$
$$y = A + \frac{y^p}{x^r}, \tag{2.9}$$

then

$$\frac{x - A}{y - A} = \frac{x^{p+r}}{y^{p+r}}. \tag{2.10}$$

Assume that $x \ne y$, then $x > y$ or $x < y$.

In case $x > y$, we have $(x - A)/(y - A) > x/y \ge x^{p+r}/y^{p+r}$, which contradicts with (2.10).

In case $x < y$, we have $(x - A)/(y - A) < x/y \le x^{p+r}/y^{p+r}$, again a contradiction.

Thus $x = y = \overline{x}$. By Lemma 2.4, the required result follows. □

Theorem 2.6. *Let $0 < p \le r < 1$ and $A^{r-p+1} \ge p/r$. Then every positive solution to* (1.7) *converges to the unique equilibrium $\overline{x}$.*

Proof. By Lemma 2.3, every positive solution $\{x_n\}$ to (1.7) is bounded, which implies that there are finite $\liminf x_n = I$ and $\limsup x_n = S$. Assume that $I \neq S$ $(I < S)$. Taking the lim inf and lim sup in (1.7), it follows that

$$A + \frac{I^p}{S^r} \leq I < S \leq A + \frac{S^p}{I^r}. \tag{2.11}$$

From this and $r \in (0,1)$, it follows that

$$AS^r + I^p \leq IS^r < SI^r \leq AI^r + S^p, \tag{2.12}$$

yielding

$$AS^r - S^p < AI^r - I^p. \tag{2.13}$$

Define function $f(x) = Ax^r - x^p$, $x \in (A, \infty)$. Since

$$f'(x) = Arx^{r-1} - px^{p-1} = x^{p-1}(Arx^{r-p} - p) > x^{p-1}\left(rA^{r-p+1} - p\right) \geq 0, \tag{2.14}$$

we deduce that f is increasing, and thus (2.13) cannot hold. Therefore we have $I = S$, which implies the result. □

Theorem 2.7. *Let $0 < p < 1$, $r \geq 1$, and $A^{r-p+1} \geq r + p - 1$. Then every positive solution to* (1.7) *converges to the unique equilibrium $\overline{x}$.*

Proof. From (2.11) we have

$$AI^{r-1}S^r + I^{p+r-1} \leq I^r S^r \leq AI^r S^{r-1} + S^{p+r-1}. \tag{2.15}$$

Consequently, we obtain $(AI^{r-1}S^{r-1})(S - I) \leq (S^{r+p-1} - I^{r+p-1})$. Suppose that $I \neq S$, we get

$$AI^{r-1}S^{r-1} \leq \frac{S^{r+p-1} - I^{r+p-1}}{S - I} = (r + p - 1)\gamma^{p+r-2}, \tag{2.16}$$

where $\gamma \in (I, S)$, leading to

$$A^r S^{r-1} \leq AI^{r-1}S^{r-1} \leq (r + p - 1)\gamma^{p+r-2} < (r + p - 1)A^{p-1}S^{r-1}. \tag{2.17}$$

This implies that $A^{r-p+1} < r + p - 1$, which is a contradiction. Hence, $I = S = \overline{x}$. □

3. Oscillation Character

In this section we investigate the oscillation character of the positive solutions to (1.7).

Theorem 3.1. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a positive solution to* (1.7)*. Then either* $\{x_n\}_{n=-k}^{\infty}$ *consists of a single semicycle or* $\{x_n\}_{n=-k}^{\infty}$ *oscillates about the equilibrium* $\overline{x}$ *with semicycles having at most* $k-1$ *terms.*

Proof. Suppose that $\{x_n\}_{n=-k}^{\infty}$ has at least two semicycles. Then there exists $N \geq -k$ such that either $x_N < \overline{x} \leq x_{N+1}$ or $x_{N+1} < \overline{x} \leq x_N$. Assume that $x_N < \overline{x} \leq x_{N+1}$. (The argument for the case $x_{N+1} < \overline{x} \leq x_N$ is similar and is omitted). Now suppose that the positive semicycle beginning with the term x_{N+1} has $k-1$ terms. Then $x_N < \overline{x} \leq x_{N+k-1}$ and so

$$x_{N+k} = A + \frac{x_N^p}{x_{N+k-1}^r} < A + \frac{\overline{x}^p}{\overline{x}^r} = A + \overline{x}^{p-r} = \overline{x}. \tag{3.1}$$

This completes the proof. □

Theorem 3.2. *Suppose that* k *is even and let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution to* (1.7)*, which has* $k-1$ *consecutive semicycles of length one, then every semicycle after this point is of length one.*

Proof. There exists $N \geq -k$ such that either

$$x_N, x_{N+2}, \ldots, x_{N+k-2} < \overline{x} \leq x_{N+1}, x_{N+3}, \ldots, x_{N+k-1} \tag{3.2}$$

or

$$x_{N+1}, x_{N+3}, \ldots, x_{N+k-1} < \overline{x} \leq x_N, x_{N+2}, \ldots, x_{N+k-2}. \tag{3.3}$$

We prove the former case. The proof for the latter is similar and is omitted. Now, we have

$$\begin{aligned} x_{N+k} &= A + \frac{x_N^p}{x_{N+k-1}^r} < A + \frac{\overline{x}^p}{\overline{x}^r} = A + \overline{x}^{p-r} = \overline{x}, \\ x_{N+k+1} &= A + \frac{x_{N+1}^p}{x_{N+k}^r} > A + \frac{\overline{x}^p}{\overline{x}^r} = A + \overline{x}^{p-r} = \overline{x}. \end{aligned} \tag{3.4}$$

The result then follows by induction. □

Lemma 3.3. *Let* $0 < p < r+1$*. Then* (1.7) *has no nontrivial periodic solutions of (not necessarily prime) period* $k-1$*.*

Proof. Suppose that $\{x_n\}_{n=-k}^{\infty}$ is a positive solution to (1.7) satisfying $x_{n-1} = x_{n-k}$ for all $n \geq 1$, then $x_n = A + x_{n-k}^p / x_{n-1}^r = A + x_{n-1}^{p-r}$ implies that $x_{n-1} = x_n = \overline{x}$ for all $n > -k$. The proof is complete. □

Theorem 3.4. *Assume that* $p \leq r$*. Let* $\{x_n\}_{n=-k}^{\infty}$ *be a positive solution to* (1.7)*, which consists of a single semicycle, then* $\{x_n\}_{n=-k}^{\infty}$ *converges to the equilibrium* $\overline{x}$*.*

Proof. Suppose $x_n \geq \overline{x}$ (the case for $x_n \leq \overline{x}$ is similar and is omitted) for all $n \geq -k$, then

$$x_{n+1} = A + \frac{x_{n-(k-1)}^p}{x_n^r} \geq \overline{x} = A + \overline{x}^{p-r}, \tag{3.5}$$

implying that

$$x_{n-(k-1)} \geq \overline{x}^{(p-r)/p} x_n^{r/p} \geq x_n^{(p-r)/p} x_n^{r/p} = x_n, \tag{3.6}$$

and so

$$x_{n-(k-1)} \geq x_n \geq \overline{x} \quad \text{for } n = 1, 2, \ldots. \tag{3.7}$$

From here it is clear that for $i = 0, \ldots, k-2$ there exists α_{i} such that

$$\lim_{n\to\infty} x_{n(k-1)+i} = \alpha_i. \tag{3.8}$$

But then $\alpha_0, \alpha_1, \ldots, \alpha_{k-2}$ is a periodic solution of (not necessarily prime) period $k-1$. By Lemma 3.3 the result holds. □

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