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Research Article

Twin Positive Solutions of a Nonlinear m-Point Boundary Value Problem for Third-Order p-Laplacian Dynamic Equations on Time Scales

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Several existence theorems of twin positive solutions are established for a nonlinear m-point boundary value problem of third-order p-Laplacian dynamic equations on time scales by using a fixed point theorem. We present two theorems and four corollaries which generalize the results of related literature. As an application, an example to demonstrate our results is given. The obtained conditions are different from some known results.

1. Introduction

A time scale $\mathbf{T}$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that 0 and T are points in $\mathbf{T}$. By an interval $(0,T)$, we always mean the intersection of the real interval $(0,T)$ with the given time scale, that is, $(0,T)\cap\mathbf{T}$.

In this paper, we will be concerned with the existence of positive solutions of the p-Laplacian dynamic equations on time scales:

$$(\phi_p(u^{\Delta\nabla}))^{\nabla} + a(t)f(t,u(t)) = 0, \quad t\in(0,T), \tag{1.1}$$

$$\phi_p(u^{\Delta\nabla}(0)) = \sum_{i=1}^{m-2} a_i\phi_p(u^{\Delta\nabla}(\xi_i)), \quad u^{\Delta}(0)=0, \quad u(T)=\sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $\phi_p(s)$ is p-Laplacian operator; that is, $\phi_p(s)=|s|^{p-2}s,\ p>1,\ \phi_p^{-1}=\phi_q,\ 1/p+1/q=1,\ 0<\xi_1<\cdots<\xi_{m-2}<\rho(T)$, and

(H$_1$) $a_i, b_i\in[0,+\infty),\ i=1,2,\dots,$ satisfy $0<\sum_{i=1}^{m-2}a_i<1$ and $\sum_{i=1}^{m-2}b_i<1$;

(H_2) $a(t) \in C_{ld}([0,T],[0,+\infty))$ and there exists $t_0 \in (\xi_{m-2},T)$ such that $a(t_0) > 0$;

(H_3) $f \in C([0,T] \times [0,+\infty),[0,+\infty))$.

We point out that the Δ-derivative and the ∇-derivative in (1.2) and the C_{ld} space in (H_2) are defined in Section 2.

Recently, there has been much attention paid to the existence of positive solutions for third-order nonlinear boundary value problems of differential equations. For example, see [1–10] and the listed references. Anderson [2] considered the following third-order nonlinear problem:

$$\begin{aligned} &x'''(t) = f(t,x(t)), \quad t_1 \le t \le t_3, \\ &x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0. \end{aligned} \tag{1.3}$$

He used the Krasnoselskii and the Leggett and Williams fixed-point theorems to prove the existence of solutions to the nonlinear problem (1.3). Li [6] considered the existence of single and multiple positive solutions to the nonlinear singular third-order two-point boundary value problem:

$$\begin{aligned} &u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(1) = 0. \end{aligned} \tag{1.4}$$

Under various assumptions on a and f, they established intervals of the parameter λ which yield the existence of at least two and infinitely many positive solutions of the boundary value problem by using Krasnoselski's fixed-point theorem of cone expansion-compression type. Liu et al. [7] discussed the existence of at least one or two nondecreasing positive solutions for the following singular nonlinear third-order differential equations:

$$\begin{aligned} &x'''(t) + \lambda \alpha(t) f(t,x(t)) = 0, \quad a < t < b, \\ &x(a) = x''(a) = x'(b) = 0. \end{aligned} \tag{1.5}$$

Green's function and the fixed-point theorem of cone expansion-compression type are utilized in their paper. In [8], Sun considered the following nonlinear singular third-order three-point boundary value problem:

$$\begin{aligned} &u'''(t) - \lambda a(t) F(t,u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u'(\eta) = u''(1) = 0. \end{aligned} \tag{1.6}$$

He obtained various results on the existence of single and multiple positive solutions to the boundary value problem (1.6) by using a fixed-point theorem of cone expansion-compression type due to Krasnosel'skii. In [10], Zhou and Ma studied the existence and iteration of positive solutions for the following third-order generalized right-focal boundary value problem with p-Laplacian operator:

$$\begin{aligned} &(\phi_p(u''))'(t) = q(t) f(t,u(t)), \quad 0 \le t \le 1, \\ &u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i). \end{aligned} \tag{1.7}$$

They established a corresponding iterative scheme for (1.7) by using the monotone iterative technique.

On the other hand, the existence of positive solutions for third-order nonlinear boundary value problems of difference equations is also extensively studied by a number of authors (see [1, 3, 5, 9] and the listed references). The present work is motivated by a recent paper [4]. In [4], Henderson and Yin considered the existence of solutions for a third-order boundary value problem on a time-scale equation of the form

$$u^{\Delta^3} = f(t, u, u^{\Delta}, u^{\Delta\Delta}), \quad t \in \mathbf{T}, \tag{1.8}$$

which is uniform for the third-order difference equation and the third-order differential equation.

2. Preliminaries and lemmas

For convenience, we list the following definitions which can be found in [4, 11–15].

Definition 2.1. Let $\mathbf{T}$ be a time scale. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and the backward jump operator ρ, respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \\ \rho(r) &= \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T} \end{aligned} \tag{2.1}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right-scattered, and if $\rho(r) < r$, r is said to be left-scattered; if $\sigma(t) = t$, t is said to be right-dense, and if $\rho(r) = r$, r is said to be left-dense. If $\mathbf{T}$ has a right-scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left-scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided that it exists), with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \tag{2.2}$$

for all $s \in U$.

For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t is denoted by $f^{\nabla}(t)$ (provided that it exists), with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s| \tag{2.3}$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e., ld-continuous) if f is continuous at each left-dense point in $\mathbf{T}$, and its right-sided limit exists at each right-dense point in $\mathbf{T}$.

