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Research Article

On the Nonoscillation of Second-Order Neutral Delay Differential Equation with Forcing Term

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This paper is concerned with nonoscillation of second-order neutral delay differential equation with forcing term. By using contraction mapping principle, some sufficient conditions for the existence of nonoscillatory solution are established. The criteria obtained in this paper complement and extend several known results in the literature. Some examples illustrating our main results are given.

1. Introduction

During the last two decades, there has been much research activity concerning the oscillation and nonoscillation of solutions of neutral type delay differential equations (see [1–9]). Investigation of such equations or systems, besides of their theoretical interest, have some importance in modelling of the networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar and also in population dynamics, and so forth (see [1, 2, 8, 10] and the references cited therein).

In this paper, we consider the second-order neutral delay differential equation with forcing term of the form

$$[x(t)+P(t)x(t-\tau)]''+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=h(t),\quad t\ge t_0,\tag{1.1}$$

where

$$\tau>0,\qquad \sigma_1\ge 0,\qquad \sigma_2\ge 0,\qquad P,Q_1,Q_2,h\in C([t_0,\infty),\mathbb{R}).\tag{1.2}$$

Let $\varphi \in C([t_0 - \sigma, t_0), \mathbb{R})$, where $\sigma = \max\{\tau, \sigma_1, \sigma_2\}$, be a given function and let x_0 be a given constant. By the method of steps (see [2]), it is easy to know that (1.1) has a unique solution $x \in C([t_0 - \sigma, \infty), \mathbb{R})$ in the sense that $x(t) + P(t)x(t-\tau)$ is twice continuously differential for $t \geq t_0$, $x(t)$ satisfies (1.1) and

$$x(s) = \varphi(s) \quad \text{for } s \in [t_0 - \sigma, t_0],$$
$$\left[x(t) + P(t)x(t-\tau)\right]'_{t=t_0} = x_0. \tag{1.3}$$

As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory. Equation (1.1) is oscillatory if all its solutions are oscillatory.

When $P(t) = p$ and the forcing term $h(t) \equiv 0$, (1.1) reduces to

$$\left[x(t) + px(t-\tau)\right]'' + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad t \geq t_0, \tag{1.4}$$

where $p \in \mathbb{R}(p \neq \pm 1)$ and $\int^{\infty} tQ_i(t)dt < \infty,\ i = 1, 2$. The first global result of (1.4) (with respect to p), which is a sufficient condition for the existence of a nonoscillatory solution for all values of $p \neq \pm 1$, have been examined by Kulenović and Hadžiomerspahić [4].

Recently, Parhi and Rath [7] studied oscillation behaviors for forced first-order neutral differential equations as follows

$$\left[x(t) - P(t)x(t-\tau)\right]' + Q(t)G\left(x(t-\sigma)\right) = h(t), \quad t \geq t_0. \tag{1.5}$$

Necessary and sufficient conditions are obtained in various ranges for $P(t) \neq \pm 1$ so that every solution of (1.5) is oscillatory or tends to zero or to $\pm\infty$ as $t \to \infty$.

Motivated by the idea of [4, 7], in present paper we establish sufficient conditions for existence of a nonoscillatory solution to (1.1) depending on various ranges of $P(t) \neq \pm 1$. Hereinafter, we assume that the following conditions hold,

(H1) $Q_i \geq 0$, and $\int^{\infty} tQ_i(t)dt < \infty,\ i = 1, 2$.

(H2) There exists a function $H(t) \in C^2([t_0, \infty), \mathbb{R})$ such that $H''(t) = h(t)$ and $\lim_{t \to \infty} H(t) = M \in \mathbb{R}$.

2. Main results

Theorem 2.1. *Suppose that conditions* $(H1)$ *and* $(H2)$ *hold. If* $P(t)$ *is in one of the following ranges:*

$$\begin{aligned} &\text{(i)} \quad 0 < P(t) \leq p_1 < 1, \\ &\text{(ii)} \quad 1 < p_2 \leq P(t) \leq p_1, \\ &\text{(iii)} \quad -1 < -p_2 \leq P(t) < 0, \\ &\text{(iv)} \quad -p_2 \leq P(t) \leq -p_1 < -1, \end{aligned} \tag{2.1}$$

then (1.1) *has a nonoscillatory solution.*

Proof. The proof of this theorem will be divided into four cases in terms of the four different ranges of $P(t)$.

