Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2008, Article ID 629620, 17 pages
doi:10.1155/2008/629620

Research Article

Permanence and Global Attractivity of a Delayed Discrete Predator-Prey System with General Holling-Type Functional Response and Feedback Controls

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Received 8 June 2008; Accepted 11 September 2008

Recommended by Leonid Berezansky

This paper discusses a delayed discrete predator-prey system with general Holling-type functional response and feedback controls. Firstly, sufficient conditions are obtained for the permanence of the system. After that, under some additional conditions, we show that the periodic solution of the system is global stable.

1. Introduction

The following predator-prey system with Holling-type II functional response and delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau_1(t)) - \frac{a_{12}(t)x_2(t)}{1+mx_1(t)}\right], \\ \dot{x}_2(t) &= x_2(t)\left[-r_2(t) - \frac{a_{21}(t)x_1(t-\tau_2(t))}{1+mx_1(t-\tau_2(t))} - a_{22}(t)x_2(t-\tau_3(t))\right], \end{aligned} \tag{1.1}$$

and some generalized systems of general Holling-type functional response have been studied by many scholars (see [1–3] and the references cited therein). It has been found that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations (see [4–12]). In [4], Yang considered the following delayed discrete predator-prey system with general Holling-type functional response:

$$
\begin{aligned}
N_1(k+1) &= N_1(k)\exp\left\{r(k) - b(k)N_1^{\theta}(k-[\tau_1(t)]) - \frac{\alpha(k)N_1^{p-1}(k)}{1+mN_1^{p}(k)}N_2(k-[\sigma(t)])\right\},\\
N_2(k+1) &= N_2(k)\exp\left\{-d(k) - a(k)N_2(k-[\tau_2(t)]) + \frac{\beta(k)N_1^{p}(k-[\tau_3(t)])}{1+mN_1^{p}(k-[\tau_3(t)])}\right\}.
\end{aligned}
\tag{1.2}
$$

Sufficient conditions which guarantee the existence of at least one positive periodic solution are obtained by using the continuation theorem of coincidence degree theory. But Yang did not consider the permanence and globally attractivity of system (1.2), which are two of the most important topics in the study of population dynamics.

On the other hand, as was pointed out by Huo and Li [13], ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables (for more discussion on this section, one could refer to [12–16] for more details). Though much works dealt with the continuous time model. However, to the best of the author's knowledge, up to this day, there are still no scholars that propose and study the system (1.2) with feedback control. Therefore, the main purpose of this paper is to study the following delayed discrete predator-prey system with general Holling-type functional response and feedback control:

$$
\begin{aligned}
x_1(k+1) &= x_1(k)\exp\left\{r_1(k) - b_1(k)x_1^{\theta}(k-\tau_1) - \frac{\alpha_1(k)x_1^{p-1}(k)}{1+mx_1^{p}(k)}x_2(k-\tau_3) - e_1(k)u_1(k)\right\},\\
x_2(k+1) &= x_2(k)\exp\left\{-r_2(k) - b_2(k)x_2(k-\tau_2) + \frac{\alpha_2(k)x_1^{p}(k-\tau_4)}{1+mx_1^{p}(k-\tau_4)} - e_2(k)u_2(k)\right\},\\
\Delta u_1(k) &= -\eta_1(k)u_1(k) + q_1(k)x_1(k),\\
\Delta u_2(k) &= -\eta_2(k)u_2(k) + q_2(k)x_2(k),
\end{aligned}
\tag{1.3}
$$

where $x_1(k)$ is the density of prey species at kth generation, $x_2(k)$ is the density of predator species at kth generation, $u_1(k)$ and $u_2(k)$ are control variables. Also, $r_1(k)$, $b_1(k)$ denote the intrinsic growth rate and density-dependent coefficient of the prey, respectively, $r_2(k)$, $b_2(k)$ denote the death rate and density-dependent coefficient of the predator, $\alpha_1(k)$ denote the capturing rate of the predator, $\alpha_2(k)/\alpha_1(k)$ represent the rate of conversion of nutrients into the reproduction of the predator. Further, τ_i $(i = 1,2,3,4)$ are nonnegative constants and m,p are positive constants. In this paper, we always assume that $\{r_i(k)\}, \{b_i(k)\}, \{\alpha_1(k)\}, \{e_i(k)\}, \{\eta_i(k)\}, \{q_i(k)\}$, $i = 1,2$, are bounded nonnegative sequences and

$$
0 < \eta_i^{L} \le \eta_i^{M} < 1, \quad i = 1,2. \tag{1.4}
$$

Here, for any bounded sequence $\{a(k)\}$, $a^M = \sup_{k\in N}\{a(k)\}$, and $a^L = \inf_{k\in N}\{a(k)\}$, where $N = \{0,1,2,\ldots\}$.

This paper is organized as follows. In Section 2, we will introduce some definition and establish several useful lemma. The permanence of system (1.3) is then studied in Section 3. In Section 4, based on the permanence result, under the assumption that all the delays are equal to zero and the coefficients of the system are periodic sequences, we obtain a set of sufficient conditions which guarantee the existence and stability of a unique globally attractive positive periodic solution of the system.

By the biological meaning, we will focus our discussion on the positive solution of system (1.3). So it is assumed that the initial conditions of (1.3) are of the form

$$x_i(-k) \ge 0, \quad u_i(-k) \ge 0, \quad k \in N \cap (0,\tau], \quad x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1,2, \tag{1.5}$$

where $\tau = \max\{\tau_1,\tau_2,\tau_3,\tau_4\}$.

One can easily show that the solutions of (1.3) with the initial condition (1.5) are defined and remain positive for all $k \in N$.

2. Preliminaries

In this section, we will introduce the definition of permanence and several useful lemmas.

