Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2008, Article ID 820629, 11 pages
doi:10.1155/2008/820629

Research Article

On the Periodicity of a Difference Equation with Maximum

Ali Gelisken, Cengiz Cinar, and Ibrahim Yalcinkaya

Department of Mathematics, Education Faculty, Selcuk University, 42099 Konya, Turkey

Correspondence should be addressed to Ibrahim Yalcinkaya, iyalcinkaya1708@yahoo.com

Received 20 January 2008; Revised 8 April 2008; Accepted 18 May 2008

Recommended by Yong Zhou

We investigate the periodic nature of solutions of the max difference equation $x_{n+1} = \max\{x_n, A\}/(x_n x_{n-1})$, $n = 0, 1, \ldots$, where A is a positive real parameter, and the initial conditions $x_{-1} = A^{r_{-1}}$ and $x_0 = A^{r_0}$ such that r_{-1} and r_0 are positive rational numbers. The results in this paper answer the Open Problem 6.2 posed by Grove and Ladas (2005).

1. Introduction

Recently, there has been a great interest in studying the periodic nature of nonlinear difference equations. Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the periodic behavior of their solutions. The periodic nature of nonlinear difference equations of the max type has been investigated by many authors. See, for example, [1–20] (see also the references therein).

In [7], the following open problems were posed.

Problem 1.1 (open problem [7, page 217, Open Problem 6.1]). Assume that $A \in (0, \infty)$ and that r_1 and r_2 are positive rational numbers.

Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

Problem 1.2 (open problem [7, page 217, Open Problem 6.2]). Assume that $A \in (0, \infty)$ and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where the initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

Problem 1.3 (open problem [7, page 217, Open Problem 6.3]). Assume that $A \in (0, \infty)$ and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where the initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

Problem 1.4 (open problem [7, page 218, Open Problem 6.4]). Assume that $A \in (0, \infty)$ and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where the initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

Problem 1.5 (open problem [7, page 218, Open Problem 6.5]). Assume that $A \in (0, \infty)$, k and l are natural numbers, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.5}$$

where the initial conditions $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

In [6], we solved the open problem 6.1. And, in [20] we solved the open problem 6.4. Now, in this paper we give answer to the open problem 6.2.

2. The case $A < 1$

We consider (1.2) where $A < 1$. It is clear that the change of variables

$$x_n = A^{r_n} \quad \text{for } n \geq -1 \tag{2.1}$$

reduces (1.2) to the difference equation

$$r_{n+1} = \min\{0, 1 - r_n\} - r_{n-1}, \quad n = 0, 1, \ldots, \tag{2.2}$$

where the initial conditions are positive rational numbers.

In this section, we consider the behavior of the solutions of (2.2) (or equivalently of (1.2)) where $A < 1$. We give the following lemmas which give us explicit solutions of (2.2) for some consecutive terms and show us the pattern of the behavior of solutions of (2.2) (or equivalently of (1.2)). The proofs of some lemmas and theorems in this section are similar. So, some will be proved and the proofs of the others will be omitted.

Lemma 2.1. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2). *If at least one of the initial conditions of* (2.2) *is less than or equal to one and* $\max\{r_{-1}, r_0\} = r$, *then the following statements are true for some positive integer* N.

(a) *If* $r_N = -r$, *then* $|r_{N-1}| \leq |r_N|$.

(b) *If* $r \leq 1$, *then* $r_n = r_{n+4}$ *for all* $n \geq -1$.

(c) *If* $r > 1, r_N = -r$, *then* $r_N = r_{N+3}$ *or* $r_N = r_{N+4}$.

(d) *If* $r > 1, r_N = r_{N+3} = -r$ *and* $r_{N-1} < -1$, *then*

$$r_N \leq r_{N-1}, \qquad r_{N+2} = 1 - r_N + r_{N-1}. \tag{2.3}$$

(e) *If* $r > 1, r_N = r_{N+4} = -r$ *and* $-1 < r_{N-1}$, *then*

$$r_{N-1} \leq -r_N, \qquad r_{N+3} = 1 + r_N + r_{N-1}. \tag{2.4}$$

Proof. (a) Let $1 < r$. From (2.2), if $r_0 = r$, we have $r_1 = 1 - r_0 - r_{-1}$ and $r_2 = -r_0$. Thus, we get $r_2 \leq r_1$ and $r_3 = -r_1$. If $r_1 \leq -1$, then we get $r_4 = 2 - r_{-1}$, $r_5 = -r_0$, and $r_4 \leq -r_5$. If $-1 < r_1$, then we get $r_4 = r_0$, $r_5 = 2 - 2r_0 - r_{-1}$, $r_6 = -r_0$, and $r_6 < r_5$.

