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Research Article

Permanence for a Discrete Model with Feedback Control and Delay

Yong-Hong Fan[1,2] and Lin-Lin Wang[1,2]

[1] *School of Mathematics and Information, Ludong University, Yantai, Shandong 264025, China*
[2] *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China*

Correspondence should be addressed to Lin-Lin Wang, llwang@bit.edu.cn

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The permanence of a single-species population discrete model with feedback control is considered. We found that if we use the method of comparison theorem, then the additional condition, to some extent, is necessary. But for the system itself, this condition may not be necessary. Here, we use new methods instead of the comparison theorem to get the permanence of the system in consideration; the additional condition in Chen's paper (2007) is deleted.

1. Introduction

In [1], Li and Zhu investigated the single-species population discrete model with feedback control:

$$\begin{aligned} N(n+1) &= N(n)\exp\left[r(n)\left(1-\frac{N(n-m)}{k(n)}-c(n)\mu(n)\right)\right], \\ \Delta\mu(n) &= -a(n)\mu(n)+b(n)N(n-m), \end{aligned} \tag{1.1}$$

which is a difference form of the single model with feedback control:

$$\begin{aligned} \frac{dN(t)}{dt} &= r(t)N(t)\left(1-\frac{N(t-\tau(t))}{k(t)}-c(t)\mu(t)\right), \\ \frac{d\mu(n)}{dt} &= -a(t)\mu(t)+b(t)N(t-\tau(t)). \end{aligned} \tag{1.2}$$

In [1, Theorem 2.1], under the assumptions that $a : Z \to (0,1)$, $c,k,r,b : Z \to R^+ := \{x \mid x > 0\}$ are all ω-periodic sequences, m is a nonnegative integer, which may be zero, and Δ is the first forward difference operator: $\Delta\mu(n) = \mu(n+1) - \mu(n)$. They obtained what follows.

Lemma 1.1. *System* (1.1) *has at least one positive* ω*-periodic solution.*

The following notations and definition will be useful to our discussion.

Let C denote the set of all bounded sequences $f : \mathbb{Z} \to \mathbb{R}$, and let C_+ be the set of all $f \in C$ such that $f > 0$. Given $f \in C_+$, denote

$$f^u = \sup_{k\in[0,+\infty]} f(k), \qquad f^l = \inf_{k\in[0,+\infty]} f(k). \tag{1.3}$$

Also for $f \in C_\omega := \{f \in C_+ \mid f(k+\omega) = f(k)\}$, set

$$\overline{f} = \frac{1}{\omega}\sum_{i=0}^{\omega-1} f(i). \tag{1.4}$$

Definition 1.2. System (1.1) is said to be *permanent* if there exist two positive constants λ_1, λ_2 such that

$$\lambda_1 \le \liminf_{k\to\infty} N_i(k) \le \limsup_{k\to\infty} N_i(k) \le \lambda_2, \quad i = 1,2, \tag{1.5}$$

for any solution $(N_1(k), N_2(k))$ of (1.1).

Recently, by using the comparison theorem, Chen [2] investigated the permanence of system (1.1) under the basic assumptions that

(H) $a : Z \to (0,1)$, $c,k,r,b : Z \to R^+ := \{x \mid x > 0\}$ are all bounded sequences.

He obtained what follows.

Lemma 1.3. *Assume that (H) and*

$$c^u M_2 < 1 \tag{1.6}$$

hold, then system (1.1) *is permanent, where*

$$M_2 = \frac{b^u M_1}{a^l}, \quad M_1 = \frac{k^u \exp\left\{r^u(m+1)-1\right\}}{r^u}. \tag{1.7}$$

In [3, 4], we studied the discrete predator-prey system which takes the form

$$\begin{aligned} N_1(k+1) &= N_1(k)\exp\left\{b_1(k) - a_1(k)N_1(k-[\tau_1]) - \frac{\alpha_1(k)N_1(k)N_2(k)}{N_1^2(k)+m^2N_2^2(k)}\right\},\\ N_2(k+1) &= N_2(k)\exp\left\{-b_2(k) + \frac{\alpha_2(k)N_1^2(k-[\tau_2])}{N_1^2(k-[\tau_2])+m^2N_2^2(k-[\tau_2])}\right\}. \end{aligned} \tag{1.8}$$

Using the method of comparison theorem, we obtained the following lemma.