Definition 2.1. System (1.3) is said to be permanent if there exist positive constants $x_i^*, u_i^*, x_{i*}, u_{i*}$, which are independent of the solution of the system, such that for any positive solution $(x_1(k), x_2(k), u_1(k), u_2(k))$ of system (1.3) satisfies

$$\begin{aligned} x_{i*} &\le \liminf_{k\to\infty} x_i(k) \le \limsup_{k\to\infty} x_i(k) \le x_i^*, \\ u_{i*} &\le \liminf_{k\to\infty} u_i(k) \le \limsup_{k\to\infty} u_i(k) \le u_i^*, \end{aligned} \tag{2.1}$$

for $i = 1,2$.

Lemma 2.2. *Assume that $x(k)$ satisfies*

$$x(k+1) \le x(k)\exp\left\{a(k) - b(k)x^{\theta}(k)\right\} \quad \forall\, k \ge k_0, \tag{2.2}$$

where $\{a(k)\}$ and $\{b(k)\}$ are positive sequences, $x(k_0) > 0$, θ is a positive constant, and $k_0 \in N$. Then one has

$$\limsup_{k\to\infty} x(k) \le D, \tag{2.3}$$

where $D = (1/\theta b^L)^{1/\theta}\exp(a^M - 1/\theta)$.

Lemma 2.3. *Assume that $x(k)$ satisfies*

$$x(k+1) \ge x(k)\exp\left\{a(k) - b(k)x^{\theta}(k)\right\} \quad \forall\, k \ge k_0, \tag{2.4}$$

where $\{a(k)\}$ *and* $\{b(k)\}$ *are positive sequences,* $x(k_0) > 0$*,* θ *is a positive constant, and* $k_0 \in N$*. Also,* $\limsup_{k\to\infty} x(k) \le D$ *and* $b^M D^\theta / a^L > 1$*. Then one has*

$$\liminf_{k\to\infty} x(k) \ge C, \tag{2.5}$$

where $C = (a^L/b^M)^{1/\theta} \exp(a^L - b^M D^\theta)$*.*

Proof. The proofs of Lemmas 2.2 and 2.3 are very similar to those of [6, Propositions 2.1 and 2.2], respectively. So we omit the detail here. □

Lemma 2.4. *Assume that* $x(k)$ *satisfies*

$$x(k+1) \le x(k) \exp\left\{a(k) - b(k) x^\theta(k-\tau)\right\} \quad \forall k \ge k_0 > \tau, \tag{2.6}$$

where $\{a(k)\}$ *and* $\{b(k)\}$ *are positive sequences,* $x(k_0) > 0$*,* θ *and* τ *are positive constants, and* $k_0 \in N$*. Then one has*

$$\limsup_{k\to\infty} x(k) \le B, \tag{2.7}$$

where $B = (1/\theta\beta^L)^{1/\theta} \exp(a^M - 1/\theta)$ *and* $\beta(k) = b(k) \exp\{-\theta \sum_{i=k-\tau}^{k-1} a(i)\}$*.*

Proof. From the above equation, one has

$$x(k+1) \le x(k) \exp\{a(k)\} \quad \forall\, k \ge k_0. \tag{2.8}$$

Sequently we can easily obtain that

$$x(k-\tau) \ge x(k) \exp\left\{ - \sum_{i=k-\tau}^{k-1} a(i) \right\}. \tag{2.9}$$

So one has

$$\begin{aligned} x(k+1) &\le x(k) \exp\left\{ a(k) - b(k) \exp\left\{ -\theta \sum_{i=k-\tau}^{k-1} a(i) \right\} x^\theta(k) \right\} \\ &= x(k) \exp\left\{ a(k) - \beta(k) x^\theta(k) \right\}. \end{aligned} \tag{2.10}$$

By Lemma 2.2, we can complete the proof of Lemma 2.4. □

Lemma 2.5. *Assume that* $x(k)$ *satisfies*

$$x(k+1) \ge x(k) \exp\left\{a(k) - b(k) x^\theta(k-\tau)\right\} \quad \forall k \ge k_0 > \tau, \tag{2.11}$$

where $\{a(k)\}$ *and* $\{b(k)\}$ *are positive sequences,* $x(k_0) > 0$*,* θ *and* τ *are positive constants, and* $k_0 \in N$*. Also,* $\limsup_{k\to\infty} x(k) \le B$ *and* $\gamma^M B^\theta / a^L > 1$*, where* $\gamma(k) = b(k)\exp\{-\theta\sum_{i=k-\tau}^{k-1}(a(i) - b(i)B^\theta)\}$*. Then one has*

$$\liminf_{k\to\infty} x(k) \ge A, \tag{2.12}$$

where $A = (a^L/\gamma^M)^{1/\theta}\exp(a^L - \gamma^M B^\theta)$.

Proof. From the above equation, one has

$$x(k+1) \ge x(k)\exp\left\{a(k) - b(k)D^\theta\right\} \quad \forall k \ge k_0. \tag{2.13}$$

Sequently we can easily obtain that

$$x(k-\tau) \le x(k)\exp\left\{-\sum_{i=k-\tau}^{k-1}(a(i) - b(i)D^\theta)\right\}. \tag{2.14}$$

So one has

$$\begin{aligned} x(k+1) &\ge x(k)\exp\left\{a(k) - b(k)\exp\left\{-\theta\sum_{i=k-\tau}^{k-1}(a(i) - b(i)D^\theta)\right\}x^\theta(k)\right\} \\ &= x(k)\exp\left\{a(k) - \gamma(k)x^\theta(k)\right\}. \end{aligned} \tag{2.15}$$

By Lemma 2.3, we can complete the proof of Lemma 2.5. □

Lemma 2.6 is a direct corollary of [17, Theorem 6.2, page 125] by L. Wang and M. Q. Wang.

Lemma 2.6. *Consider the following first-order difference equation:*

$$y(k+1) = Ay(k) + B, \quad k = 1, 2\ldots, \tag{2.16}$$

where A, B *are positive constants. Assuming* $A < 1$*, for any solution* $\{y(k)\}$ *of the above system, one has*

$$\lim_{k\to\infty} y(k) = \frac{B}{1-A}. \tag{2.17}$$

The following comparison theorem for the difference equation is of [17, Theorem 2.1, page 241] by L. Wang and M. Q. Wang.