Similarly, if $r_{-1} = r$, we get that $r_1 = -r_{-1}$, $r_0 < -r_1$, and then $r_2 = -r_0$, $r_3 = r_{-1}$, $r_4 = 1 + r_0 - r_{-1}$, $r_5 = -r_{-1}$, and $r_5 < r_4$. If this proceeds, we have $|r_{N-1}| \leq |r_N|$ for $r_N = -r$ and $1 < r$.

Now, let $r \leq 1$. If $r_0 = r$, from (2.2), we have $r_1 = -r_{-1}$ and $r_2 = -r_0$. So, we obtain $r_2 < r_1$.

If $r_{-1} = r$, we have $r_1 = -r_{-1}$. So, we obtain $r_0 < -r_1$. If this proceeds, we have $|r_{N-1}| \leq |r_N|$ if $r_N = -r$ and $r \leq 1$. So, the proof of (a) is complete.

(b) Let $r \leq 1$. Then, we have $0 < r_{-1} \leq 1$ and $0 < r_0 \leq 1$. So, the proof of (b) follows directly from (2.2).

(c) Let $1 < r$ and $r_N = -r$. From (2.2), we get $r_{N+1} = -r_{N-1}$.

If $r_{N-1} < -1$, then $r_{N+2} = \min\{0, 1 - r_{N+1}\} - r_N$, $r_{N+2} = 1 - r_N + r_{N-1}$, and $r_{N+3} = \min\{0, 1 - r_{N+2}\} - r_{N+1}$, $r_{N+3} = r_N$ from (a).

If $-1 < r_{N-1}$, then $r_{N+2} = -r_N$, $r_{N+3} = 1 + r_N + r_{N-1}$, $r_{N+4} = \min\{0, -r_N - r_{N-1}\} - (-r_N)$, and $r_{N+4} = r_N$ from (a). So, the proof of (c) is complete.

(d) Let $r > 1$ and $r_N = r_{N+3} = -r$ for some positive integer N. Suppose that $r_{N-1} < -1$. From (2.2), we get that $r_{N+1} = -r_{N-1}$, $r_{N+2} = 1 - r_N + r_{N-1}$, and $r_{N+3} = \min\{0, r_N - r_{N-1}\} - (-r_{N-1})$. From $r_N = r_{N+3}$, we have $r_N \leq r_{N-1}$. So, the proof of (d) is complete.

(e) Let $r > 1$ and $r_N = r_{N+4} = -r$ for some positive integer N. Suppose that $-1 < r_{N-1}$. From (2.2), we get that $r_{N+1} = -r_{N-1}$, $r_{N+2} = -r_N$, $r_{N+3} = 1 + r_N + r_{N-1}$, and $r_{N+4} = \min\{0, -r_N - r_{N-1}\} - (-r_N)$. From $r_N = r_{N+4}$, we have $r_{N-1} \leq -r_N$. So, the proof of (e) is complete. □

The proof of the following lemma is similar and will be omitted.

Lemma 2.2. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2), *where* $r_{-1}, r_0 > 1$. *If* $r_{-1} + r_0 = r$, *then the following statements are true for some positive integer* N.

(a) *If* $r_N = 1 - r$, *then* $|r_{N-1}| \leq |r_N|$.

(b) *If* $r_N = 1 - r$, *then* $r_N = r_{N+3}$ *or* $r_N = r_{N+4}$.

(c) *If* $r_N = r_{N+3} = 1 - r$ *and* $r_{N-1} < -1$, *then*

$$r_N \leq r_{N-1}, \qquad r_{N+2} = 1 - r_N + r_{N-1}. \tag{2.5}$$

(d) *If* $r_N = r_{N+4} = 1 - r$ *and* $-1 < r_{N-1}$, *then*

$$r_{N-1} \leq -r_N, \qquad r_{N+3} = 1 + r_N + r_{N-1}. \tag{2.6}$$

Remark 2.3. In view of Lemma 2.1 for $r > 1$, it is clear that $r_j = -r$ and $r_j = r_{j+4}$ for $j = 1$ or $j = 2$. But, it is not clear $r_{j+4} = r_{j+7}$ or $r_{j+4} = r_{j+8}$. If $r_{j+4} = r_{j+7}$, then $r_{j+7} = r_{j+10}$ or $r_{j+7} = r_{j+11}$. If $r_{j+4} = r_{j+8}$, then $r_{j+8} = r_{j+11}$ or $r_{j+8} = r_{j+12}$. So, we have $r_{N+3} = -r$ or $r_{N+4} = -r$, if $r_N = -r$ for $N > 2$.