Definition 2.4. If $\phi^{\Delta}(t) = f(t)$, then one defines the delta integral by

$$\int_a^b f(t)\Delta t = \phi(b) - \phi(a). \tag{2.4}$$

If $F^{\nabla}(t) = f(t)$, then one defines the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a). \tag{2.5}$$

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear BVP

$$\left(\phi_p(u^{\Delta\nabla})\right)^{\nabla} + h(t) = 0, \quad t \in (0,T), \tag{2.6}$$

$$\phi_p(u^{\Delta\nabla}(0)) = \sum_{i=1}^{m-2} a_i\phi_p(u^{\Delta\nabla}(\xi_i)), \qquad u^{\Delta}(0) = 0, \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{2.7}$$

Lemma 2.5. *If $\sum_{i=1}^{m-2} a_i \neq 1$ and $\sum_{i=1}^{m-2} b_i \neq 1$, then for $h \in C_{ld}[0,T]$ the BVP* (2.6)-(2.7) *has the unique solution*

$$u(t) = -\int_0^t (t-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + C, \tag{2.8}$$

where

$$\begin{gathered} A = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}, \\ C = \frac{\int_0^T (T-s)\phi_q(\int_0^s h(\tau)\nabla\tau - A)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi_q(\int_0^s h(\tau)\nabla\tau - A)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}. \end{gathered} \tag{2.9}$$

Proof. (i) Let u be a solution, then we will show that (2.8) holds. By taking the nabla integral of problem (2.6) on $(0,t)$, we have

$$\phi_p(u^{\Delta\nabla}(t)) = -\int_0^t h(\tau)\nabla\tau + A \tag{2.10}$$

then

$$u^{\Delta\nabla}(t) = \phi_q\left(-\int_0^t h(\tau)\nabla\tau + A\right) = -\phi_q\left(\int_0^t h(\tau)\nabla\tau - A\right). \tag{2.11}$$

By taking the nabla integral of (2.11) on $(0,t)$, we can get

$$u^{\Delta}(t) = -\int_0^t \phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + B. \tag{2.12}$$

By taking the delta integral of (2.12) on $(0, t)$, we can get

$$u(t) = -\int_0^t (t-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + Bt + C. \tag{2.13}$$

Similarly, let $t = 0$ on (2.10), then we have $\phi_p(u^{\Delta\nabla}(0)) = A$; let $t = \xi_i$ on (2.10), then we have

$$\phi_p(u^{\Delta\nabla}(\xi_i)) = -\int_0^{\xi_i} h(\tau)\nabla\tau + A. \tag{2.14}$$

Let $t = 0$ on (2.12), then we have

$$u^{\Delta}(0) = B. \tag{2.15}$$

Let $t = T$ on (2.13), then we have

$$u(T) = -\int_0^T (T-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + BT + C. \tag{2.16}$$

Similarly, let $t = \xi_i$ on (2.13), then we have

$$u(\xi_i) = -\int_0^{\xi_i} (\xi_i-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + B\xi_i + C. \tag{2.17}$$

By the boundary condition (2.7), we can get

$$B = 0, \tag{2.18}$$

$$A = \sum_{i=1}^{m-2} a_i\left(-\int_0^{\xi_i} h(\tau)\nabla\tau + A\right). \tag{2.19}$$

Solving (2.19), we get

$$A = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}. \tag{2.20}$$

By the boundary condition (2.7), we can obtain

$$-\int_0^T (T-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + C = \sum_{i=1}^{m-2} b_i\left[-\int_0^{\xi_i} (\xi_i-s)\phi_q\left(\int_0^s h(\tau)\nabla\tau - A\right)\nabla s + C\right]. \tag{2.21}$$

Substituting (2.20) in the above expression, one has

$$C=\frac{\int_0^T(T-s)\phi_q(\int_0^s h(\tau)\nabla\tau-A)\nabla s-\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}(\xi_i-s)\phi_q(\int_0^s h(\tau)\nabla\tau-A)\nabla s}{1-\sum_{i=1}^{m-2}b_i}. \tag{2.22}$$

(ii) We show that the function u given in (2.8) is a solution.

Let u be as in (2.8). By [12, Theorem 2.10(iii)] and taking the delta derivative of (2.8), we have

$$u^{\Delta}(t)=-\int_0^t\phi_q\left(\int_0^s h(\tau)\nabla\tau-A\right)\nabla s; \tag{2.23}$$

moreover, we get

$$\begin{aligned} u^{\Delta\nabla}(t)&=-\phi_q\left(\int_0^t h(\tau)\nabla\tau-A\right),\\ \phi_p(u^{\Delta\nabla})&=-\left(\int_0^t h(\tau)\nabla\tau-A\right). \end{aligned} \tag{2.24}$$

Taking the nabla derivative of this expression yields $(\phi_p(u^{\Delta\nabla}))^{\nabla}=-h(t)$. Also, routine calculation verifies that u satisfies the boundary value conditions in (2.7) so that u given in (2.8) is a solution of (2.6) and (2.7). The proof is complete. □

Lemma 2.6. *Assume (H_1) holds. For $h\in C_{ld}[0,T]$ and $h\geq 0$, the unique solution u of* (2.6) *and* (2.7) *satisfies*

$$u(t)\geq 0 \quad \text{for } t\in[0,T]. \tag{2.25}$$

Proof. Let

$$\varphi_0(s)=\phi_q\left(\int_0^s h(\tau)\nabla\tau-A\right). \tag{2.26}$$

Since

$$\int_0^s h(\tau)\nabla\tau-A=\int_0^s h(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\geq 0, \tag{2.27}$$

then $\varphi_0(s)\geq 0$.