Lemma 2.7. *Let* $k \in \{k_0, k_0+1, \ldots, k_0+l, \ldots\}$, $r \geq 0$. *For fixed* k, $g(k,r)$ *is a nondecreasing function with respect to* r, *and for* $k \geq k_0$, *the following inequalities hold:*

$$y(k+1) \leq g(k, y(k)),$$
$$u(k+1) \geq g(k, u(k)). \tag{2.18}$$

If $y(k_0) \leq u(k_0)$, *then* $y(k) \leq u(k)$ *for all* $k \geq k_0$.

3. Permanence

In this section, we establish a permanent result for system (1.3).

Proposition 3.1. *In addition to* (1.4), *assume further that*

(H_1)

$$\left(\frac{\alpha_2(k)}{m} - r_2(k)\right)^L > 0; \tag{3.1}$$

for any positive solution $(x_1(k), x_2(k), u_1(k), u_2(k))$ *of system* (1.3), *one has*

$$\limsup_{k\to\infty} x_i(k) \leq x_i^*, \quad \limsup_{k\to\infty} u_i(k) \leq u_i^*, \quad i = 1,2, \tag{3.2}$$

where

$$x_1^* = \left(\frac{1}{\theta\beta_1^L}\right)^{1/\theta} \exp\left(r_1^M - \frac{1}{\theta}\right), \qquad \beta_1(k) = b_1(k)\exp\left\{-\theta\sum_{i=k-\tau_1}^{k-1} r_1(i)\right\},$$
$$x_2^* = \frac{1}{\beta_2^L}\exp\left(\left(\frac{\alpha_2(k)}{m} - r_2(k)\right)^M - 1\right), \qquad \beta_2(k) = b_2(k)\exp\left\{-\sum_{i=k-\tau_2}^{k-1}\left(\frac{\alpha_2(i)}{m} - r_2(i)\right)\right\},$$
$$u_1^* = \frac{q_1^M x_1^*}{\eta_1^L}, \qquad u_2^* = \frac{q_2^M x_2^*}{\eta_2^L}. \tag{3.3}$$

Proof. Let $(x_1(k), x_2(k), u_1(k), u_2(k))$ be any positive solution of system (1.3), from the first equation of (1.3), it follows that

$$x_1(k+1) \leq x_1(k)\exp\left\{r_1(k) - b_1(k)x_1^\theta(k-\tau_1)\right\}. \tag{3.4}$$

By applying Lemmas 2.4 and 2.7, we obtain

$$\limsup_{k\to\infty} x_1(k) \leq x_1^*, \tag{3.5}$$

where

$$x_1^* = \left(\frac{1}{\theta\beta_1^L}\right)^{1/\theta} \exp\left(r_1^M - \frac{1}{\theta}\right), \qquad \beta_1(k) = b_1(k)\exp\left\{-\theta\sum_{i=k-\tau_1}^{k-1} r_1(i)\right\}. \tag{3.6}$$

Similarly, from the second equation of (1.3), it follows that

$$x_2(k+1) \le x_2(k)\exp\left\{\frac{\alpha_2(k)}{m} - r_2(k) - b_2(k)x_2(k-\tau_2)\right\}. \tag{3.7}$$

Under the assumption (H_1), by applying Lemmas 2.4 and 2.7, we obtain

$$\limsup_{k\to\infty} x_2(k) \le x_2^*, \tag{3.8}$$

where

$$x_2^* = \frac{1}{\beta_2^L}\exp\left(\left(\frac{\alpha_2(k)}{m} - r_2(k)\right)^M - 1\right), \qquad \beta_2(k) = b_2(k)\exp\left\{-\sum_{i=k-\tau_2}^{k-1}\left(\frac{\alpha_2(i)}{m} - r_2(i)\right)\right\}. \tag{3.9}$$

For any positive constant ε small enough, it follows from (3.5) and (3.8) that there exists large enough $K_1 > \tau$ such that

$$x_1(k) \le x_1^* + \varepsilon, \quad x_2(k) \le x_2^* + \varepsilon \quad \forall k \ge K_1. \tag{3.10}$$

Then the third equation of (1.3) leads to

$$\Delta u_1(k) \le -\eta_1(k)u_1(k) + q_1(k)(x_1^* + \varepsilon). \tag{3.11}$$

And so

$$u_1(k+1) \le (1-\eta_1^L)u_1(k) + q_1^M(x_1^* + \varepsilon) \quad \forall k \ge K_1. \tag{3.12}$$

By applying Lemmas 2.6 and 2.7, it follows from (3.12) that

$$\limsup_{k\to\infty} u_1(k) \le \frac{q_1^M(x_1^* + \varepsilon)}{\eta_1^L}. \tag{3.13}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\limsup_{k\to\infty} u_1(k) \le u_1^*, \tag{3.14}$$

where $u_1^* = q_1^M x_1^* / \eta_1^L$. Similarly, we can obtain

$$\limsup_{k \to \infty} u_2(k) \le u_2^*, \tag{3.15}$$

where $u_2^* = q_2^M x_2^* / \eta_2^L$. Thus we complete the proof of Proposition 3.1. □

Proposition 3.2. *In addition to* (1.4)*, assume further that*

(H_2)

$$r_1^L - \alpha_1^M (x_1^*)^{p-1} x_2^* - e_1^M u_1^* > 0, \tag{3.16}$$

(H_3)

$$-r_2^M + \frac{\alpha_2^L x_{1*}^p}{1 + m x_{1*}^p} - e_2^M u_2^* > 0, \tag{3.17}$$

for any positive solution $(x_1(k), x_2(k), u_1(k), u_2(k))$ *of system* (1.3)*, there exist positive constants* x_{i*}, u_{i*}*, such that*

$$\liminf_{k \to \infty} x_i(k) \ge x_{i*}, \quad \liminf_{k \to \infty} u_i(k) \ge u_{i*}, \quad i = 1, 2. \tag{3.18}$$