In view of Lemma 2.1, for $r > 1$, from Lemma 2.1(c), we obtain

$$r_{N+2} = 1 - r_N + r_{N-1}, \qquad r_{N+3} = r_N = -r.$$

Also, from Lemma 2.1(d), we have

$$r_{N+6} = 2 + r_{N-1}, \qquad r_{N+7} = r_{N+3} = -r. \tag{2.7}$$

Now, applying Lemma 2.1(d) and then we get

$$r_{N+3} = 1 + r_N + r_{N-1}, \qquad r_{N+4} = r_N = -r.$$

Moreover, from Lemma 2.1(c), we have

$$r_{N+6} = 2 + r_{N-1}, \qquad r_{N+7} = r_{N+4} = -r.$$

It shows that the last corresponding two terms are the same in each two cases. So, we can apply Lemma 2.1(c) or Lemma 2.1(d) for getting the last two terms we need. Furthermore, it is reality there are infinite number of integers N satisfying Lemma 2.1(b). If we determine exactly the number of integers N, we can apply Lemmas 2.1(c) and 2.1(d), consecutively. Also, in view of Lemma 2.1,

$$\begin{aligned} &\text{if } r_N = -r, \quad r_{N-1} < -1, \quad \text{then } r_N = r_{N+3}, \\ &\text{if } r_N = -r, \quad -1 < r_{N-1}, \quad \text{then } r_N = r_{N+4}. \end{aligned} \tag{2.8}$$

Remark 2.4. In view of Lemma 2.2, we can get similar results as Remark 2.3. So, we have the following:

$$\begin{aligned} &\text{if } r_N = 1 - r, \quad r_{N-1} < -1, \quad \text{then } r_N = r_{N+3}, \\ &\text{if } r_N = 1 - r, \quad -1 < r_{N-1}, \quad \text{then } r_N = r_{N+4}. \end{aligned} \tag{2.9}$$

Also, we can apply Lemma 2.2(c) or Lemma 2.2(d) firstly for getting the last two terms we need.

Clearly, there are infinite number of integers N satisfying Lemma 2.1 or Lemma 2.2. We give the following two lemmas about the number of integers N.

Lemma 2.5. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2) *satisfying Lemma 2.1. If* $\max\{r_{-1}, r_0, 1\} = k/m$ *and* $\mathrm{GCD}(k, m) = 1$, *then the following statements are true.*

(a) *If* $k+m$ *is an even integer, then the number of integers* N *satisfying Lemma 2.1(c) is* $(k-m)/2$ *and the number of integers* N *satisfying Lemma 2.1(d) is* $(k+m)/2$ *for* $N < (7k+m)/2$.

(b) *If* $k+m$ *is an odd integer, then the number of integers* N *satisfying Lemma 2.1(c) is* $k-m$ *and the number of integers* N *satisfying Lemma 2.1(d) is* $k+m$ *for* $N < 7k+m$.

Proof. (a) Assume that $k+m$ is an even integer. So, both of k and m are even or odd integers. From $\mathrm{GCD}(k, m) = 1$, k and m are odd integers. We have $\max\{r_{-1}, r_0, 1\} = k/m \geq 1$.

Let $k/m = 1$. Then, we have $k = m = 1$, $0 < r_{-1} \leq 1$, and $0 < r_0 \leq 1$. If $r_{-1} = r$, we get $r_1 = -r$ from (2.2), then we get $r_1 = r_5 = -r$ from $-1 < r_0$, Lemma 2.1(d) and Remark 2.3. Similarly, If $r_0 = r$, we get $r_2 = -r$ from (2.2), then we get $r_2 = r_6 = -r$ from $-1 < r_1 = -r_{-1}$, Lemma 2.1(d) and Remark 2.3. So, the number of integers N satisfying Lemma 2.1(d) is $(1+1)/2 = 1$ for $N < (7.1+1)/2 = 4$, such that $N = 1$ or $N = 2$. There are not any integers N satisfying Lemma 2.1(c). So, the claim is true for $k/m = 1$.