Lemma 1.4. *Assume that the following conditions hold:*

(H1) $2m\overline{b}_1 > \overline{\alpha}_1$,

(H2) $\overline{\alpha}_2 > \overline{b}_2$.

Furthermore, assume that

$$\left(b_2(k)\right)^u < 1 \tag{1.9}$$

holds. Then, system (1.8) is permanent.

We should point out that conditions (H1) and (H2) are sufficient for the permanence of system (1.8) (the reader can refer to [5]); that is, condition (1.9) is not necessary for its permanence.

We found that if we use the method of comparison theorem, then the additional condition, to some extent, is necessary. But for the system itself, this condition may not be necessary. Motivated by the above problem, we discuss the permanence of system (1.1) again; our investigation shows that condition (1.6) is also not necessary.

2. Main results

In the remainder of this paper, for biological reasons, we only consider the solution $(x(k), \mu(k))$ of system (1.1) with initial condition

$$x(-m), x(-m+1), \ldots, x(-1) \geq 0, x(0), \mu(0) > 0. \tag{2.1}$$

One can easily show that any solution of system (1.1) with initial condition (2.1) remains positive.

First, we state our main result below.

Theorem 2.1. *Under the basic assumptions (H), the system* (1.1) *is permanent.*

In order to prove our main result, firstly we give some lemmas which will be useful for the following discussion.

Lemma 2.2. *Assume that* $A > 0$ *and* $y(0) > 0$, *and further suppose that*

(1)

$$y(n+1) \leq Ay(n) + B(n), \quad n = 1, 2, \ldots. \tag{2.2}$$

Then for any integer $k \leq n$,

$$y(n) \leq A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \tag{2.3}$$

Especially, if $A < 1$ *and* B *is bounded above with respect to* M, *then*

$$\limsup_{n\to\infty} y(n) \leq \frac{M}{1-A}, \tag{2.4}$$

(2)

$$y(n+1) \geq Ay(n) + B(n), \quad n = 1, 2, \ldots. \tag{2.5}$$

Then for any integer $k \leq n$,

$$y(n) \geq A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \tag{2.6}$$

Especially, if $A < 1$ and B is bounded below with respect to m^*, then

$$\liminf_{n\to\infty} y(n) \geq \frac{m^*}{1-A}. \tag{2.7}$$

Proof. Since the proof of (2) is similar to that of (1), we only need to prove (1). For any integer $k \leq n$,

$$\begin{gathered} y(n) \leq Ay(n-1) + B(n-1), \\ y(n-1) \leq Ay(n-2) + B(n-2), \\ \cdots \\ y(n-k+1) \leq Ay(n-k) + B(n-k), \end{gathered} \tag{2.8}$$

then

$$\begin{gathered} Ay(n-1) \leq A^2 y(n-2) + AB(n-2), \\ \cdots \\ A^{k-1} y(n-k+1) \leq A^k y(n-k) + A^{k-1} B(n-k), \end{gathered} \tag{2.9}$$

which implies that

$$\begin{aligned} y(n) &\leq A^k y(n-k) + B(n-1) + AB(n-2) + A^{k-1} B(n-k) \\ &= A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \end{aligned} \tag{2.10}$$

From the above inequality, (2.4) is obvious and the proof is complete. □

The following lemma can be found in [1, 6].