Proof. Let $(x_1(k), x_2(k), u_1(k), u_2(k))$ be any positive solution of system (1.3). From (H_2) and (H_3), there exists a small enough positive constant ε such that

$$r_1^L - \alpha_1^M (x_1^* + \varepsilon)^{p-1} (x_2^* + \varepsilon) - e_1^M (u_1^* + \varepsilon) > 0, \tag{3.19}$$

$$-r_2^M + \frac{\alpha_2^L (x_{1*} - \varepsilon)^p}{1 + m (x_{1*} - \varepsilon)^p} - e_2^M (u_2^* + \varepsilon) > 0. \tag{3.20}$$

Also, according to Proposition 3.1, for the above ε, there exists $K_2 > K_1$ such that for $k \ge K_2$,

$$\begin{aligned} x_1(k) &\le x_1^* + \varepsilon, \qquad x_2(k) \le x_2^* + \varepsilon, \\ u_1(k) &\le u_1^* + \varepsilon, \qquad u_2(k) \le u_2^* + \varepsilon. \end{aligned} \tag{3.21}$$

Then from the first equation of (1.3), one has

$$\begin{aligned} x_1(k+1) &\ge x_1(k) \exp\left\{ r_1(k) - b_1(k) x_1^\theta (k - \tau_1) - \alpha_1(k) x_1^{p-1}(k) x_2(k - \tau_3) - e_1(k) u_1(k) \right\}, \\ &\ge x_1(k) \exp\left\{ r_1(k) - \alpha_1(k) (x_1^* + \varepsilon)^{p-1} (x_2^* + \varepsilon) - e_1(k)(u_1^* + \varepsilon) - b_1(k) x_1^\theta (k - \tau_1) \right\}. \end{aligned} \tag{3.22}$$

Let $a_1(k,\varepsilon) = r_1(k) - \alpha_1(k)(x_1^* + \varepsilon)^{p-1}(x_2^* + \varepsilon) - e_1(k)(u_1^* + \varepsilon)$, so the above inequality follows that

$$x_1(k+1) \geq x_1(k)\exp\{a_1(k,\varepsilon) - b_1(k)x_1^{\theta}(k-\tau_1)\}. \tag{3.23}$$

Consequently, let $\gamma_1(k,\varepsilon) = b_1(k)\exp\{-\theta\sum_{i=k-\tau_1}^{k-1}(a_1(i,\varepsilon) - b_1(i)x_1^{*\theta})\}$. Because $\gamma_1^M > \beta_1^L$, one has

$$\frac{\gamma_1^M}{a_1^L}(x_1^*)^{\theta} = \frac{\gamma_1^M}{a_1^L}\frac{\exp(\theta r_1^M - 1)}{\theta\beta_1^L} > 1. \tag{3.24}$$

Here we use the fact that $\exp(\theta r_1^M - 1) > \theta r_1^M > \theta a_1^M > \theta a_1^L$. From (3.19) and (3.23), by Lemmas 2.5 and 2.7, one has

$$\liminf_{k\to\infty} x_1(k) \geq \left(\frac{a_1^L(\varepsilon)}{\gamma_1^M(\varepsilon)}\right)^{1/\theta}\exp\left\{a_1^L(\varepsilon) - \gamma_1^M(\varepsilon)(x_1^*)^{\theta}\right\}. \tag{3.25}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\liminf_{k\to\infty} x_1(k) \geq x_{1*}, \tag{3.26}$$

where

$$\begin{gathered} x_{1*} = \left(\frac{a_1^L}{\gamma_1^M}\right)^{1/\theta}\exp\left\{a_1^L - \gamma_1^M(x_1^*)^{\theta}\right\}, \\ a_1(k) = r_1(k) - \alpha_1(k)(x_1^*)^{p-1}x_2^* - e_1(k)u_1^*, \end{gathered} \tag{3.27}$$

and $\gamma_1(k) = b_1(k)\exp\{-\theta\sum_{i=k-\tau_1}^{k-1}(a_1(i) - b_1(i)x_1^{*\theta})\}$.

Similarly, from the second equation of (1.3), one has

$$x_2(k+1) \geq x_2(k)\exp\left\{-r_2(k) + \frac{\alpha_2(k)(x_{1*}-\varepsilon)^p}{1+m(x_{1*}-\varepsilon)^p} - e_2(k)(u_2^* + \varepsilon) - b_2(k)x_2(k-\tau_2)\right\}. \tag{3.28}$$

Let $a_2(k,\varepsilon) = -r_2(k) + \alpha_2(k)(x_{1*}-\varepsilon)^p/(1+m(x_{1*}-\varepsilon)^p) - e_2(k)(u_2^*+\varepsilon)$, so the above inequality leads to

$$x_2(k+1) \geq x_2(k)\exp\{a_2(k,\varepsilon) - b_2(k)x_2(k-\tau_2)\}. \tag{3.29}$$

Consequently, let $\gamma_2(k,\varepsilon) = b_2(k)\exp\{-\sum_{i=k-\tau_2}^{k-1}(a_2(i,\varepsilon) - b_2(i)x_2^*)\}$. Because $\gamma_2^M > \beta_2^L$, one has

$$\frac{\gamma_2^M}{a_2^L}x_2^* = \frac{\gamma_2^M}{a_2^L}\frac{\exp\{(\alpha_2(k)/m - r_2(k))^M - 1\}}{\beta_2^L} > 1. \tag{3.30}$$