Let $k/m > 1$ and the number of integers N satisfying Lemma 2.1(c) is $((k-m)/2+1)$ for $N < (7k+m)/2$. From Remark 2.3, observe that j is the smallest integer of the integers N satisfying Lemma 2.1(b), such that $j = 1$ or $j = 2$. This assumption and Remark 2.3 allow us that Lemma 2.1(c) can be applied consecutively for iterated $((k-m)/2+1)$ times such that

$$r_{j+4+3l_1} = r_{j+4+3(l_1+1)} = -r, \quad r_{j+4+3l_1-1} < -1 \quad \text{for } l_1 = 0, 1, \ldots, \frac{k-m}{2}. \tag{2.10}$$

So, we have

$$\begin{aligned} r_{j+4+3((k-m)/2)-1} &< -1, \\ r_{j+3} &= 1 + r_j + r_{j-1}, \\ r_{j+6} &= 1 - r_{j+4} + r_{j+3} = 1 + r_j + (1 - r_j) + r_{j-1}, \\ r_{j+9} &= 1 - r_{j+7} + r_{j+6} = 1 + r_j + 2(1 - r_j) + r_{j-1}, \\ &\vdots \\ r_{j+4+3((k-m)/2)-1} &= 1 + r_j + \left(\frac{k-m}{2}\right)(1 - r_j) + r_{j-1}. \end{aligned} \tag{2.11}$$

Thus,

$$\begin{gathered} 1 + r_j + \left(\frac{k-m}{2}\right)(1 - r_j) + r_{j-1} = 1 - \frac{k}{m} + \left(\frac{k-m}{2}\right)\left(1 - \left(-\frac{k}{m}\right)\right) + r_{j-1} < -1, \\ r_{j-1} < -2 + \frac{k}{m} - \left(\frac{k-m}{2}\right)\left(1 + \frac{k}{m}\right) \leq -\frac{k}{m}. \end{gathered} \tag{2.12}$$

This shows that $r_{j-1} < r_j$. But, it contradicts Lemma 2.1(a). This means Lemma 2.1(c) cannot be applied consecutively for iterated $((k-m)/2+1)$ times. So, the number of integers N satisfying Lemma 2.1(c) is not more than $(k-m)/2$ for $N < (7k+m)/2$.

Similarly, we assume that the number of integers N satisfying Lemma 2.1(d) is $((k+m)/2+1)$ for $N < (7k+m)/2$. So, we can apply Lemma 2.1(d) consecutively for iterated $((k+m)/2+1)$ times such that

$$r_{j+4l_2} = r_{j+4(l_2+1)} = -r, \quad r_{j+4l_2-1} > -1 \quad \text{for } l_2 = 0, 1, \ldots, \frac{k+m}{2}. \tag{2.13}$$

So, we have

$$\begin{aligned} r_{j+4((k+m)/2)-1} &> -1, \\ r_{j+3} &= 1 + r_j + r_{j-1}, \\ r_{j+8} &= 1 + r_{j+4} + r_{j+3} = 2(1 + r_j) + r_{j-1}, \\ &\vdots \\ r_{j+4((k+m)/2)-1} &= \left(\frac{k+m}{2}\right)(1 + r_j) + r_{j-1}. \end{aligned} \tag{2.14}$$

Thus, we have

$$\begin{gathered} \left(\frac{k+m}{2}\right)(1 + r_j) + r_{j-1} = \left(\frac{k+m}{2}\right)\left(1 + \left(-\frac{k}{m}\right)\right) + r_{j-1} > -1, \\ r_{j-1} > -1 - \left(\frac{k+m}{2}\right)\left(1 - \frac{k}{m}\right) \geq \frac{k}{m}. \end{gathered} \tag{2.15}$$

This means that $r_{j-1} > -r_j$. But, it contradicts Lemma 2.1(a) So, the number of integers N satisfying Lemma 2.1(c) is not more than $(k+m)/2$.