According to Lemma 2.5, we get

$$
\begin{aligned}
u(0) = C &= \frac{\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&\geq \frac{\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T - s)\varphi_0(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&= \frac{\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \left(\int_0^{T} (T - s)\varphi_0(s)\nabla s - \int_{\xi_i}^{T} (T - s)\varphi_0(s)\nabla s\right)}{1-\sum_{i=1}^{m-2} b_i} \\
&= \int_0^T (T-s)\varphi_0(s)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^{T} (T - s)\varphi_0(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \geq 0, \\
u(T) &= -\int_0^T (T-s)\varphi_0(s)\nabla s + C \\
&= -\int_0^T (T-s)\varphi_0(s)\nabla s + \frac{\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&\geq -\int_0^T (T-s)\varphi_0(s)\nabla s + \frac{\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T - s)\varphi_0(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^{T} (T - s)\varphi_0(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \geq 0.
\end{aligned}
\tag{2.28}
$$

If $t \in (0,T)$, we have

$$
\begin{aligned}
u(t) &= -\int_0^t (t-s)\varphi_0(s)\nabla s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\varphi_0(s)\nabla s\right] \\
&\geq -\int_0^T (T-s)\varphi_0(s)\nabla s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[\int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T - s)\varphi_0(s)\nabla s\right] \\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[-\left(1-\sum_{i=1}^{m-2} b_i\right)\int_0^T (T-s)\varphi_0(s)\nabla s + \int_0^T (T-s)\varphi_0(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T - s)\varphi_0(s)\nabla s\right] \\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^{T} (T-s)\varphi_0(s)\nabla s \geq 0.
\end{aligned}
\tag{2.29}
$$

So $u(t) \geq 0$, $t \in [0,T]$. □

Lemma 2.7. *Assume* (H_1) *holds. If* $h \in C_{ld}[0,T]$ *and* $h \geq 0$*, then the unique solution* u *of* (2.6) *and* (2.7) *satisfies*

$$
\inf_{t\in[0,T]} u(t) \geq \gamma \|u\|, \tag{2.30}
$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{T-\sum_{i=1}^{m-2} b_i\xi_i}, \qquad \|u\| = \max_{t\in[0,T]}|u(t)|. \tag{2.31}$$

Proof. It is easy to check that $u^{\Delta}(t) = -\int_0^t \varphi_0(s)\nabla s \le 0$; this implies that

$$\|u\| = u(0), \quad \min_{t\in[0,T]} u(t) = u(T). \tag{2.32}$$

It is easy to see that $u^{\Delta}(t_2) \le u^{\Delta}(t_1)$ for any $t_1, t_2 \in [0,T]$ with $t_1 \le t_2$. Hence, $u^{\Delta}(t)$ is a decreasing function on $[0,T]$. This means that the graph of $u(t)$ is concave down on $(0,T)$.

For each $i \in \{1,2,\dots,m-2\}$, we have

$$\frac{u(T)-u(0)}{T-0} \ge \frac{u(T)-u(\xi_i)}{T-\xi_i}, \tag{2.33}$$

that is,

$$Tu(\xi_i) - \xi_i u(T) \ge (T-\xi_i)u(0), \tag{2.34}$$

so that

$$T\sum_{i=1}^{m-2} b_i u(\xi_i) - \sum_{i=1}^{m-2} b_i\xi_i u(T) \ge \sum_{i=1}^{m-2} b_i(T-\xi_i)u(0). \tag{2.35}$$

With the boundary condition $u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, we have

$$u(T) \ge \frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{T-\sum_{i=1}^{m-2} b_i\xi_i} u(0). \tag{2.36}$$

This completes the proof. □

Let the norm on $C_{ld}[0,T]$ be the maximum norm. Then, the $C_{ld}[0,T]$ is a Banach space. It is easy to see that BVP (1.1)-(1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator

$$(Au)(t) = -\int_0^t (t-s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\nabla s + \widetilde{C}, \tag{2.37}$$

where

$$\widetilde{A} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i},$$

$$\widetilde{C} = \frac{\int_0^T (T-s)\phi_q(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A})\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi_q(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A})\nabla s}{1-\sum_{i=1}^{m-2} b_i}. \tag{2.38}$$

Denote

$$K = \left\{u \mid u \in C_{ld}[0,T],\ u(t) \ge 0,\ \inf_{t\in[0,T]} u(t) \ge \gamma\|u\|\right\}, \tag{2.39}$$

where γ is the same as in Lemma 2.7. It is obvious that K is a cone in $C_{ld}[0,T]$. By Lemma 2.7, $A(K) \subset K$. So by applying Arzela-Ascoli theorem on time scales [16], we can obtain that $A(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [13], it is easy to prove that A is continuous. Hence, $A : K \to K$ is completely continuous.

Lemma 2.8. *$A : K \to K$ is completely continuous.*

Proof. First, we show that A maps bounded set into bounded set.

Assume $c > 0$ is a constant and $u \in \overline{K_c} = \{u \in K : \|u\| \leq c\}$. Note that the continuity of f guarantees that there is $c' > 0$ such that $f(t, u(t)) \leq \phi_p(c')$ for $t \in [0, T]$. So

$$\begin{aligned}\|Au\| &= \max_{t\in[0,T]}|Au(t)| \leq \widetilde{C}\\ &\leq \frac{\int_0^T (T-s)\phi_q(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A})\nabla s}{1-\sum_{i=1}^{m-2} b_i}\\ &\leq \frac{c'\int_0^T (T-s)\phi_q(\int_0^s a(\tau)\nabla\tau + \sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\nabla\tau/(1-\sum_{i=1}^{m-2} a_i))\nabla s}{1-\sum_{i=1}^{m-2} b_i}.\end{aligned} \tag{2.40}$$

That is, $A\overline{K_c}$ is uniformly bounded.

In addition, notice that for any $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned}&|Au(t_1) - Au(t_2)|\\ &\quad= \left|\int_0^{t_1}(t_2-t_1)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau-\widetilde{A}\right)\nabla s+\int_{t_1}^{t_2}(t_2-s)\phi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau-\widetilde{A}\right)\nabla s\right|\\ &\quad\leq c'|t_1-t_2|\left[\int_0^T\phi_q\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right)\nabla s\right.\\ &\quad\quad\left.+T\max_{s\in[0,T]}\phi_q\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\right].\end{aligned} \tag{2.41}$$

So, by applying Arzela-Ascoli theorem on time scales [16], we obtain that $A\overline{K_c}$ is relatively compact.

