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Research Article

On a Max-Type Difference Equation

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We study the behaviour of the solutions of the following difference equation with the max operator: $x_{n+1} = \max\{1/x_n, Ax_{n-1}\}$, $n \in \mathbb{N}_0$, where parameter $A \in \mathbb{R}$ and initial values x_{-1} and x_0 are nonzero real numbers. In the most of the cases we determine the behaviour of the solutions in the terms of the initial values x_{-1} and x_0 and the parameter A.

1. Introduction

Let us consider the following difference equation with the max operator:

$$x_{n+1} = \max\left\{\frac{1}{x_n}, Ax_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where parameter $A \in \mathbb{R}$, and initial values x_{-1} and x_0 are nonzero real numbers. In this paper we study the behaviour of the solutions of (1.1). The paper studies not only positive solutions of the equation, but also all defined solutions.

Some closely related equations were investigated in [1–12] (see also the references cited therein). For example, the investigation of the difference equation:

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \dots, \frac{A_k}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where A_i, $i = 0, 1, \dots, k$, are real numbers, such that at least one of them is different from zero and initial values $x_0, x_{-1}, \dots, x_{-k}$, are different from zero, was proposed in [13, 14].

A special case of the max operator in (1.2) arises naturally in certain models in automatic control theory (see, e.g., [15, 16]).

For some other recent results concerning, among other problems, the periodic nature of scalar nonlinear difference equations see, for example, [17–32], and the references therein.

Before we formulate and prove the main results, note that when $A = 0$, (1.1) becomes $x_{n+1} = 1/x_n$, from which it follows that every solution in this case is periodic with period equal to two. Hence in the sequel we exclude the case $A = 0$.

2. The case $A < 0$

In this section we consider the behaviour of the solutions of (1.1) in the case $A < 0$. A somewhat surprising fact is that in this case the behaviour of the solutions of (1.1) is quite simple. The reason for this is in the fact that when $A < 0$, each solution of (1.1) is eventually positive. The following theorem completely describes the behaviour of the solutions of (1.1) in this case.

Theorem 2.1. *Consider (1.1) where $A < 0$.*

(a) *If $x_{-1} < 0$, $x_0 > 0$, and $x_1 = 1/x_0$, then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{2.1}$$

(b) *If $x_{-1} < 0$, $x_0 > 0$, and $x_1 = Ax_{-1}$, then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \dots, Ax_{-1}, \frac{1}{Ax_{-1}}, \dots\right). \tag{2.2}$$

(c) *If $x_{-1} > 0$, $x_0 < 0$, $x_1 = 1/x_0$, and $A \in (-\infty, -1]$, then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \frac{x_0}{A}, \dots, \frac{A}{x_0}, \frac{x_0}{A}, \dots\right). \tag{2.3}$$

(d) *If $x_{-1} > 0$, $x_0 < 0$, $x_1 = 1/x_0$, and $A \in (-1, 0)$, then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{1}{Ax_0}, \dots, Ax_0, \frac{1}{Ax_0}, \dots\right). \tag{2.4}$$

(e) *If $x_{-1} > 0$, $x_0 < 0$, $x_1 = Ax_{-1}$, and $1/(Ax_0) \geq A^2 x_{-1}$, then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \dots, Ax_0, \frac{1}{Ax_0}, \dots\right). \tag{2.5}$$

(f) *If $x_{-1} > 0$, $x_0 < 0$, $x_1 = Ax_{-1}$, and $1/(Ax_0) < A^2 x_{-1}$, then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, Ax_0, A^2 x_{-1}, \frac{1}{A^2 x_{-1}}, \dots, A^2 x_{-1}, \frac{1}{A^2 x_{-1}}, \dots\right). \tag{2.6}$$

(g) *If $x_{-1}, x_0 > 0$, then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{2.7}$$

(h) *If* $x_{-1}, x_0 < 0$, *and* $1/(Ax_{-1}) \leq Ax_0$, *then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots\right). \tag{2.8}$$

(i) *If* $x_{-1}, x_0 < 0$, *and* $1/(Ax_{-1}) > Ax_0$, *then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, Ax_{-1}, \ldots, \frac{1}{Ax_{-1}}, Ax_{-1}, \ldots\right). \tag{2.9}$$