We assume that the number of integers N satisfying Lemma 2.1(c) is $(k-m)/2-1$ for $N < (7k+m)/2$. We have just had the number of integers N satisfying Lemma 2.1(d) is less than $(k+m)/2+1$. We apply Lemma 2.1(d) for iterated $(k+m)/2$ times, and then we get that the first integer N satisfying Lemma 2.1(c) is $[j+4((k+m)/2)]$ such that $r_{j+4((k+m)/2)} = r_{j+4((k+m)/2)+3} = -r$. So, we apply Lemma 2.1(c) for iterated $((k-m)/2-2)$ times, and then we get that the biggest integer N satisfying Lemma 2.1(b) for $N < (7k+m)/2$ is $[j+4((k+m)/2)+3((k-m)/2-2)]$ such that $r_{j+4((k+m)/2)+3((k-m)/2-2)} = -r$. From Lemma 2.1(b), $r_{j+4((k+m)/2)+3((k-m)/2-2)+3} = -r$ or $r_{j+4((k+m)/2)+3((k-m)/2-2)+4} = -r$. But, $r_{j+4((k+m)/2)+3((k-m)/2-2)+4} = -r$ is not possible because the number of integers N satisfying Lemma 2.1(d) is not $((k+m)/2+1)$. So, it must be $r_{j+4((k+m)/2)+3((k+m)/2-2)+3} = -r$ and it contradicts our assumption. Thus, the number of integers N satisfying Lemma 2.1(c) is $(k-m)/2$ exactly for $N < (7k+m)/2$.

Similarly, the number of integers N satisfying Lemma 2.1(d) is not $((k+m)/2-1)$ and can be showed for $N < (7k+m)/2$. So, the number of integers N satisfying Lemma 2.1(d) is $(k+m)/2$ exactly for $N < (7k+m)/2$. The proof is complete.

(b) The proof of (b) is similar and will be omitted. □

The proof of the following lemma is similar and will be omitted.

Lemma 2.6. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2) *satisfying Lemma 2.2. If* $r = k/m$ *and* $\mathrm{GCD}(k, m) = 1$, *then the following statements are true.*

(a) *If k is an even integer, then the number of integers N satisfying Lemma 2.2(a) is $(k-2m)/2$ and the number of integers N satisfying Lemma 2.2(b) is $k/2$ for $N < (7k-6m)/2$.*

(b) *If k is an odd integer, then the number of integers N satisfying Lemma 2.2(a) is $k-2m$ and the number of integers N satisfying Lemma 2.2(b) is k for $N < 7k-6m$.*

We give the following two lemmas which are generalized from Lemmas 2.1 and 2.5, and Remark 2.3. It allows us to more quickly calculate terms in the solution of (2.2).

Lemma 2.7. *Suppose that $\{r_n\}_{n=-1}^{\infty}$ is a solution of (2.2) satisfying Lemmas 2.1 and 2.5. If $k+m$ is an even integer, then the following statements are true.*

(a) *If $r_N = r_{N+4l_1} = -r$, then*

$$r_{N+4l_1-1} = 1 + r_{N+4(l_1-1)} + r_{N+4(l_1-1)-1} \tag{2.16}$$

for $l_1 = 1, 2, \ldots, (k+m)/2$.

(b) *If $r_{N+4((k+m)/2)} = r_{N+4((k+m)/2)+3l_2} = -r$, then*

$$r_{N+4((k+m)/2)+3l_2-1} = 1 - r_{N+4((k+m)/2)+3(l_2-1)} + r_{N+4((k+m)/2)+3(l_2-1)-1} \tag{2.17}$$

for $l_2 = 1, 2, \ldots, (k-m)/2$.

Lemma 2.8. *Suppose that $\{r_n\}_{n=-1}^{\infty}$ is a solution of (2.2) satisfying Lemma 2.1 and 2.5. If $k+m$ is an odd integer, then the following statements are true.*

(a) *If $r_N = r_{N+4l_1} = -r$, then*

$$r_{N+4l_1-1} = 1 + r_{N+4(l_1-1)} + r_{N+4(l_1-1)-1} \tag{2.18}$$

for $l_1 = 1, 2, \ldots, k+m$.

(b) *If $r_{N+4(k+m)} = r_{N+4(k+m)+3l_2} = -r$, then*

$$r_{N+4(k+m)+3l_2-1} = 1 - r_{N+4(k+m)+3(l_2-1)} + r_{N+4(k+m)+3(l_2-1)-1} \tag{2.19}$$

for $l_2 = 1, 2, \ldots, k-m$.

The following two lemmas are generalized from Lemmas 2.2, 2.6, and Remark 2.4.

Lemma 2.9. *Suppose that $\{r_n\}_{n=-1}^{\infty}$ is a solution of (2.2) satisfying Lemmas 2.2 and 2.6. If k is an even integer, then the following statements are true.*

(a) *If $r_N = r_{N+4l_1} = 1-r$, then*

$$r_{N+4l_1-1} = 1 + r_{N+4(l_1-1)} + r_{N+4(l_1-1)-1} \tag{2.20}$$

for $l_1 = 1, 2, \ldots, k/2$.