Here we use the fact that $\exp\{(\alpha_2(k)/m - r_2(k))^M - 1\} > (\alpha_2(k)/m - r_2(k))^M > a_2^M > a_2^L$. From (3.20) and (3.29), by Lemmas 2.5 and 2.7, one has

$$\liminf_{k\to\infty} x_2(k) \geq \frac{a_2^L(\varepsilon)}{\gamma_2^M(\varepsilon)} \exp\left\{a_2^L(\varepsilon) - \gamma_2^M(\varepsilon)x_2^*\right\}. \tag{3.31}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\liminf_{k\to\infty} x_2(k) \geq x_{2*}, \tag{3.32}$$

where

$$\begin{aligned} x_{2*} &= \frac{a_2^L}{\gamma_2^M} \exp\left\{a_2^L - \gamma_2^M x_2^*\right\}, \\ a_2(k) &= -r_2(k) + \frac{\alpha_2(k)x_{1*}^p}{1 + m x_{1*}^p} - e_2(k)u_2^*, \\ \gamma_2(k) &= b_2(k)\exp\left\{-\sum_{i=k-\tau_2}^{k-1}\left(a_2(i) - b_2(i)x_2^*\right)\right\}. \end{aligned} \tag{3.33}$$

Then the third equation of (1.3) leads to

$$\Delta u_1(k) \geq -\eta_1(k)u_1(k) + q_1(k)(x_{1*} - \varepsilon). \tag{3.34}$$

And so

$$u_1(k+1) \geq (1 - \eta_1^M)u_1(k) + q_1^L(x_{1*} - \varepsilon) \quad \forall k \geq K_2. \tag{3.35}$$

By applying Lemmas 2.6 and 2.7, it follows from (3.35) that

$$\limsup_{k\to\infty} u_1(k) \geq \frac{q_1^L(x_{1*} - \varepsilon)}{\eta_1^M}. \tag{3.36}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\limsup_{k\to\infty} u_1(k) \geq u_{1*}, \tag{3.37}$$

where $u_{1*} = q_1^L x_{1*}/\eta_1^M$. Similarly, we can obtain

$$\limsup_{k\to\infty} u_2(k) \leq u_{2*}, \tag{3.38}$$

where $u_{2*} = q_2^L x_{2*}/\eta_2^M$. Thus we complete the proof of Proposition 3.2. □

Theorem 3.3. *In addition to* (1.4)*, assume further that* (H_1), (H_2), *and* (H_3) *hold, then system* (1.3) *is permanent.*

It should be noticed that, from the proofs of Propositions 3.1 and 3.2, we know that under the conditions of Theorem 3.3, the set $\Omega = \{(x_1, x_2, u_1, u_2) \mid x_{i*} \le x_i \le x_i^*,\ u_{i*} \le u_i \le u_i^*,\ i = 1,2\}$ is an invariant set of system (1.3).

4. Existence and stability of a periodic solution

In this section, we consider the stability property of system (1.3) under the assumption $\tau_i = 0$ $(i = 1,2,3,4)$, that is, we consider the following system:

$$\begin{aligned}
x_1(k+1) &= x_1(k)\exp\left\{r_1(k) - b_1(k)x_1^{\theta}(k) - \frac{\alpha_1(k)x_1^{p-1}(k)}{1+mx_1^p(k)}x_2(k) - e_1(k)u_1(k)\right\},\\
x_2(k+1) &= x_2(k)\exp\left\{-r_2(k) - b_2(k)x_2(k) + \frac{\alpha_2(k)x_1^{p}(k)}{1+mx_1^p(k)} - e_2(k)u_2(k)\right\},\\
\Delta u_1(k) &= -\eta_1(k)u_1(k) + q_1(k)x_1(k),\\
\Delta u_2(k) &= -\eta_2(k)u_2(k) + q_2(k)x_2(k),
\end{aligned} \tag{4.1}$$

which are similar to system (1.3) but do not include delays. In this section, we always assume that $\{r_i(k)\}$, $\{b_i(k)\}$, $\{\alpha_1(k)\}$, $\{e_i(k)\}$, $\{\eta_i(k)\}$, $\{q_i(k)\}$ are bounded nonnegative periodic sequences with a common period ω and satisfy

$$0 < \eta_i(k) < 1, \quad k \in N \cap [0,\omega],\ i = 1,2. \tag{4.2}$$

Also it is assumed that the initial conditions of (4.1) are of the form

$$x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1,2. \tag{4.3}$$

Using a similar way, under some conditions, we can obtain the permanence of system (4.1). As above, still let x_i^* and u_i^*, $i = 1,2$, be the upper bound of $\{x_i(k)\}$ and $\{u_i(k)\}$, x_{i*} and let u_{i*}, $i = 1,2$, be the lower bound of $\{x_i(k)\}$ and $\{u_i(k)\}$, where x_i^*, u_i^*, x_{i*}, and u_{i*} are independent of the solution of system (4.1). Our first result concerns with the existence of a periodic solution.

Theorem 4.1. *In addition to* (4.2)*, assume further that* (H_1), (H_2), *and* (H_3) *hold, then system* (4.1) *has a periodic solution denoted by* $\{\overline{x}_1(k), \overline{x}_2(k), \overline{u}_1(k), \overline{u}_2(k)\}$.

Proof. Let $\Omega = \{(x_1, x_2, u_1, u_2) \mid x_{i*} \le x_i \le x_i^*,\ u_{i*} \le u_i \le u_i^*,\ i = 1,2\}$, Ω is an invariant set of system (4.1). Thus, we can define a mapping F on Ω by

$$F(x_1(0), x_2(0), u_1(0), u_2(0)) = (x_1(\omega), x_2(\omega), u_1(\omega), u_2(\omega)) \tag{4.4}$$

for $(x_1(0), x_2(0), u_1(0), u_2(0)) \in \Omega$.

Obviously, F depends continuously on $(x_1(0), x_2(0), u_1(0), u_2(0))$. Thus F is continuous and maps a compact set Ω into itself. Therefore, F has a fixed point $(\overline{x}_1, \overline{x}_2, \overline{u}_1, \overline{u}_2)$. It is easy to see that the solution $\{\overline{x}_1(k), \overline{x}_2(k), \overline{u}_1(k), \overline{u}_2(k)\}$ passing through $(\overline{x}_1, \overline{x}_2, \overline{u}_1, \overline{u}_2)$ is a periodic solution of system (4.1). This completes the proof. □

Now, we study the globally stability property of the periodic solution obtained in Theorem 4.1.