Finally, we prove that $A : \overline{K_c} \to K$ is continuous. Suppose that $\{u_n\}_{n=1}^{\infty} \subset \overline{K_c}$ and $u_n(t)$ converges to $u^*(t)$ uniformly on $[0, T]$. Hence, $\{Au_n(t)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0, T]$. The Arzela-Ascoli theorem on time scales [16] tells us that there exists uniformly convergent subsequence in $\{Au_n(t)\}_{n=1}^{\infty}$. Let $\{Au_{n(m)}(t)\}_{m=1}^{\infty}$ be a subsequence which converges to $v(t)$ uniformly on $[0, T]$. In addition,

$$0 \leq Au_n(t) \leq \frac{c'\int_0^T (T-s)\phi_q(\int_0^s a(\tau)\nabla\tau + \sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\nabla\tau/(1-\sum_{i=1}^{m-2} a_i))\nabla s}{1-\sum_{i=1}^{m-2} b_i}. \tag{2.42}$$

Observe the expression of $\{Au_{n(m)}(t)\}$, and then letting $m \to \infty$, we obtain

$$v(t) = -\int_0^t (t-s)\phi_q\left(\int_0^s a(\tau)f(\tau,u^*(\tau))\nabla\tau - \widetilde{A}^*\right)\nabla s + \widetilde{C}^*, \tag{2.43}$$

where

$$\widetilde{A}^* = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) f(\tau, u^*(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} a_i},$$
$$\widetilde{C}^* = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\int_0^T (T-s)\phi_q\left(\int_0^s a(\tau) f(\tau, u^*(\tau)) \nabla\tau - \widetilde{A}^* \right) \nabla s \right.$$
$$\left. - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi_q\left(\int_0^s a(\tau) f(\tau, u^*(\tau)) \nabla\tau - \widetilde{A}^* \right) \nabla s \right]. \tag{2.44}$$

Here, we have used the Lebesgue dominated convergence theorem on time scales [13]. From the definition of A, we know that $v(t) = Au^*(t)$ on $[0, T]$. This shows that each subsequence of $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $Au^*(t)$. Therefore, the sequence $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $Au^*(t)$. This means that A is continuous at $u^* \in \overline{K_c}$. So, A is continuous on $\overline{K_c}$ since u^* is arbitrary. Thus, A is completely continuous. This proof is complete. □

Lemma 2.9. *Let*

$$\varphi(s) = \phi_q\left(\int_0^s a(\tau) f(\tau, u(\tau)) \nabla\tau - \widetilde{A} \right). \tag{2.45}$$

For ξ_i $(i = 1, \ldots, m-2)$,

$$\int_0^{\xi_i} (\xi_i - s)\varphi(s) \nabla s \le \frac{\xi_i}{T} \int_0^T (T-s)\varphi(s) \nabla s. \tag{2.46}$$

Proof. Since

$$\int_0^s a(\tau) f(\tau, u(\tau)) \nabla\tau - \widetilde{A} = \int_0^s a(\tau) f(\tau, u(\tau)) \nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) f(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}$$
$$\ge 0, \tag{2.47}$$

then $\varphi(s) \ge 0$. For all $t \in (0, T]$, we have

$$\left(\frac{\int_0^t (t-s)\varphi(s) \nabla s}{t} \right)^{\nabla} = \frac{t\int_0^t \varphi(s) \nabla s - \int_0^t (t-s)\varphi(s) \nabla s}{t\rho(t)} \ge 0. \tag{2.48}$$

In fact, let $\psi(t) = t\int_0^t \varphi(s) \nabla s - \int_0^t (t-s)\varphi(s) \nabla s$; taking the nabla derivative of this expression, we have

$$\psi^{\nabla}(t) = \int_0^t \varphi(s) \nabla s + t\varphi(t) - \int_0^t \varphi(s) \nabla s = t\varphi(t) \ge 0. \tag{2.49}$$

Hence, $\psi(t)$ is a nondecreasing function on $[0, T]$. That is,

$$\psi(t) \ge 0. \tag{2.50}$$

For all $t \in (0,T]$,

$$\frac{\int_0^t (t-s)\varphi(s)\nabla s}{t} \le \frac{\int_0^T (T-s)\varphi(s)\nabla s}{T}. \tag{2.51}$$

By (2.51), for ξ_i $(i=1,\dots,m-2)$, we have

$$\int_0^{\xi_i} (\xi_i - s)\varphi(s)\nabla s \le \frac{\xi_i}{T}\int_0^T (T-s)\varphi(s)\nabla s. \tag{2.52}$$

□

Lemma 2.10 (see [17]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*
$F : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that

(i) $\|Fu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Fu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Fu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Fu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then, F has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Now, we introduce the following notations. Let

$$\begin{aligned} A_0 &= \left\{ \frac{1}{1-\sum_{i=1}^{m-2} b_i} \int_0^T (T-s)\phi_q\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\nabla s \right\}^{-1}, \\ B_0 &= \left\{ \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \int_0^T (T-s)\phi_q\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\nabla s \right\}^{-1}. \end{aligned} \tag{2.53}$$

For $l > 0$, $\Omega_l = \{u \in K : \|u\| < l\}$, and $\partial\Omega_l = \{u \in K : \|u\| = l\}$,

$$\alpha(l) = \sup\{\|Au\| : u \in \partial\Omega_l\}, \qquad \beta(l) = \inf\{\|Au\| : u \in \partial\Omega_l\}, \tag{2.54}$$

by Lemma 2.6, where α and β are well defined.