(b) *If $r_{N+4(k/2)} = r_{N+4(k/2)+3l_2} = 1-r$, then*

$$r_{N+4(k/2)+3l_2-1} = 1 - r_{N+4(k/2)+3(l_2-1)} + r_{N+4(k/2)+3(l_2-1)-1} \tag{2.21}$$

for $l_2 = 1, 2, \ldots, (k-2m)/2$.

Lemma 2.10. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2) *satisfying Lemmas 2.2 and 2.6. If k is an odd integer, then the following statements are true.*

(a) *If* $r_N = r_{N+4l_1} = 1 - r$, *then*

$$r_{N+4l_1-1} = 1 + r_{N+4(l_1-1)} + r_{N+4(l_1-1)-1} \tag{2.22}$$

for $l_1 = 1, 2, \ldots, k$.

(b) *If* $r_{N+4k} = r_{N+4k+3l_2} = 1 - r$, *then*

$$r_{N+4k+3l_2-1} = 1 - r_{N+4k+3(l_2-1)} + r_{N+4k+3(l_2-1)-1} \tag{2.23}$$

for $l_2 = 1, 2, \ldots, k - 2m$.

Theorem 2.11. *Suppose that* $\{x_n\}_{n=-1}^{\infty}$ *is a solution of* (1.2) *with the initial conditions* $x_{-1} = A^{r_{-1}}$ *and* $x_0 = A^{r_0}$, *such that* $0 < A < 1$, r_{-1} *and* r_0 *are positive rational numbers. Let at least one of* r_{-1} *and* r_0 *is less than or equal to one. If* $\max\{r_{-1}, r_0, 1\} = k/m$ *and* $\mathrm{GCD}(k, m) = 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic*

$$\text{with prime period } \begin{cases} \dfrac{7k+m}{2}, & \text{if } k+m \text{ is an even integer}, \\ 7k+m, & \text{if } k+m \text{ is an odd integer}. \end{cases} \tag{2.24}$$

Proof. Let $\{r_n\}_{n=-1}^{\infty}$ is a solution of (2.2) satisfying Lemmas 2.1 and 2.5. We assume that $k + m$ is an odd integer. We must show that

$$r_n = r_{n+7k+m} \quad \forall n \geq -1. \tag{2.25}$$

From Lemma 2.8(a), we get that

$$r_N = r_{N+4(k+m)}, \qquad r_{N+4(k+m)-1} = (k+m)(1 + r_N) + r_{N-1}. \tag{2.26}$$

Then, from Lemma 2.8(b), we get that

$$\begin{gathered} r_{N+4(k+m)} = r_{N+4(k+m)+3(k-m)}, \\ r_{N+4(k+m)+3(k-m)-1} = (k+m)(1 + r_N) + r_{N-1} + (k-m)(1 - r_N). \end{gathered} \tag{2.27}$$

So, at the end of this process, we have $r_N = r_{N+7k+m}$ and $r_{N-1} = r_{N+7k+m-1}$. From $r_{n-1} = \min\{0, 1 - r_n\} - r_{n+1}$, we get immediately $r_N = r_{N+7k+m}$ for all $N \geq -1$. Also, it is easy to see that $r_{N-1} \neq r_{N+4l_1-1}$ and $r_{N-1} \neq r_{N+4(k+m)+3l_2-1}$ for $l_1 = 1, 2, \ldots, k+m$ and $l_2 = 1, 2, \ldots, k-m-1$. It shows that $7k + m$ is prime period. So, the proof is complete. The proof of the case that $k + m$ is an even integer is similar and will be omitted. □

The proof of the following theorem is similar and follows directly from Lemmas 2.9 and 2.10.

Theorem 2.12. *Suppose that* $\{x_n\}_{n=-1}^{\infty}$ *is a solution of* (1.2) *with the initial conditions* $x_{-1} = A^{r_{-1}}$ *and* $x_0 = A^{r_0}$ *such that* $0 < A < 1$, r_{-1} *and* r_0 *are positive rational numbers. Let* $1 < r_{-1}, r_0$. *If* $r_{-1} + r_0 = k/m$ *and* $\mathrm{GCD}(k, m) = 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic*

$$\text{with prime period } \begin{cases} \dfrac{7k-6m}{2} & \text{if } k \text{ is an even integer}, \\ 7k-6m & \text{if } k \text{ is an odd integer}. \end{cases} \tag{2.28}$$

3. The case $A > 1$

We consider (1.2) where $A > 1$. It is clear that the change of variables

$$x_n = A^{r_n} \quad \text{for } n \geq -1 \tag{3.1}$$

reduces (1.2) to the difference equation

$$r_{n+1} = \max\{0, 1 - r_n\} - r_{n-1}, \quad n = 0, 1, \ldots, \tag{3.2}$$

where the initial conditions are positive rational numbers.