Lemma 2.3. *Assume that $\{x(n)\}$ satisfies $x(n_1) > 0$ and*

(1)

$$x(n+1) \leq x(n) \exp\left\{r(n)\left(1 - ax(n)\right)\right\} \tag{2.11}$$

for $n \in [n_1, +\infty)$, where a is a positive constant and n_1 is a positive integer. Then,

$$\limsup_{n\to\infty} x(n) \leq \frac{1}{ar^u} \exp\left(r^u - 1\right), \tag{2.12}$$

(2)

$$x(n+1) \geq x(n) \exp\{r(n)(1 - ax(n))\} \tag{2.13}$$

for $n \in [n_1, +\infty)$, where a is a positive constant and n_1 is a positive integer. Further assume that $\limsup_{n\to\infty} x(n) \leq x^*$ and $ax^* > 1$. Then,

$$\liminf_{n\to\infty} x(n) \geq \frac{1}{a} \exp\left(r^u(1 - ax^*)\right). \tag{2.14}$$

The following two lemmas are direct conclusions of [2].

Lemma 2.4. *There exists a positive constant K_1 such that*

$$\limsup_{n\to\infty} N(n) \leq K_1. \tag{2.15}$$

In fact, one can choose

$$K_1 = \frac{k^u \exp\{r^u(m+1) - 1\}}{r^u}. \tag{2.16}$$

Lemma 2.5. *There exists a positive constant K_2 such that*

$$\limsup_{n\to\infty} \mu(n) \leq K_2. \tag{2.17}$$

Similarly, one can choose

$$K_2 = \frac{b^u K_1}{a^l}. \tag{2.18}$$

Lemma 2.6. *There exists a positive constant k_1 such that*

$$\liminf_{n\to\infty} N(n) \geq k_1. \tag{2.19}$$

Proof. By Lemmas 2.4 and 2.5 and by the first equation of system (1.1), we have

$$\begin{aligned} N(n+1) &= N(n) \exp\left[r(n)\left(1 - \frac{N(n-m)}{k(n)} - c(n)\mu(n)\right)\right] \\ &\geq N(n) \exp\left\{r(n)\left(1 - \frac{K_1}{k(n)} - c(n)K_2\right)\right\} \end{aligned} \tag{2.20}$$

for n sufficiently large; then

$$\prod_{i=n-m}^{n-1} \frac{N(i+1)}{N(i)} \geq \exp\left\{\sum_{i=n-m}^{n-1} r(i)\left(1 - \frac{K_1}{k(i)} - c(i)K_2\right)\right\}. \tag{2.21}$$

Notice that the sequence $\sum_{i=n-m}^{n-1} r(i)(1 - K_1/k(i) - c(i)K_2)$ is bounded below; thus

$$N(n-m) \le N(n) \exp\left\{ - \sum_{i=n-m}^{n-1} r(i)\left(1 - \frac{K_1}{k(i)} - c(i)K_2\right)\right\}. \tag{2.22}$$

For simplicity, we set $-r(i)(1 - K_1/k(i) - c(i)K_2) = D(i)$; then

$$N(n-m) \le N(n) \exp\left\{ \sum_{i=n-m}^{n-1} D(i)\right\}. \tag{2.23}$$

From the second equation of system (1.1), we have

$$\begin{aligned} \mu(n+1) &= \left(1 - a(n)\right)\mu(n) + b(n)N(n-m) \\ &\le \left(1 - a^l\right)\mu(n) + b(n)N(n-m) \\ &\le \left(1 - a^l\right)\mu(n) + b(n)N(n) \exp\left\{ \sum_{i=n-m}^{n-1} D(i)\right\} \\ &:= A\mu(n) + B(n). \end{aligned} \tag{2.24}$$

Then, Lemma 2.2 implies that for any $k \le n - m$,

$$\begin{aligned} \mu(n) &\le A^k \mu(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1) \\ &= A^k \mu(n-k) + \sum_{i=0}^{k-1} A^i b(n-i+1) N(n-i-1-m) \\ &\le A^k \mu(n-k) + \sum_{i=0}^{k-1} A^i b^u N(n) \exp\left\{ \sum_{j=n-(i+1+m)}^{n-1} D(j)\right\}. \end{aligned} \tag{2.25}$$

Note that

$$0 \le A^k \mu(n-k) \le K_2 A^k \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{2.26}$$

Hence, there exists a positive integer K such that for any solution $(N(n), \mu(n))$ of system (1.1), $c^u A^k \mu(n-k) < 1/2$, as $k > K$. In fact, we can choose $K = \max\{1, -\log_A 2c^u K_2\}$, then we get

$$\begin{aligned} \mu(n) &\le A^K \mu(n-K) + \sum_{i=0}^{K-1} A^i b^u N(n) \exp\left\{ \sum_{j=n-(i+1+m)}^{n-1} D(j)\right\} \\ &\le A^K K_2 + \left[\sum_{i=0}^{K-1} A^i b^u \exp\left\{(i+m+1)D^u\right\}\right] N(n) \end{aligned} \tag{2.27}$$

for $n > K$.