Theorem 4.2. *In addition to the conditions of Theorem 4.1, if system* (4.1) *satisfies*

$$\lambda_1 = \max\left\{\left|1 - \theta b_1^L x_{1*}^{\theta}\right|, \ \left|1 - \theta b_1^M (x_1^*)^{\theta} - \alpha_1^M W_1 x_1^*\right|\right\} + \alpha_1^M W_2 x_2^* + e_1^M < 1, \tag{4.5}$$

$$\lambda_2 = \max\left\{\left|1 - b_2^L x_{2*}\right|, \ \left|1 - b_2^M x_2^*\right|\right\} + \alpha_2^M W_3 x_1^* + e_2^M < 1, \tag{4.6}$$

$$\lambda_3 = 1 - \eta_1^L + q_1^M x_1^* < 1, \tag{4.7}$$

$$\lambda_4 = 1 - \eta_2^L + q_2^M x_2^* < 1, \tag{4.8}$$

where the definition of W_i, $i = 1,2,3$ can be seen in the following proof, then the ω-periodic solution $(\overline{x}_1(k), \overline{x}_2(k), \overline{u}_1(k), \overline{u}_2(k))$ obtained in Theorem 4.1 is globally attractive.

Proof. Assume that $(x_1(k), x_2(k), u_1(k), u_2(k))$ is any positive solution of system (4.1), let

$$x_i(k) = \overline{x}_i(k)\exp\{y_i(k)\}, \quad u_i(k) = \overline{u}_i(k) + v_i(k), \quad i = 1,2. \tag{4.9}$$

To complete the proof, it suffices to show that

$$\lim_{k\to\infty} y_i(k) = 0, \quad \lim_{k\to\infty} v_i(k) = 0, \quad i = 1,2. \tag{4.10}$$

Since

$$\begin{aligned}
y_1(k+1) &= y_1(k) - b_1(k)\overline{x}_1^{\theta}(k)(\exp\{\theta y_1(k)\} - 1) - e_1(k)v_1(k) \\
&\quad - \alpha_1(k)\left[\frac{\overline{x}_1^{p-1}(k)\exp\{(p-1)y_1(k)\}}{1 + m\overline{x}_1^{p}(k)\exp\{p y_1(k)\}}\overline{x}_2(k)\exp\{y_2(k)\} - \frac{\overline{x}_1^{p-1}(k)}{1 + m\overline{x}_1^{p}(k)}\overline{x}_2(k)\right] \\
&= y_1(k) - b_1(k)\overline{x}_1^{\theta}(k)\exp\{\xi_1(k)\theta y_1(k)\}\theta y_1(k) - e_1(k)v_1(k) \\
&\quad - \alpha_1(k)\left[f_1'(\xi_2'(k), x_2(k))\overline{x}_1(k)\exp\{\xi_4(k)y_1(k)\}y_1(k)\right. \\
&\qquad \left. + f_2'(\overline{x}_1(k), \xi_3'(k))\overline{x}_2(k)\exp\{\xi_5(k)y_2(k)\}y_2(k)\right],
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
f(x,y) &= \frac{x^{p-1}y}{1 + m x^{p}}, \\
\xi_2'(k) &= \overline{x}_1(k) + \xi_2(k)(x_1(k) - \overline{x}_1(k)), \\
\xi_3'(k) &= \overline{x}_2(k) + \xi_3(k)(x_2(k) - \overline{x}_2(k)),
\end{aligned} \tag{4.12}$$

and $\xi_i(k) \in (0,1)$ for $i = 1,2,3,4,5$. Because of the boundedness of $\{\overline{x}_1(k)\}$, $\{\overline{x}_2(k)\}$, $\{x_1(k)\}$, $\{x_2(k)\}$, $|f_1'(\xi_2'(k), x_2(k))|$, $|f_2'(\overline{x}_1(k), \xi_3'(k))|$ are bounded, where f_1' and f_2' mean the partial derivation of the function $f(x,y)$. Let $|f_1'(\xi_2'(k), x_2(k))| < W_1$ and $|f_2'(\overline{x}_1(k), \xi_3'(k))| < W_2$.

Similarly, we get

$$\begin{aligned} y_2(k+1) &= y_2(k) - b_2(k)\overline{x}_2(k)(\exp\{y_2(k)\} - 1) \\ &\quad + \alpha_2(k)\left[\frac{\overline{x}_1^p(k)\exp\{py_1(k)\}}{1 + m\overline{x}_1^p(k)\exp\{py_1(k)\}} - \frac{\overline{x}_1^p(k)}{1 + m\overline{x}_1^p(k)}\right] - e_2(k)v_2(k) \\ &= y_2(k) - b_2(k)\overline{x}_2(k)\exp\{\xi_5(k)y_2(k)\}y_2(k) \\ &\quad + \alpha_2(k)g'(\xi_6'(k))\overline{x}_1(k)(\exp\{y_1(k)\} - 1) - e_2(k)v_2(k) \\ &= y_2(k) - b_2(k)\overline{x}_2(k)\exp\{\xi_5(k)y_2(k)\}y_2(k) \\ &\quad + \alpha_2(k)g'(\xi_6'(k))\overline{x}_1(k)\exp\{\xi_4(k)y_1(k)\}y_1(k) - e_2(k)v_2(k), \end{aligned} \tag{4.13}$$

where

$$\xi_6'(k) = \overline{x}_1(k) + \xi_6(k)(x_1(k) - \overline{x}_1(k)), \quad \xi_6(k) \in (0,1). \tag{4.14}$$

Because of the boundedness of $\{\overline{x}_1(k)\}, \{\overline{x}_2(k)\}, \{x_1(k)\}, \{x_2(k)\}, g'(\xi_6'(k))$ is bounded, where $g(x) = x^p/(1 + mx^p)$ and g' means the derivation of the function $g(x)$. Let $|g'(\xi_6'(k))| < W_3$.