Case (i) $(0 < P(t) \leq p_1 < 1)$. Choose a $t_1 > t_0 + \sigma$ sufficiently large such that

$$\int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds < \frac{3(1-p_1)}{4}, \tag{2.2}$$

$$\int_{t_1}^{\infty} sQ_1(s)ds \leq \frac{p_1 + N_1 - 1}{N_1}, \tag{2.3}$$

$$\int_{t_1}^{\infty} sQ_2(s)ds \leq \frac{1 - p_1(1+2N_1) - 2M_1}{2N_1}, \tag{2.4}$$

$$\left|H(t) - M\right| \leq \frac{1-p_1}{4}, \tag{2.5}$$

where M_1 and N_1 are positive constants such that

$$1 - N_1 < p_1 < \frac{1-2M_1}{1+2N_1}. \tag{2.6}$$

Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$A = \left\{x \in X : M_1 \leq x(t) \leq N_1,\ t \geq t_0\right\}. \tag{2.7}$$

Define a mapping $T : A \rightarrow X$ as follows:

$$(Tx)(t) = \begin{cases} \dfrac{3(1-p_1)}{4} - P(t)x(t-\tau) + t\displaystyle\int_t^{\infty} \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds & \\ \quad + \displaystyle\int_{t_1}^{t} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds + H(t) - M, & t \geq t_1 \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases} \tag{2.8}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$, using (2.3) and (2.5), we get

$$(Tx)(t) \leq 1 - p_1 + N_1 \int_{t_1}^{\infty} sQ_1(s)ds \leq N_1. \tag{2.9}$$

Furthermore, from (2.4) and (2.5), we have

$$(Tx)(t) \geq \frac{1-p_1}{2} - p_1 N_1 - N_1 \int_{t_1}^{\infty} sQ_2(s)ds \geq M_1. \tag{2.10}$$

Thus we prove that $TA \subset A$. To apply the contraction principle, we need to prove that T is a contraction mapping on A since A is a bounded, closed, and convex subset of X.

Now, for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$\begin{aligned} \left|(Tx_1)(t) - (Tx_2)(t)\right| &\leq \|x_1 - x_2\| \left\{ p_1 + \int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds \right\} \\ &= q_1 \|x_1 - x_2\|, \end{aligned} \tag{2.11}$$

where we used sup norm. From (2.2), we obtain $q_1 < 1$, which completes the proof of Case (i).

Example 2.2. Consider the second-order neutral delay differential equation

$$\left(x(t) + e^{-t}x(t-1)\right)'' + e^{-t-1}x(t-1) - 4e^{-t}x(t-1) = h(t), \quad t \geq 1, \tag{2.12}$$

where $P(t) = e^{-t}$, $Q_1(t) = e^{-t-1}$, $Q_2(t) = 4e^{-t}$, $h(t) = e^{-t} + e^{-2t}$ such that $0 < P(t) \leq e^{-1} < 1$. Since $H(t) = e^{-t} + 1/4e^{-2t} \to 0$ as $t \to \infty$, $\int_{t_1}^{\infty} sQ_1(s)ds = 2e^{-t}$, and $\int_{t_1}^{\infty} sQ_2(s)ds = 8e^{-t}$, then the sufficient conditions—in Case (i) of Theorem 2.1—are satisfied. Therefore, the equation has a positive solution. In fact $y = e^{-t}$ is a positive solution of this equation.