In this section, we consider the behavior of the solutions of (3.2) (or equivalently of (1.2)) where $A > 1$. We omit the proofs of the following results since they can easily be obtained in a way similar to the proofs of the lemmas and theorems in the previous section.

Lemma 3.1. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (3.2). *If* $\max\{r_{-1}, r_0\} = r$, *then the following statements are true for some integer* $N \geq -1$.

(a) *If* $r_N = r$, *then* $|r_{N-1}| \leq r_N$.

(b) *If* $r_N = r$, *then* $r_N = r_{N+3}$ *or* $r_N = r_{N+4}$.

(c) *If* $r_N = r_{N+3} = r$ *and* $-1 < r_{N-1}$, *then*

$$r_{N-1} \leq r_N, \qquad r_{N+2} = 1 - r_N + r_{N-1}. \tag{3.3}$$

(d) *If* $r_N = r_{N+4} = r$ *and* $r_{N-1} < -1$, *then*

$$-r_N \leq r_{N-1}, \qquad r_{N+3} = 1 + r_N + r_{N-1}.$$

Lemma 3.2. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (3.2) *satisfying Lemma 3.1. If* $\max\{r_{-1}, r_0, 1\} = k/m$ *and* $\mathrm{GCD}(k, m) = 1$, *then the following statements are true.*

(a) *If* $k+m$ *is an even integer, then the number of integers* N *satisfying Lemma 3.1(a) is* $(k+m)/2$ *and the number of integers* N *satisfying Lemma 3.1(b) is* $(k-m)/2$ *for* $N < (7k - m)/2$.

(b) *If* $k + m$ *is an odd integer, then the number of integers* N *satisfying Lemma 3.1(a) is* $k + m$ *and the number of integers* N *satisfying Lemma 3.1(b) is* $k - m$ *for* $N < 7k - m$.

The following two lemmas are generalized from Lemmas 3.1 and 3.2.

Lemma 3.3. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (3.2) *satisfying Lemmas 3.1 and 3.2. If* $k + m$ *is an even integer, then the following statements are true.*

(a) *If* $r_N = r_{N+4l_1} = r$, *then*

$$r_{N+4l_1-1} = 1 + r_{N+4(l_1-1)} + r_{N+4(l_1-1)-1} \tag{3.4}$$

for $l_1 = 1, 2, \ldots, (k - m)/2$.

(b) *If* $r_{N+4((k-m)/2)} = r_{N+4((k-m)/2)+3l_2} = r$, *then*

$$r_{N+4((k-m)/2)+3l_2-1} = 1 - r_{N+4((k-m)/2)+3(l_2-1)} + r_{N+4((k-m)/2)+3(l_2-1)-1} \tag{3.5}$$

for $l_2 = 1, 2, \ldots, (k+m)/2$.

Lemma 3.4. *Suppose that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (3.2) *satisfying Lemmas 3.1 and 3.2. If* $k+m$ *is an odd integer, then the following statements are true.*

(a) *If* $r_N = r_{N+4l_1} = r$, *then*

$$r_{N+4l_1-1} = 1 + r_{N+4(l_1-1)} + r_{N+4(l_1-1)-1} \tag{3.6}$$

for $l_1 = 1, 2, \ldots, k-m$.

(b) *If* $r_{N+4(k-m)} = r_{N+4(k-m)+3l_2} = r$, *then*

$$r_{N+4(k-m)+3l_2-1} = 1 - r_{N+4(k-m)+3(l_2-1)} + r_{N+4(k-m)+3(l_2-1)-1} \tag{3.7}$$

for $l_2 = 1, 2, \ldots, k+m$.

Theorem 3.5. *Suppose that* $\{x_n\}_{n=-1}^{\infty}$ *is a solution of* (1.2) *with the initial conditions* $x_{-1} = A^{r_{-1}}$ *and* $x_0 = A^{r_0}$ *such that* $1 < A$, r_{-1}, *and* r_0 *are positive rational numbers. If* $\max\{r_{-1}, r_0, 1\} = k/m$ *and* $\mathrm{GCD}(k, m) = 1$, *then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic*

$$\text{with prime period } \begin{cases} \dfrac{7k-m}{2} & \text{if } k+m \text{ is an even integer,} \\ 7k-m & \text{if } k+m \text{ is an odd integer.} \end{cases} \tag{3.8}$$

4. Conclusion

In this paper, we have solved the open problem 6.2 which was proposed in [7]. We think that the method used in this work may solve the open problems 6.3 and 6.5 which were proposed in [7].