Also, one has

$$\begin{aligned} v_1(k+1) &= (1 - \eta_1(k))v_1(k) + q_1(k)\overline{x}_1(k)(\exp\{y_1(k)\} - 1) \\ &= (1 - \eta_1(k))v_1(k) + q_1(k)\overline{x}_1(k)\exp\{\xi_4(k)y_1(k)\}y_1(k), \\ v_2(k+1) &= (1 - \eta_2(k))v_2(k) + q_2(k)\overline{x}_2(k)\{(\exp\{y_2(k)\} - 1\}) \\ &= (1 - \eta_2(k))v_2(k) + q_2(k)\overline{x}_2(k)\exp\{\xi_5(k)y_2(k)\}y_2(k). \end{aligned} \tag{4.15}$$

In view of (4.5)–(4.8), we can choose a $\varepsilon > 0$ such that

$$\begin{aligned} &\lambda_1^\varepsilon = \max\left\{\left|1 - \theta b_1^L(x_{1*} - \varepsilon)^\theta\right|, \left|1 - \theta b_1^M(x_1^* + \varepsilon)^\theta - \alpha_1^M W_1(x_1^* + \varepsilon)\right|\right\} + \alpha_1^M W_2(x_2^* + \varepsilon) + e_1^M < 1, \\ &\lambda_2^\varepsilon = \max\left\{\left|1 - b_2^L(x_{2*} - \varepsilon)\right|, \left|1 - b_2^M(x_2^* + \varepsilon)\right|\right\} + \alpha_2^M W_3(x_1^* + \varepsilon) + e_2^M < 1, \\ &\lambda_3^\varepsilon = 1 - \eta_1^L + q_1^M(x_1^* + \varepsilon) < 1, \\ &\lambda_4^\varepsilon = 1 - \eta_2^L + q_2^M(x_2^* + \varepsilon) < 1. \end{aligned} \tag{4.16}$$

Also, from Propositions 3.1 and 3.2, there exist $K_3 > K_2$ such that

$$x_{i*} - \varepsilon \le x_i(k), \quad x_i^*(k) \le x_i^* + \varepsilon \quad \forall k \ge K_3, \; i = 1,2. \tag{4.17}$$

Then from (4.11), for $k > K_3$, one has

$$\begin{aligned}|y_1(k+1)| \le \max\left\{\left|1-\theta b_1^L(x_{1*}-\varepsilon)^{\theta}\right|, \left|1-\theta b_1^M(x_1^*+\varepsilon)^{\theta}-\alpha_1^M W_1(x_1^*+\varepsilon)\right|\right\} \\ \cdot\left|y_1(k)\right| + \alpha_1^M W_2(x_2^*+\varepsilon)\left|y_2(k)\right| + e_1^M\left|v_1(k)\right|.\end{aligned} \tag{4.18}$$

So from (4.13), for $k > K_3$, one has

$$\begin{aligned}|y_2(k+1)| \le \max\left\{\left|1-b_2^L(x_{2*}-\varepsilon)\right|, \left|1-b_2^M(x_2^*+\varepsilon)\right|\right\}|y_2(k)| \\ + \alpha_2^M W_3(x_1^*+\varepsilon)|y_1(k)| + e_2^M|v_2(k)|.\end{aligned} \tag{4.19}$$

Also, for $k > K_3$, one has

$$|v_1(k+1)| \le (1-\eta_1^L)|v_1(k)| + q_1^M(x_1^*+\varepsilon)|y_1(k)|, \tag{4.20}$$

$$|v_2(k+1)| \le (1-\eta_2^L)|v_2(k)| + q_2^M(x_2^*+\varepsilon)|y_2(k)|. \tag{4.21}$$

Let $\lambda = \max\{\lambda_1^{\varepsilon}, \lambda_2^{\varepsilon}, \lambda_3^{\varepsilon}, \lambda_4^{\varepsilon}\}$, then $0 < \lambda < 1$. In view of (4.18)–(4.21), one has

$$\max\{|y_1(k+1)|, |y_2(k+1)|, |v_1(k+1)|, |v_2(k+1)|\} \le \lambda \max\{|y_1(k)|, |y_2(k)|, |v_1(k)|, |v_2(k)|\} \tag{4.22}$$

for $k > K_3$. This implies

$$\max\{|y_1(k)|, |y_2(k)|, |v_1(k)|, |v_2(k)|\} \le \lambda^{k-K_3} \max\{|y_1(K_3)|, |y_2(K_3)|, |v_1(K_3)|, |v_2(K_3)|\}. \tag{4.23}$$

Therefore

$$\lim_{k\to\infty} y_i(k) = 0, \quad \lim_{k\to\infty} v_i(k) = 0, \quad i = 1, 2. \tag{4.24}$$

This completes the proof. □

5. Examples

The following two examples show the feasibility of our main results.

Example 5.1. Consider system (1.3) with

$$\begin{gathered} r_1(k) = 0.14 + 0.01\ \cos(k), \quad b_1(k) = 0.1, \quad \alpha_1(k) = 0.001, \quad e_1(k) = 0.03 + 0.01\ \sin(k), \\ r_2(k) = 0.18 + 0.02\ \cos\left(\sqrt{2}k\right), \quad b_2(k) = 1.8 + 0.1\sin(k), \quad \alpha_2(k) = 1.4, \\ e_2(k) = 0.008 + 0.002\ \sin(k), \quad \eta_1(k) = 0.7, \quad q_1(k) = 0.2 + 0.1\sin(k), \quad \eta_2(k) = 0.8, \\ q_2(k) = 0.1, \quad \tau_1 = \tau_2 = \tau_3 = \tau_4 = 1, \quad p = 1.3, \quad \theta = 1.2, \quad m = 0.8, \end{gathered} \tag{5.1}$$