Proof. (a), (b) Let first $x_{-1} < 0$ and $x_0 > 0$, then $x_1 = \max\{1/x_0, Ax_{-1}\} > 0$. Using induction it is easy to see that $x_n > 0$ for every $n \geq 0$, and consequently

$$x_{n+1} = \max\left\{\frac{1}{x_n}, Ax_{n-1}\right\} = \frac{1}{x_n}, \quad n \geq 1. \tag{2.10}$$

Hence, if $x_1 = 1/x_0 \geq Ax_{-1}$, then every solution is eventually two-periodic, moreover (x_n) can be written as follows:

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots\right). \tag{2.11}$$

If $x_1 = Ax_{-1}$, then

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots\right). \tag{2.12}$$

(c)–(f) If $x_{-1} > 0$ and $x_0 < 0$, then $x_1 = \max\{1/x_0, Ax_{-1}\} < 0$, $x_2 = \max\{1/x_1, Ax_0\} = Ax_0 > 0$, and

$$x_3 = \max\left\{\frac{1}{x_2}, Ax_1\right\} = \max\left\{\frac{1}{Ax_0}, \min\left\{\frac{A}{x_0}, A^2x_{-1}\right\}\right\} > 0. \tag{2.13}$$

By induction we obtain $x_n > 0, \text{for all}\, n \geq 2$. Hence $x_{n+1} = 1/x_n, \text{for all}\, n \geq 3$. Consequently, in this case, every solution is eventually two-periodic.

If $x_1 = 1/x_0$ and $A \in (-\infty, -1]$, then $Ax_{-1} \leq 1/x_0$ which implies $A^2x_{-1} \geq A/x_0$ and, consequently, $x_3 = \max\{1/(Ax_0), A/x_0\} = A/x_0$. Hence

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \frac{x_0}{A}, \ldots, \frac{A}{x_0}, \frac{x_0}{A}, \ldots\right). \tag{2.14}$$

If $x_1 = 1/x_0$ and $A \in (-1, 0)$, then $x_2 = Ax_0$ and $x_3 = \max\{1/Ax_0, A/x_0\} = 1/(Ax_0)$. Thus

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots\right). \tag{2.15}$$

If $x_1 = Ax_{-1} \geq 1/x_0$, then $A^2x_{-1} \leq A/x_0$; and if $1/(Ax_0) \geq A^2x_{-1}$, then $x_3 = \max\{1/(Ax_0), A^2x_{-1}\}$, so that

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots\right). \tag{2.16}$$

If $x_1 = Ax_{-1}$ and $1/(Ax_0) < A^2x_{-1}$, then $x_3 = A^2x_{-1}$ and

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, Ax_0, A^2x_{-1}, \frac{1}{A^2x_{-1}}, \dots, A^2x_{-1}, \frac{1}{A^2x_{-1}}, \dots\right). \tag{2.17}$$

(g) If $x_{-1}, x_0 > 0$, then $x_1 = 1/x_0 > 0$. By induction we have $x_n > 0$, for all $n \geq -1$ and, consequently, $x_{n+1} = 1/x_n$, for all $n \geq 0$. Thus, in this case

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{2.18}$$

(h), (i) If $x_{-1}, x_0 < 0$, then $x_1 = Ax_{-1} > 0$, and

$$x_2 = \max\left\{\frac{1}{x_1}, Ax_0\right\} = \max\left\{\frac{1}{Ax_{-1}}, Ax_0\right\} > 0. \tag{2.19}$$

Using induction we have $x_n > 0$, for all $n \geq 1$, which implies $x_{n+1} = 1/x_n$, for all $n \geq 2$. Therefore, if $1/(Ax_{-1}) \leq Ax_0$, we have

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \dots, Ax_0, \frac{1}{Ax_0}, \dots\right). \tag{2.20}$$

On the other hand, if $1/(Ax_{-1}) > Ax_0$, we have

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, Ax_{-1}, \dots, \frac{1}{Ax_{-1}}, Ax_{-1}, \dots\right). \tag{2.21}$$

□

Remark 2.2. As we have already mentioned, a reason for the facility in finding an explicit solution of (1.1) in the case $A < 0$ is in the fact that each solution of the equation is eventually positive. Using this fact, we can multiply (1.1) by x_n and use the change of variable $y_n = x_n x_{n-1}$. We obtain the equation $y_{n+1} = \max\{Ay_n, 1\}, n \geq n_0$, with $y_{n_0} > 0$. Hence, $y_{n_0+1} = 1$. By induction we obtain $y_n = 1$, for all $n \geq n_0 + 1$, which is equivalent to $x_{n+1} = 1/x_n$, for all $n \geq n_0$. This implies that each solution of (1.1), in this case, is eventually two-periodic.