Since $\sum_{i=0}^{K-1} A^i b^u \exp\{(i+m+1)D^u\}$ is bounded above, set $M_1 = [\sum_{i=0}^{K-1} A^i b^u \exp\{(i+m+1)D^u\}]^u$; then

$$\mu(n) \le A^K K_2 + M_1 N(n). \tag{2.28}$$

Considering the first equation of system (1.1), we have

$$N(n+1) \geq N(n)\exp\left\{r(n)\left[\left(1-c^u A^K K_2\right)-\left(c^u M_1+\frac{m\exp\{D^u\}}{k^l}\right)N(n)\right]\right\}$$
$$\geq N(n)\exp\left\{\frac{1}{2}r(n)\left[1-2\left(c^u M_1+\frac{m\exp\{D^u\}}{k^l}\right)N(n)\right]\right\}. \tag{2.29}$$

In order to prove this lemma, by Lemma 2.2(2), for the rest we only need to prove

$$2\left(c^u M_1+\frac{m\exp\{D^u\}}{k^l}\right)K_1 > 1. \tag{2.30}$$

Notice that

$$2\left(c^u M_1+\frac{m\exp\{D^u\}}{k^l}\right)K_1 > \frac{1}{k^l}K_1 = \frac{k^u}{k^l}\frac{\exp\{r^u(m+1)-1\}}{r^u} > m+1 > 1. \tag{2.31}$$

Here, we use the Bernoulli inequality $e^x > 1+x$ for $x > -1$.

Thus,

$$\liminf_{n\to\infty} N(n) \geq k_1, \tag{2.32}$$

where k_1 can be chosen as

$$k_1 = \frac{\exp\{(1/2)r^u(1-M_2K_1)\}}{M_2}, \quad M_2 = 2\left(c^u M_1+\frac{mD^u}{k^l}\right). \tag{2.33}$$

This completes the proof. □

Lemma 2.7. *There exists a positive constant k_2 such that*

$$\liminf_{n\to\infty} \mu(n) \geq k_2. \tag{2.34}$$

Proof. Without loss of generality, for any $0 < \varepsilon < (1/2)k_1$, there exists a positive integer $N_2 > K$ such that

$$N(n) \geq m_1 - \varepsilon \quad \forall n > N_2. \tag{2.35}$$

Then, the second equation implies that

$$\Delta\mu(n) \geq -a^u\mu(n) + b^l(m_1-\varepsilon) \quad \forall n > N_2. \tag{2.36}$$

By Lemma 2.2(2), we have

$$\liminf_{n\to\infty} \mu(n) \geq k_2, \tag{2.37}$$

where k_2 can be chosen as

$$k_2 = \frac{b^l k_1}{a^u}. \tag{2.38}$$

The proof is complete. □

Proof of Theorem 2.1. From Lemmas 2.4–2.7 and the definition of permanence, the conclusion is obvious. □

3. Discussion

In many situations, as our investigations show, the permanence and the existence of periodic solutions of a system are closely related with each other. But sometimes when we want to get the permanence of the system under the precondition that the periodic solution exists, we need some additional conditions. This mainly charges upon the method we used, especially when we use the comparison theorem. To make the comparison theorem holds, some additional conditions must be given, while to the system itself, these conditions may not be necessary. Just as in [2, 3], the conditions for the permanence of the system include some additional conditions besides the conditions for the existence of periodic solutions, while in this text and in [5], the conditions for the permanence of the system are exactly the same as the conditions for the existence of periodic solutions. This is because in this text and in [5] we use new methods instead of the comparison theorem.

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