3. Main results

Theorem 3.1. *Assume (H_1), (H_2), and (H_3) hold, and assume that the following conditions hold:*

(A_1) $p_i \in C([0,+\infty),[0,+\infty))$, $i=1,2$, *and*

$$\overline{\lim_{l\to 0^+}} \frac{p_1(l)}{l^{p-1}} < A_0^{p-1}, \qquad \overline{\lim_{l\to\infty}} \frac{p_2(l)}{l^{p-1}} < A_0^{p-1}; \tag{3.1}$$

(A_2) $k_i \in L^1([0,T],[0,+\infty))$, $i=1,2$;

(A_3) *there exist $0 < c_1 \le c_2$ and $0 \le \lambda_2 < p-1 < \lambda_1$ such that*

$$\begin{aligned} f(t,l) &\le p_1(l) + k_1(t)l^{\lambda_1}, \quad (t,l) \in [0,T]\times[0,c_1], \\ f(t,l) &\le p_2(l) + k_2(t)l^{\lambda_2}, \quad (t,l) \in [0,T]\times[c_2,+\infty); \end{aligned} \tag{3.2}$$

(A_4) *there exists* $b > 0$ *such that*

$$\min\{f(t,l) : (t,l) \in [0,T] \times [\gamma b, b]\} \geq (bB_0)^{p-1}. \tag{3.3}$$

Then, problem (1.1)-(1.2) *has at least two positive solutions* u_1^*, u_2^* *satisfying* $0 < \|u_1^*\| < b < \|u_2^*\|$.

Theorem 3.2. *Assume* (H_1), (H_2), *and* (H_3) *hold, and assume that the following conditions hold:*

(B_1) $p_i \in C([0,+\infty),[0,+\infty))$, $i = 3,4$, *and*

$$\underline{\lim}_{l\to 0^+} \frac{p_3(l)}{l^{p-1}} > \left(\frac{B_0}{\gamma}\right)^{p-1}, \qquad \underline{\lim}_{l\to \infty} \frac{p_4(l)}{l^{p-1}} > \left(\frac{B_0}{\gamma}\right)^{p-1}; \tag{3.4}$$

(B_2) $k_i \in L^1([0,T],[0,+\infty))$, $i = 3,4$;

(B_3) *there exist* $0 < c_3 \leq c_4$ *and* $0 \leq \lambda_4 < p-1 < \lambda_3$ *such that*

$$\begin{aligned} f(t,l) &\geq p_3(l) - k_3(t)l^{\lambda_3}, \quad (t,l) \in [0,T] \times [0,c_3], \\ f(t,l) &\geq p_4(l) - k_4(t)l^{\lambda_4}, \quad (t,l) \in [0,T] \times [c_4,+\infty); \end{aligned} \tag{3.5}$$

(B_4) *there exists* $a > 0$ *such that*

$$\max\{f(t,l) : (t,l) \in [0,T] \times [0,a]\} \leq (aA_0)^{p-1}. \tag{3.6}$$

Then, problem (1.1)-(1.2) *has at least two positive solutions* u_3^*, u_4^* *satisfying* $0 < \|u_3^*\| < a < \|u_4^*\|$.

Proof of Theorem 3.1. Let

$$\epsilon = \frac{1}{2}\min\left[A_0^{p-1} - \overline{\lim_{l\to 0^+}} \frac{p_1(l)}{l^{p-1}},\ A_0^{p-1} - \overline{\lim_{l\to \infty}} \frac{p_2(l)}{l^{p-1}}\right], \tag{3.7}$$

then there exist $0 < \overline{a}_1 \leq c_1$ and $c_2 \leq \overline{a}_2 < +\infty$ such that

$$\begin{aligned} p_1(l) &\leq (A_0^{p-1} - \epsilon)l^{p-1}, \quad 0 \leq l \leq \overline{a}_1, \\ p_2(l) &\leq (A_0^{p-1} - \epsilon)l^{p-1}, \quad \overline{a}_2 \leq l \leq +\infty. \end{aligned} \tag{3.8}$$

If $0 \leq l \leq \overline{a}_1$, $u \in \partial\Omega_l$, then $0 \leq u(t) \leq l$, $0 \leq t \leq T$. By condition (A_3), we have

$$\begin{aligned} f(t,u(t)) &\leq p_1(u(t)) + k_1(t)u^{\lambda_1}(t) \\ &\leq (A_0^{p-1} - \epsilon)u^{p-1}(t) + k_1(t)u^{\lambda_1}(t) \\ &\leq (A_0^{p-1} - \epsilon)\|u\|^{p-1} + k_1(t)\|u\|^{\lambda_1} \\ &= (A_0^{p-1} - \epsilon)l^{p-1} + k_1(t)l^{\lambda_1} \end{aligned} \tag{3.9}$$

so that

$$\begin{aligned}
&\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau-\widetilde{A}\\
&\quad=\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\\
&\quad\le\int_0^s a(\tau)[(A_0^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)[(A_0^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}.
\end{aligned}\tag{3.10}$$

Therefore,

$$\begin{aligned}
\|Au\|\le\widetilde{C}&=\frac{1}{1-\sum_{i=1}^{m-2}b_i}\left(\int_0^T(T-s)\varphi(s)\nabla s-\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}(\xi_i-s)\varphi(s)\nabla s\right)\\
&\le\frac{1}{1-\sum_{i=1}^{m-2}b_i}\int_0^T(T-s)\varphi(s)\nabla s\\
&\le\frac{1}{1-\sum_{i=1}^{m-2}b_i}\\
&\quad\times\int_0^T(T-s)\phi_q\Bigg\{\int_0^s a(\tau)[(A_0^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau\\
&\qquad+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)[(A_0^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\Bigg\}\nabla s.
\end{aligned}\tag{3.11}$$

It follows that

$$\begin{aligned}
\frac{\alpha(l)}{l}&\le\frac{1}{1-\sum_{i=1}^{m-2}b_i}\\
&\quad\times\int_0^T(T-s)\phi_q\Bigg\{\int_0^s a(\tau)[A_0^{p-1}-\epsilon+k_1(\tau)l^{\lambda_1-p+1}]\nabla\tau\\
&\qquad+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)[A_0^{p-1}-\epsilon+k_1(\tau)l^{\lambda_1-p+1}]\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\Bigg\}\nabla s.
\end{aligned}\tag{3.12}$$

Noticing $\lambda_1-p+1>0$, we have

$$\begin{aligned}
\overline{\lim_{l\to0^+}}\frac{\alpha(l)}{l}&\le\frac{1}{1-\sum_{i=1}^{m-2}b_i}\int_0^T(T-s)\phi_q\left[\int_0^s a(\tau)(A_0^{p-1}-\epsilon)\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)(A_0^{p-1}-\epsilon)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\nabla s\\
&=\frac{(A_0^{p-1}-\epsilon)^{1/(p-1)}}{1-\sum_{i=1}^{m-2}b_i}\int_0^T(T-s)\phi_q\left[\int_0^s a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\nabla s\\
&=(A_0^{p-1}-\epsilon)^{1/(p-1)}A_0^{-1}\\
&=(1-A_0^{-(p-1)}\epsilon)^{1/(p-1)}<1.
\end{aligned}\tag{3.13}$$

Therefore, there exist $0<a_1<\overline{a}_1$ such that $\alpha(a_1)<a_1$. It implies that $\|Au\|<\|u\|,\ u\in\partial\Omega_{a_1}$.