3. The case $A > 0$

In this section we consider the behaviour of the solutions of (1.1) in the case $A > 0$. Prior to investigating the behaviour of the solutions of (1.1) in this case, we prove two auxiliary results, which are also of independent interest.

LEMMA 3.1. *Consider the difference equation*

$$y_{n+1} = \max\{Ay_n, 1\}, \tag{3.1}$$

where $y_0 > 0$. Then the following statements are true.

(a) *Let $A \in (0, 1]$, then each solution (y_n) of (3.1) is eventually constant. Moreover, if $A \in (0, 1)$, or $A = 1$ and $y_0 \in (0, 1]$, then $y_n = 1$ eventually, and if $A = 1$ and $y_0 > 1$, then $y_n = y_0$ eventually.*

(b) *Let* $A > 1$, *then each solution* (y_n) *of (3.1) eventually satisfies the difference equation* $y_{n+1} = Ay_n$.

Proof. (a) Let first $A \in (0,1)$. If $y_0 \in (0,1/A]$, then $y_1 = 1$, since $y_0 A \leq 1$. It follows that $y_1 A < 1$, which implies $y_2 = 1$. By induction we have $y_n = 1$, for all $n \geq 1$.

If $y_0 > 1/A$, then $y_1 = Ay_0$. If $A^2 y_0 \leq 1$, then $y_2 = 1$ and consequently $y_n = 1$, for all $n \geq 2$. Otherwise, $y_2 = A^2 y_0$. Since $A \in (0,1)$, we have that $A^n \to 0$ as $n \to \infty$, hence there is an index $n_0 \in \mathbb{N}$ such that $A^{n_0} y_0 \leq 1$ and $A^{n_0 - 1} y_0 > 1$. It is easy to see that $y_n = 1$ for all $n \geq n_0$, as desired.

If $A = 1$, then for $y_0 \in (0,1]$ we have $y_1 = 1$ and, consequently, $y_n = 1$ for all $n \geq n_0$. If $y_0 > 1$, then $y_1 = y_0 > 1$ and by induction $y_n = y_0$, for all $n \geq 0$, and the result is proven.

(b) If $y_0 \in (0,1/A]$, then $y_1 = 1$. Further $y_2 = \max\{Ay_1, 1\} = Ay_1 = A > 1 = y_1$. By induction we obtain that $y_{n+1} \geq y_n$, for all $n \geq 1$, which implies $y_{n+1} = Ay_n$, for all $n \geq 1$.

If $y_0 > 1/A$, then $y_1 = Ay_0 > 1$. From this it easily follows that $y_{n+1} = Ay_n$, for all $n \geq 0$, finishing the proof. □

The following lemma can be considered as a dual result of Lemma 3.1.

Lemma 3.2. *Consider the difference equation*

$$y_{n+1} = \min\{Ay_n, 1\}, \tag{3.2}$$

where $y_0 > 0$. *Then the following statements are true.*

(a) *Let* $A \in (0,1)$, *then each solution* (y_n) *of (3.2) eventually satisfies the difference equation* $y_{n+1} = Ay_n$.

(b) *Let* $A \geq 1$, *then each solution* (y_n) *of (3.2) is eventually constant. Moreover, if* $A > 1$, *or* $A = 1$ *and* $y_0 > 1$, *then* $y_n = 1$ *eventually, and if* $A = 1$ *and* $y_0 \in (0,1]$, *then* $y_n = y_0$ *eventually.*

Proof. (a) If $y_0 \in (0,1/A]$, then $y_1 = Ay_0 \leq 1$. Hence, by induction we have $y_{n+1} = Ay_n$, for all $n \geq 0$. If $y_0 > 1/A$, then $y_1 = 1$ and $y_2 = A < 1$. Thus by induction $y_{n+1} = Ay_n$, for all $n \geq 1$.

(b) If $A = 1$, then if $y_0 \in (0,1]$ it follows that $y_1 = y_0$ and thus $y_n = y_0$, for all $n \geq 0$. If $y_0 > 1$, then $y_1 = 1$, from which it follows that $y_n = 1$, for all $n \geq 1$, as desired.