Case (ii) $(1 < p_2 \leq P(t) \leq p_1)$. Choose a $t_1 > t_0 + \sigma$ sufficiently large such that

$$\int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds < \frac{3(p_2 - 1)}{4}, \tag{2.13}$$

$$\int_{t_1}^{\infty} sQ_1(s)ds \leq \frac{1 - p_2(1 - N_2)}{N_2}, \tag{2.14}$$

$$\int_{t_1}^{\infty} sQ_2(s)ds \leq \frac{p_2(p_2 - 1) - 2p_1(N_2 + p_2 M_2)}{2p_1 N_2}, \tag{2.15}$$

$$\left|H(t) - M\right| \leq \frac{p_2 - 1}{4}, \tag{2.16}$$

where M_2 and N_2 are positive constants such that

$$1 - \frac{1}{p_2} < N_2 < \frac{p_2(p_2 - 1 - 2p_1 M_2)}{2p_1}. \tag{2.17}$$

Let X be the set as in Case (i). Set

$$A = \left\{x \in X : M_2 \leq x(t) \leq N_2,\ t \geq t_0\right\}. \tag{2.18}$$

Define a mapping $T : A \to X$ as follows:

$$(Tx)(t) = \begin{cases} \dfrac{p_2-1}{4P(t+\tau)} - \dfrac{x(t+\tau)}{P(t+\tau)} + \dfrac{t+\tau}{P(t+\tau)}\displaystyle\int_{t+\tau}^{\infty}\left[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)\right]ds \\ \quad +\dfrac{1}{P(t+\tau)}\displaystyle\int_{t_1}^{t+\tau} s\left[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)\right]ds + \dfrac{H(t+\tau)-M}{P(t+\tau)}, & t \ge t_1 \\ (Tx)(t_1), & t_0 \le t \le t_1. \end{cases} \tag{2.19}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \ge t_1$, using (2.14) and (2.16), we get

$$(Tx)(t) \le 1 - \frac{1}{p_2} + \frac{N_2}{p_2}\int_{t_1}^{\infty} sQ_1(s)ds \le N_2. \tag{2.20}$$

Furthermore, from (2.15) and (2.16) we have

$$(Tx)(t) \ge \frac{p_2-1}{2p_1} - \frac{N_2}{p_2} - \frac{N_2}{p_2}\int_{t_1}^{\infty} sQ_2(s)ds \ge M_2. \tag{2.21}$$

Thus we prove that $TA \subset A$. To apply the contraction principle, we need to prove that T is a contraction mapping on A since A is a bounded, closed, and convex subset of X.

Now, for $x_1, x_2 \in A$ and $t \ge t_1$, we have

$$\begin{aligned} \left|(Tx_1)(t) - (Tx_2)(t)\right| &\le \frac{1}{p_2}\left\|x_1 - x_2\right\|\left\{1 + \int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right]ds\right\} \\ &= q_2\left\|x_1 - x_2\right\|, \end{aligned} \tag{2.22}$$

where we used sup norm. From (2.13), we obtain $q_2 < 1$, which completes the proof of Case (ii).

Example 2.3. Consider the second-order neutral delay differential equation

$$\left(x(t) + \left(2 + e^{-t}\right)x(t-1)\right)'' + e^{-t-1}x(t-1) - 4e^{-t}x(t-1)) = h(t), \quad t \ge 1, \tag{2.23}$$

where $P(t) = 2 + e^{-t}$, $Q_1(t) = e^{-t-1}$, $Q_2(t) = 4e^{-t}$, $h(t) = e^{-t} + 2e^{-t+1} + e^{-2t}$ such that $P(t) \ge 2 + e^{-1} > 1$. Since $H(t) = e^{-t} + 1/4e^{-2t} + 2e^{-t+1} \to 0$ as $t \to \infty$, $\int_{t_1}^{\infty} sQ_1(s)ds = 2e^{-t}$, and $\int_{t_1}^{\infty} sQ_2(s)ds = 8e^{-t}$, then the sufficient conditions—in Case (ii) of Theorem 2.1—are satisfied. Therefore, the equation has a positive solution. In fact $y = e^{-t}$ is a positive solution of this equation.