If $\overline{a}_2 \leq l < +\infty$ and $u \in \partial\Omega_l$, then $0 \leq u(t) \leq l$. Similar to the above argument, noticing that $\lambda_2 - p + 1 < 0$, we can get $\overline{\lim}_{l\to\infty}(\alpha(l)/l) < 1$. Therefore, there exist $0 < \overline{a}_2 < a_2$ such that $\alpha(a_2) < a_2$. It implies that $\|Au\| < \|u\|$, $u \in \partial\Omega_{a_2}$.

On the other hand, since $f : [0,T] \times [0,+\infty) \to [0,+\infty)$ is continuous, by condition (A_4), there exist $a_1 < b_1 < b < b_2 < a_2$ such that

$$\min\{f(t,l) : (t,l) \in [0,T] \times [\gamma b_i, b_i]\} \geq (b_i B_0)^{p-1}, \quad i = 1,2. \tag{3.14}$$

If $u \in \partial\Omega_{b_1}$, then $\gamma b_1 \leq u(t) \leq b_1$, $0 \leq t \leq T$. Applying Lemma 2.9, it follows that

$$\begin{aligned}
\|Au\| &= \max_{0\leq t\leq T}|(Au)(t)| \\
&\geq -\int_0^T (T-s)\varphi(s)\nabla s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left(\int_0^T (T-s)\varphi(s)\nabla s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\varphi(s)\nabla s\right) \\
&= \frac{\sum_{i=1}^{m-2} b_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^T (T-s)\varphi(s)\nabla s - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\varphi(s)\nabla s}{1-\sum_{i=1}^{m-2} b_i} \\
&\geq \frac{\sum_{i=1}^{m-2} b_i}{1-\sum_{i=1}^{m-2} b_i}\int_0^T (T-s)\varphi(s)\nabla s - \frac{\sum_{i=1}^{m-2} b_i \xi_i}{T(1-\sum_{i=1}^{m-2} b_i)}\int_0^T (T-s)\varphi(s)\nabla s \\
&= \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)}\int_0^T (T-s)\varphi(s)\nabla s \\
&\geq \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)}\int_0^T (T-s)\phi_q\left[\int_0^s a(\tau)(b_1B_0)^{p-1}\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)(b_1B_0)^{p-1}\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\nabla s \\
&= \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} b_1 B_0 \int_0^T (T-s)\phi_q\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\nabla s \\
&= b_1 B_0 B_0^{-1} = b_1 = \|u\|.
\end{aligned} \tag{3.15}$$

In the same way, we can prove that if $u \in \partial\Omega_{b_2}$, then $\|Au\| \geq \|u\|$. □

Now, we consider the operator A on $\overline{\Omega_{b_1}} \setminus \Omega_{a_1}$ and $\overline{\Omega_{a_2}} \setminus \Omega_{b_2}$, respectively. By Lemma 2.10, we assert that the operator A has two fixed points $u_1^*, u_2^* \in K$ such that $a_1 \leq \|u_1^*\| \leq b_1$ and $b_2 \leq \|u_2^*\| \leq a_2$. Therefore, u_i^*, $i = 1,2$, are positive solutions of problem (1.1)-(1.2).

Proof of Theorem 3.2. Let

$$\epsilon = \frac{1}{2}\min\left[\underline{\lim}_{l\to 0^+}\frac{p_3(l)}{l^{p-1}} - \left(\frac{B_0}{\gamma}\right)^{p-1}, \ \underline{\lim}_{l\to\infty}\frac{p_4(l)}{l^{p-1}} - \left(\frac{B_0}{\gamma}\right)^{p-1}\right], \tag{3.16}$$

then there exist $0 < \overline{b}_3 \leq c_3$ and $c_4 \leq \overline{b}_4 < +\infty$ such that

$$\begin{aligned} p_3(l) &\geq \left[\left(\frac{B_0}{\gamma}\right)^{p-1} + \epsilon\right] l^{p-1}, \quad 0 \leq l \leq \overline{b}_3, \\ p_4(l) &\geq \left[\left(\frac{B_0}{\gamma}\right)^{p-1} + \epsilon\right] l^{p-1}, \quad \overline{b}_4 \leq l \leq +\infty. \end{aligned} \tag{3.17}$$

If $0 \leq l \leq \overline{b}_3,\ u \in \partial\Omega_l$, then $\gamma l \leq u(t) \leq l,\ 0 \leq t \leq T$. By Lemma 2.9 and condition (B_3), we have

$$\begin{aligned} \|Au\| &= \max_{0\leq t\leq T} |(Au)(t)| \\ &\geq \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \int_0^T (T-s)\varphi(s)\nabla s \\ &\geq \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \\ &\quad \times \int_0^T (T-s)\phi_q\Bigg\{\int_0^s a(\tau)[p_3(u(\tau)) - k_3(\tau)u^{\lambda_3}]\nabla\tau \\ &\qquad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)[p_3(u(\tau)) - k_3(\tau)u^{\lambda_3}]\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Bigg\}\nabla s \\ &\geq \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \\ &\quad \times \int_0^T (T-s)\phi_q\Bigg\{\int_0^s a(\tau)\left[\left(\left(\frac{B_0}{\gamma}\right)^{p-1} + \epsilon\right)(\gamma l)^{p-1} - k_3(\tau)l^{\lambda_3}\right]\nabla\tau \\ &\qquad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)[((B_0/\gamma)^{p-1}+\epsilon)(\gamma l)^{p-1} - k_3(\tau)l^{\lambda_3}]\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Bigg\}\nabla s. \end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned} \frac{\beta(l)}{l} &\geq \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \\ &\quad \times \int_0^T (T-s)\phi_q\Bigg\{\int_0^s a(\tau)[B_0^{p-1} + \gamma^{p-1}\epsilon - k_3(\tau)l^{\lambda_3-p+1}]\nabla\tau \\ &\qquad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)[B_0^{p-1} + \gamma^{p-1}\epsilon - k_3(\tau)l^{\lambda_3-p+1}]\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Bigg\}\nabla s. \end{aligned} \tag{3.19}$$