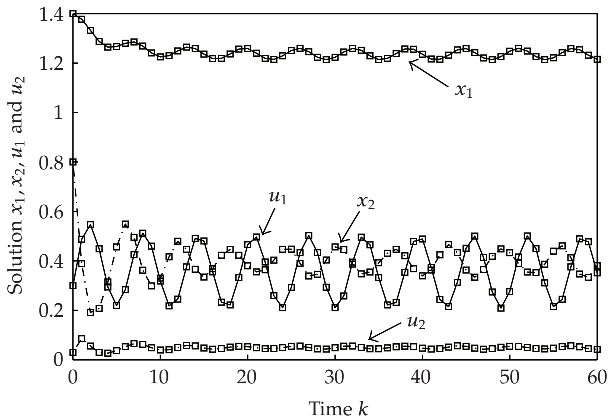

Figure 1: Dynamics behaviors of system (1.3) with initial condition $(x_1(p), x_2(p), u_1(p), u_2(p)) = (1.4, 0.8, 0.3, 0.03)$ $(P = -1, 0)$.

for all $k \in N$. One can easily see that

$$x_1^* \approx 3.7945, \qquad x_{1*} \approx 0.2882, \qquad x_2^* \approx 5.2037, \qquad u_1^* \approx 1.6262, \qquad u_2^* \approx 0.6505, \tag{5.2}$$

which means that

$$r_1^L - \alpha_1^M (x_1^*)^{p-1} x_2^* - e_1^M u_1^* \approx 0.0872, \tag{5.3}$$

$$-r_2^M + \frac{\alpha_2^L x_{1*}^p}{1 + m x_{1*}^p} - e_2^M u_2^* \approx 0.0333. \tag{5.4}$$

Also, one has

$$\left(\frac{\alpha_2(k)}{m} - r_2(k)\right)^L \approx 1.55. \tag{5.5}$$

Inequalities (5.3)–(5.5) show that (H_1)–(H_3) are fulfilled. From Theorem 3.3, system (1.3) is permanent. Figure 1 is the numeric simulation of the solution of system (1.3) with initial condition $(x_1(p), x_2(p), u_1(p), u_2(p)) = (1.4, 0.8, 0.3, 0.03)$ $(P = -1, 0)$.

Example 5.2. Consider system (4.1) with

$$\begin{gathered} r_1(k) = 0.13 + 0.02\cos(k), \qquad b_1(k) = 0.1, \qquad \alpha_1(k) = 0.01, \qquad e_1(k) = 0.03 + 0.01\sin(k), \\ r_2(k) = 0.16 + 0.02\cos(k), \qquad b_2(k) = 0.7, \qquad \alpha_2(k) = 0.6, \qquad e_2(k) = 0.015 + 0.005\sin(k), \\ \eta_1(k) = 0.7, \qquad q_1(k) = 0.2 + 0.1\sin(k), \qquad \eta_2(k) = 0.8, \qquad q_2(k) = 0.2 + 0.1\sin(k), \end{gathered} \tag{5.6}$$

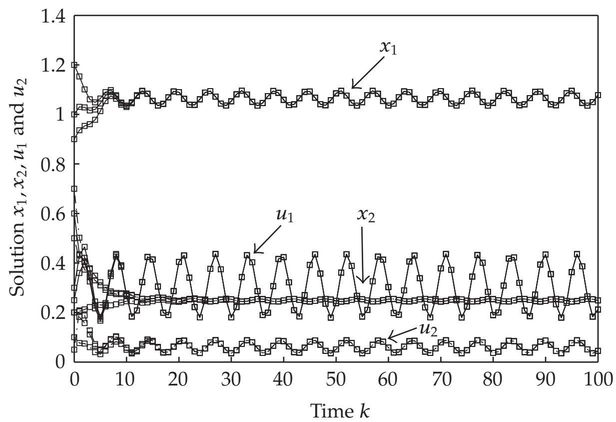

Figure 2: Dynamics behaviors of system (4.1) with initial values $(x_1(0), x_2(0), u_1(0), u_2(0)) = (0.9, 0.7, 0.6, 0.05)$, $(1.2, 0.5, 0.3, 0.2)$, and $(1, 0.2, 0.25, 0.1)$.

for all $k \in N$. One can easily see that

$$x_1^* \approx 1.2808, \quad x_{1*} \approx 0.8195, \quad x_2^* \approx 0.9672, \quad x_{2*} \approx 0.1157, \quad u_1^* \approx 0.5489, \quad u_2^* \approx 0.3627, \tag{5.7}$$

which means that

$$r_1^L - \alpha_1^M (x_1^*)^{p-1} x_2^* - e_1^M u_1^* \approx 0.0801, \tag{5.8}$$

$$-r_2^M + \frac{\alpha_2^L x_{1*}^p}{1 + m x_{1*}^p} - e_2^M u_2^* \approx 0.1387. \tag{5.9}$$

Also, one has

$$\left(\frac{\alpha_2(k)}{m} - r_2(k) \right)^L \approx 0.5700. \tag{5.10}$$

Inequalities (5.8)–(5.10) show that (H_1)–(H_3) are fulfilled. We can obtain that

$$W_1 \approx 0.7258, \quad W_2 \approx 0.6630, \quad W_3 \approx 0.075, \tag{5.11}$$

which means that

$$\lambda_1 \approx 0.8861, \quad \lambda_2 \approx 0.9966, \quad \lambda_3 \approx 0.6842, \quad \lambda_4 \approx 0.4902. \tag{5.12}$$

So (4.5)–(4.8) are fulfilled. From Theorem 4.2, system (4.1) is globally attractive. Figure 2 is the numeric simulation of the solution of system (4.1) with initial condition $(x_1(0), x_2(0), u_1(0), u_2(0)) = (0.9, 0.7, 0.6, 0.05), (1.2, 0.5, 0.3, 0.2)$, and $(1, 0.2, 0.25, 0.1)$.

Acknowledgment

This work was supported by the Foundation of Fujian Education Bureau (JB05042).

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