Acknowledgment

The authors are grateful to the anonymous referees for their valuable suggestions that improved the quality of this study.

References

[1] R. M. Abu-Saris and F. M. Allan, "Periodic and nonperiodic solutions of the difference equation $x_{n+1} = \max\{x_n^2, A\}/x_n x_{n-1}$," in *Advances in Difference Equations (Veszprém, 1995)*, pp. 9–17, Gordon and Breach, Amsterdam, The Netherlands, 1997.

[2] R. M. Abu-Sarris and F. M. Allan, "Rational recursive sequences involving the maximum function," *Far East Journal of Mathematical Sciences*, vol. 1, no. 3, pp. 335–342, 1999.

[3] K. Berenhaut, J. Foley, and S. Stevic, "Boundedness character of positive solutions of a max difference equation," *Journal of Difference Equations and Applications*, vol. 12, pp. 1193–1199, 2006.

[4] J. Feuer, "On the eventual periodicity of $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ with a period-four parameter," *Journal of Difference Equations and Applications*, vol. 12, no. 5, pp. 467–486, 2006.

[5] J. Feuer, "Periodic solutions of the Lyness max equation," *Journal of Mathematical Analysis and Applications*, vol. 288, no. 1, pp. 147–160, 2003.

[6] A. Gelisken, C. Cinar, and R. Karatas, "A note on the periodicity of the Lyness max equation," *Advances in Difference Equations*, vol. 2008, Article ID 651747, 5 pages, 2008.

[7] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, vol. 4 of *Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2005.

[8] E. J. Janowski, V. L. Kocic, G. Ladas, and S. W. Schultz, "Global behavior of solutions of $x_{n+1} = \max\{x_n, A\}/(x_{n-1})$," in *Proceedings of the 1st International Conference on Difference Equations (San Antonio, Tex, USA 1994)*, pp. 273–282, Gordon and Breach, May 1994.

[9] G. Ladas, "Open problems and conjectures," *Journal of Difference Equations and Applications*, vol. 2, no. 3, pp. 339–341, 1996.

[10] G. Ladas, "Open problems and conjectures," *Journal of Difference Equations and Applications*, vol. 4, no. 3, p. 312, 1998.

[11] D. P. Mishev, W. T. Patula, and H. D. Voulov, "A reciprocal difference equation with maximum," *Computers & Mathematics with Applications*, vol. 43, no. 8-9, pp. 1021–1026, 2002.

[12] D. P. Mishev, W. T. Patula, and Hristo D. Voulov, "Periodic coefficients in a reciprocal difference equation with maximum," *PanAmerican Mathematical Journal*, vol. 13, no. 3, pp. 43–57, 2003.

[13] W. T. Patula and H. D. Voulov, "On a max type recurrence relation with periodic coefficients," *Journal of Difference Equations and Applications*, vol. 10, no. 3, pp. 329–338, 2004.

[14] M. Golomb, W. Lafayette, and D. Callan, "Problem E3437," *The American Mathematical Monthly*, vol. 99, no. 9, pp. 882–883, 1992.

[15] S. Stevic, "On the recursive sequence $x_{n+1} = \max\{c, x_n^p/x_{n-1}^p\}$," *Applied Mathematics Letters*. In press.

[16] I. Yalçinkaya, B. D. Iričanin, and C. Çinar, "On a max-type difference equation," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 47264, 10 pages, 2007.

[17] S. Stevic, "On the recursive sequence $x_{n+1} = A + x_n^p/x_{n-1}^r$," *Journal of Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 40963, 9 pages, 2007.

[18] S. Stevic, "Boundedness Character of a class of difference equations," *Journal of Nonlinear Analysis*, In press.

[19] C. Çinar, S. Stević, and I. Yalçinkaya, "On positive solutions of a reciprocal difference equation with minimum," *Journal of Applied Mathematics & Computing*, vol. 17, no. 1-2, pp. 307–314, 2005.

[20] I. Yalçinkaya, C. Cinar, and A. Gelisken, "On the periodicity of a max-type difference equation," to appear.