Noticing $\lambda_3 - p + 1 > 0$, we get

$$\begin{aligned}
\underline{\lim}_{l\to 0^+} \frac{\beta(l)}{l} &\geq \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \\
&\quad\times \int_0^T (T-s)\phi_q\left[\int_0^s a(\tau)(B_0^{p-1}+\gamma^{p-1}\epsilon)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)(B_0^{p-1}+\gamma^{p-1}\epsilon)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\nabla s \\
&= (B_0^{p-1}+\gamma^{p-1}\epsilon)^{1/(p-1)} B_0^{-1} \\
&= (1+\gamma^{p-1}B_0^{-(p-1)}\epsilon)^{1/(p-1)} > 1.
\end{aligned} \tag{3.20}$$

Therefore, there exists b_3 with $0 < b_3 < a$ such that $\beta(b_3) > b_3$. It implies that $\|Au\| > \|u\|$ for $u \in \partial\Omega_{b_3}$.

If $\overline{b}_4 \leq \gamma l < +\infty$ and $u \in \partial\Omega_l$, then $\overline{b}_4 \leq \gamma l \leq u(t) \leq l,\ 0 \leq t \leq T$. Similar to the above argument, noticing that $\lambda_4 - p + 1 < 0$, we can get $\underline{\lim}_{l\to+\infty}(\beta(l)/l) > 1$.

Therefore, there exist b_4 with $0 < b_4 < +\infty$ such that $\beta(b_4) > b_4$. It implies that $\|Au\| > \|u\|$ for $u \in \partial\Omega_{b_4}$.

By condition (B_4), we can see that there exist $b_3 < a_3 < a < a_4 < b_4$ such that

$$\max\{f(t,l) : (t,l) \in [0,T]\times[0,a_i]\} \leq (a_iA_0)^{p-1}, \quad i = 3,4. \tag{3.21}$$

If $u \in \partial\Omega_{a_3}$, then $0 \leq u(t) \leq a_3$, $0 \leq t \leq T$, and $f(t,u(t)) \leq (a_3A_0)^{p-1}$. It follows that

$$\begin{aligned}
\|Au\| &\leq \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T (T-s)\varphi(s)\nabla s \\
&\leq \frac{1}{1-\sum_{i=1}^{m-2} b_i} a_3A_0\int_0^T (T-s)\phi_q\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right]\nabla s \\
&= a_3 = \|u\|.
\end{aligned} \tag{3.22}$$

Similarly, if $u \in \partial\Omega_{a_4}$, then $\|Au\| \leq \|u\|$. □

Now, we study the operator A on $\overline{\Omega_{a_3}} \setminus \Omega_{b_3}$ and $\overline{\Omega_{b_4}} \setminus \Omega_{a_4}$, respectively. By Lemma 2.10, we assert that the operator A has two fixed points u_3^*, $u_4^* \in K$ such that $b_3 \leq \|u_3^*\| \leq a_3$ and $a_4 \leq \|u_4^*\| \leq b_4$. Therefore, $u_i^*,\ i = 3,4$, are positive solutions of problem (1.1)-(1.2).

4. Further discussion

If the conditions of Theorems 3.1 and 3.2 are weakened, we will get the existence of single positive solution of problem (1.1)-(1.2).

Corollary 4.1. *Assume* (H_1), (H_2), *and* (H_3) *hold, and assume that the following conditions hold:*

(C_1) $p_1 \in C([0,+\infty),[0,+\infty))$, *and* $\overline{\lim}_{l\to 0^+}(p_1(l)/l^{p-1}) < A_0^{p-1}$;

(C_2) $k_1 \in L^1([0,T],[0,+\infty))$;

(C_3) *there exist* $c_1 > 0$ *and* $\lambda_1 > p-1$ *such that*

$$f(t,l) \le p_1(l) + k_1(t)l^{\lambda_1}, \quad (t,l) \in [0,T] \times [0,c_1]; \tag{4.1}$$

(C_4) *there exists* $b > 0$ *such that*

$$\min\{f(t,l) : (t,l) \in [0,T] \times [\gamma b, b]\} \ge (bB_0)^{p-1}. \tag{4.2}$$

Then, problem (1.1)-(1.2) *has at least one positive solution.*

Corollary 4.2. *Assume* (H_1), (H_2), *and* (H_3) *hold, and assume that the following conditions hold:*

(D_1) $p_2 \in C([0,+\infty),[0,+\infty))$, *and* $\overline{\lim}_{l\to\infty}(p_2(l)/l^{p-1}) < A_0^{p-1}$;

(D_2) $k_2 \in L^1([0,T],[0,+\infty))$;

(C_3) *there exist* $c_2 > 0$ *and* $0 \le \lambda_2 < p-1$ *such that*

$$f(t,l) \le p_2(l) + k_2(t)l^{\lambda_2}, \quad (t,l) \in [0,T] \times [c_2,+\infty); \tag{4.3}$$

(D_4) *there exists* $b > 0$ *such that*

$$\min\{f(t,l) : (t,l) \in [0,T] \times [\gamma b, b]\} \ge (bB_0)^{p-1}. \tag{4.4}$$

Then, problem (1.1)-(1.2) *has at least one positive solution.*

Corollary 4.3. *Assume* (H_1), (H_2), *and* (H_3) *hold, and assume that the following conditions hold:*