Case (iii) $(-1 < -p_2 \le P(t) < 0)$. Choose a $t_1 > t_0 + \sigma$ sufficiently large such that

$$\int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds < \frac{3(1-p_2)}{4}, \tag{2.24}$$

$$\int_{t_1}^{\infty} sQ_1(s) ds \le \frac{(1-p_2)(N_3-1)}{N_3}, \tag{2.25}$$

$$\int_{t_1}^{\infty} sQ_2(s) ds \le \frac{(1-p_2) - 2M_3}{2N_3}, \tag{2.26}$$

$$\left|H(t) - M\right| \le \frac{1-p_2}{4}, \tag{2.27}$$

where M_3 and N_3 are positive constants such that

$$2M_3 + p_2 < 1 < N_3. \tag{2.28}$$

Let X be the set as in Case (i). Set

$$A = \left\{x \in X : M_3 \le x(t) \le N_3,\ t \ge t_0\right\}. \tag{2.29}$$

Define a mapping $T : A \to X$ as follows:

$$(Tx)(t) = \begin{cases} \dfrac{3(1-p_2)}{4} - P(t)x(t-\tau) + t\displaystyle\int_{t_1}^{\infty} \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds & \\ \quad + \displaystyle\int_{t_1}^{t} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds + H(t) - M, & t \ge t_1 \\ (Tx)(t_1), & t_0 \le t \le t_1. \end{cases} \tag{2.30}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \ge t_1$, using (2.25) and (2.27), we get

$$(Tx)(t) \le 1 - p_2 + p_2 N_3 + N_3 \int_{t_1}^{\infty} sQ_1(s) ds \le N_3. \tag{2.31}$$

Furthermore, from (2.26) and (2.27), we have

$$(Tx)(t) \ge \frac{1-p_2}{2} - N_3 \int_{t_1}^{\infty} sQ_2(s) ds \ge M_3. \tag{2.32}$$

Thus we prove that $TA \subset A$. To apply the contraction principle, we need to prove that T is a contraction mapping on A since A is a bounded, closed, and convex subset of X.

Now, for $x_1, x_2 \in A$ and $t \geq t_1$, we have

$$\begin{aligned} \left|(Tx_1)(t) - (Tx_2)(t)\right| &\leq \|x_1 - x_2\| \left\{ p_2 + \int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds \right\} \\ &= q_3 \|x_1 - x_2\|, \end{aligned} \tag{2.33}$$

where we used sup norm. From (2.24), we obtain $q_3 < 1$, which completes the proof of Case (iii).

Example 2.4. Consider the second-order neutral delay differential equation

$$\left(x(t) - e^{-t}x(t-1)\right)'' + e^{-t-1}x(t-1) - 4e^{-t}x(t-1) = h(t), \quad t \geq 1, \tag{2.34}$$

where $h(t) = e^{-t} + e^{-2t} - 8e^{-2t+1}$. This equation has a nonoscillatory solution $y = e^{-t}$ since the sufficient conditions—in Case (iii) of Theorem 2.1—are satisfied.

Case (iv) $(-p_2 \leq P(t) \leq -p_1 < -1)$. Choose a $t_1 > t_0 + \sigma$ sufficiently large such that

$$\int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds < \frac{3(p_1 - 1)}{4}, \tag{2.35}$$

$$\int_{t_1}^{\infty} sQ_1(s) ds \leq \frac{N_4(p_1 - 1) - p_1}{N_4}, \tag{2.36}$$

$$\int_{t_1}^{\infty} sQ_2(s) ds \leq \frac{p_2 - p_1(1 + M_4)(p_2 - 1)}{p_2 N_4}, \tag{2.37}$$

$$\left|H(t) - M\right| \leq p_1 - 1, \tag{2.38}$$

where M_4 and N_4 are positive constants such that

$$N_4 > \frac{p_1}{p_1 - 1}, \qquad M_4 < \frac{p_2 - p_1(p_2 - 1)}{p_1(p_2 - 1)}. \tag{2.39}$$