(E_1) $p_3 \in C([0,+\infty),[0,+\infty))$, *and* $\underline{\lim}_{l\to 0^+}(p_3(l)/l^{p-1}) > (B_0/\gamma)^{p-1}$;

(E_2) $k_3 \in L^1([0,T],[0,+\infty))$;

(E_3) *there exist* $c_3 > 0$ *and* $\lambda_3 > p-1$ *such that*

$$f(t,l) \ge p_3(l) - k_3(t)l^{\lambda_3}, \quad (t,l) \in [0,T] \times [0,c_3]; \tag{4.5}$$

(E_4) *there exists* $a > 0$ *such that*

$$\max\{f(t,l) : (t,l) \in [0,T] \times [0,a]\} \le (aA_0)^{p-1}. \tag{4.6}$$

Then, problem (1.1)-(1.2) *has at least one positive solution.*

Corollary 4.4. *Assume* (H_1), (H_2), *and* (H_3) *hold, and assume that the following conditions hold:*

(F_1) $p_4 \in C([0,+\infty),[0,+\infty))$, *and* $\underline{\lim}_{l\to\infty}(p_4(l)/l^{p-1}) > (B_0/\gamma)^{p-1}$;

(F_2) $k_4 \in L^1([0,T],[0,+\infty))$;

(F_3) *there exist* $c_4 > 0$ *and* $0 \le \lambda_4 < p-1$ *such that*

$$f(t,l) \ge p_4(l) - k_4(t)l^{\lambda_4}, \quad (t,l) \in [0,T] \times [c_4,+\infty); \tag{4.7}$$

(F_4) *there exists* $a > 0$ *such that*

$$\max\{f(t,l) : (t,l) \in [0,T] \times [0,a]\} \le (aA_0)^{p-1}. \tag{4.8}$$

Then, problem (1.1)-(1.2) *has at least one positive solution.*

The proof of the above results is similar to those of Theorems 3.1 and 3.2; thus we omit it.

5. Some examples

In this section, we present a simple example to explain our results. We only study the case $\mathbf{T} = R, (0, T) = (0, 1)$.

Let $f(t, 0) \equiv 0$. Consider the following BVP:

$$\begin{gathered}(\phi_3(u''))' + f(t, u(t)) = 0, \quad t \in (0, 1), \\ \phi_3(u''(0)) = \frac{1}{2}\phi_3\left(u''\left(\frac{1}{2}\right)\right), \qquad u'(0) = 0, \qquad u(1) = \frac{1}{2}u\left(\frac{1}{2}\right),\end{gathered} \tag{5.1}$$

where

$$f(t, u) = \begin{cases} 223u^3 + \min\left\{\dfrac{1}{\sqrt{t(1-t)}}, \dfrac{2}{u}\right\}\sqrt{u^5}, & (t, u) \in [0, 1] \times [0, 1], \\ 225, & (t, u) \in [0, 1] \times [1, 3], \\ 74u + \dfrac{\sqrt{3}}{6}\min\left\{\dfrac{1}{\sqrt{t(1-t)}}, \dfrac{2u}{3}\right\}\sqrt{u^3}, & (t, u) \in [0, 1] \times [3, +\infty). \end{cases} \tag{5.2}$$

It is easy to check that $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. In this case, $p = 3$, $a(t) \equiv 1$, $m = 3$, $a_1 = b_1 = 1/2$, and $\xi_1 = 1/2$, and it follows from a direct calculation that

$$\begin{gathered}A_0 = \left[\frac{1}{1 - 1/2}\int_0^1 (1 - s)\phi_q\left(s + \frac{(1/2)\cdot(1/2)}{1 - 1/2}\right)ds\right]^{-1} = 1.1062, \\ \gamma = \frac{b_1(1 - \xi_1)}{1 - b_1\xi_1} = \frac{(1/2)(1 - 1/2)}{1 - (1/2)\cdot(1/2)} = \frac{1}{3}.\end{gathered} \tag{5.3}$$

We have

$$\begin{aligned}B_0 &= \left[\frac{b_1 - b_1\xi_1}{1 - b_1}\int_0^T (1 - s)\phi_q\left(s + \frac{a_1\xi_1}{1 - a_1}\right)ds\right]^{-1} \\ &= \left[\frac{1/2 - (1/2)\cdot(1/2)}{1 - 1/2}\int_0^1 (1 - s)\left(s + \frac{(1/2)\cdot(1/2)}{1 - 1/2}\right)^{1/2} ds\right]^{-1} \\ &= 4.4248 < 5.\end{aligned} \tag{5.4}$$

Choosing $c_1 = 1$, $c_2 = 3$, $b = 3$, $\lambda_1 = 5/2$, $\lambda_2 = 3/2$, $p_1(u) = 223u^3$, $p_2(u) = 74u$, and $k_1(t) = k_2(t) = 1/\sqrt{t(1 - t)}$, it is easy to check that

$$\begin{aligned}& f(t, u) \le p_1(u) + k_1(t)u^{5/2}, \quad (t, u) \in [0, 1] \times [0, 1], \\ & f(t, u) \le p_2(u) + k_2(t)u^{3/2}, \quad (t, u) \in [0, 1] \times [3, +\infty), \\ & \overline{\lim_{u\to 0}}\frac{p_1(u)}{u^2} = \overline{\lim_{l\to 0}}\frac{223u^3}{u^2} = 0 < A_0^2 = (1.1062)^2, \\ & \overline{\lim_{u\to\infty}}\frac{p_2(u)}{u^2} = \overline{\lim_{l\to\infty}}\frac{74u}{u^2} = 0 < A_0^2 = (1.1062)^2, \\ & \min\{f(t, u) : (t, u) \in [0, 1] \times [1, 3]\} = 225 > (13.274)^2 = (bB_0)^2.\end{aligned} \tag{5.5}$$

It follows that f satisfies the conditions (A_1)–(A_4) of Theorem 3.1; then problem (5.1) has at least two positive solutions.

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