Let X be the set as in Case (i). Set

$$A = \left\{x \in X : M_4 \leq x(t) \leq N_4,\ t \geq t_0\right\}. \tag{2.40}$$

Define a mapping $T : A \to X$ as follows

$$(Tx)(t) = \begin{cases} -\dfrac{1}{P(t+\tau)} - \dfrac{x(t+\tau)}{P(t+\tau)} + \dfrac{t+\tau}{P(t+\tau)}\displaystyle\int_{t+\tau}^{\infty}\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right]ds \\ \quad + \dfrac{1}{P(t+\tau)}\displaystyle\int_{t_1}^{t+\tau} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right]ds + \dfrac{H(t+\tau)-M}{P(t+\tau)}, & t \geq t_1 \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases} \tag{2.41}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$, using (2.37) and (2.38), we get

$$\begin{aligned} (Tx)(t) &\leq -\frac{1+x(t+\tau)}{p(t+\tau)} - \frac{1}{p(t+\tau)}\int_{t_1}^{\infty} sQ_2(s)x(s-\sigma_2)ds + \frac{H(t+\tau)-M}{p(t+\tau)} \\ &\leq \frac{1+N_4}{p_1} + \frac{N_4}{p_1}\int_{t_1}^{\infty} sQ_2(s)ds + \frac{p_1-1}{p_1} \leq N_4. \end{aligned} \tag{2.42}$$

Furthermore, from (2.36) and (2.38), we have

$$\begin{aligned} (Tx)(t) &\geq -\frac{1+x(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)}\int_{t_1}^{\infty} sQ_1(s)x(s-\sigma_1)ds + \frac{H(t+\tau)-M}{p(t+\tau)} \\ &\geq \frac{1+M_4}{p_2} - \frac{N_4}{p_1}\int_{t_1}^{\infty} sQ_1(s)ds - \frac{p_1-1}{p_1} \geq M_4. \end{aligned} \tag{2.43}$$

Thus we prove that $TA \subset A$. To apply the contraction principle, we need to prove that T is a contraction mapping on A since A is a bounded, closed, and convex subset of X.

Now, for $x_1, x_2 \in A$ and $t \geq t_1$, we have

$$\begin{aligned} \left|(Tx_1)(t) - (Tx_2)(t)\right| &\leq \frac{1}{p_1}\|x_1 - x_2\|\left\{1 + \int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right]ds\right\} \\ &= q_4\|x_1 - x_2\|, \end{aligned} \tag{2.44}$$

where we used sup norm. From (2.35), we obtain $q_4 < 1$, which completes the proof of Case (iv).

Example 2.5. Consider the second-order neutral delay differential equation

$$\left(x(t) - \left(1+e^{-t}\right)x(t-1)\right)'' + e^{-t-1}x(t-1) - e^{-t-1}x(t-1) = h(t), \quad t \geq 1, \tag{2.45}$$

where $h(t) = e^{-t} + 4e^{-2t+1} - e^{-t+1}$. This equation has a nonoscillatory solution $y = e^{-t}$ since the sufficient conditions—in Case (iv) of Theorem 2.1—are satisfied. □

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References

[1] J. Hale, *Theory of Functional Differential Equations*, vol. 3 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1977.
[2] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, Clarendon Press, New York, NY, USA, 1991.
[3] H. J. Li and W. L. Liu, "Oscillations of second order neutral differential equations," *Mathematical and Computer Modelling*, vol. 22, no. 1, pp. 45–53, 1995.
[4] M. R. S. Kulenović and S. Hadžiomerspahić, "Existence of nonoscillatory solution of second order linear neutral delay equation," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 2, pp. 436–448, 1998.
[5] Á. Elbert, "Oscillation/nonoscillation criteria for linear second order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 226, no. 1, pp. 207–219, 1998.
[6] W.-T. Li, "Positive solutions of second order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 221, no. 1, pp. 326–337, 1998.
[7] N. Parhi and R. N. Rath, "Oscillation criteria for forced first order neutral differential equations with variable coefficients," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 525–541, 2001.
[8] I. R. Al-Amri, "On the oscillation of first-order neutral delay differential equations with real coefficients," *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 4, pp. 245–249, 2002.
[9] J. Manojlović, Y. Shoukaku, T. Tanigawa, and N. Yoshida, "Oscillation criteria for second order differential equations with positive and negative coefficients," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 853–863, 2006.
[10] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, vol. 74 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.