If $A > 1$ and $y_0 > 1/A$, then $y_1 = 1$, which implies $y_n = 1$, for all $n \geq 1$. If $y_0 \in (0,1/A]$, then $y_1 = Ay_0$. If $A^2 y_0 > 1$, then $y_2 = 1$ and, consequently, $y_n = 1$, for all $n \geq 2$. Otherwise, $y_2 = Ay_1 = A^2 y_0 \leq 1$. Since $A^n \to \infty$ as $n \to \infty$, there is a number $m_0 \in \mathbb{N}$ such that $A^{m_0} y_0 > 1$ and $A^{m_0 - 1} y_0 \leq 1$. For such chosen index m_0 we have $y_{m_0} = 1$, which implies $y_n = 1$, for all $n \geq m_0$, finishing the proof. □

We are now in a position to formulate and prove the main result in this section.

Theorem 3.3. *Consider (1.1), where* $A > 0$.

(a) *If* $x_{-1} < 0$, $x_0 > 0$, *and* $A \in (0,1]$, *then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{3.3}$$

(b) *If* $x_{-1} < 0$, $x_0 > 0$, *and* $A > 1$, *then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \dots, A^n x_0, \frac{A^n}{x_0}, \dots\right). \tag{3.4}$$

(c) *If* $x_{-1} > 0$, $x_0 < 0$, *and* $A \in (0,1]$, *then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \dots, Ax_{-1}, \frac{1}{Ax_{-1}}, \dots\right). \tag{3.5}$$

(d) *If* $x_{-1} > 0$, $x_0 < 0$, *and* $A > 1$, *then*

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, A^2 x_{-1}, \frac{1}{x_{-1}}, \dots, \frac{A^{n-2}}{x_{-1}}, A^{n+1} x_{-1}, \dots\right). \tag{3.6}$$

(e) *If* $x_{-1}, x_0 > 0$, $x_1 = 1/x_0$, *and* $A \in (0,1]$, *then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{3.7}$$

(f) *If* $x_{-1}, x_0 > 0$, $x_1 = 1/x_0$, *and* $A > 1$, *then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \dots, A^n x_0, \frac{A^n}{x_0}, \dots\right). \tag{3.8}$$

(g) *If* $x_{-1}, x_0 > 0$, $x_1 = Ax_{-1}$, *and* $A \geq 1$, *then*

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2 x_{-1}, A^2 x_0, \dots, A^n x_{-1}, A^n x_0, \dots). \tag{3.9}$$

(h) *If* $x_{-1}, x_0 > 0$, $x_1 = Ax_{-1}$, *and* $A \in (0,1)$, *then* (x_n) *is eventually two-periodic.*

(i) *If* $x_{-1}, x_0 < 0$, $x_1 = 1/x_0$, *and* $A \in (0,1]$, *then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \dots, A^n x_0, \frac{A^n}{x_0}, \dots\right). \tag{3.10}$$

(j) *If* $x_{-1}, x_0 < 0$, $x_1 = 1/x_0$, *and* $A > 1$, *then*

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{3.11}$$

(k) *If* $x_{-1}, x_0 < 0$, $x_1 = Ax_{-1}$, *and* $A \in (0,1]$, *then*

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2 x_{-1}, \dots, A^n x_0, A^{n+1} x_{-1}, \dots). \tag{3.12}$$

(l) *If* $x_{-1}, x_0 < 0$, $x_1 = Ax_{-1}$, *and* $A > 1$, *then* (x_n) *is eventually two-periodic.*

Proof. (a), (b) Let $x_{-1} < 0, x_0 > 0$, then $x_1 = 1/x_0 > 0$ and

$$x_2 = \max\{x_0, Ax_0\} = x_0 \max\{1, A\} > 0. \tag{3.13}$$

Hence if $A \in (0,1]$, then $x_2 = x_0$, and $x_3 = \max\{1/x_0, A/x_0\} = 1/x_0$. By induction we obtain $x_{2n} = \max\{x_0, Ax_0\} = x_0$ and $x_{2n-1} = \max\{1/x_0, A/x_0\} = 1/x_0$, for all $n \geq 2$, that is,

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \dots, A^n x_0, \frac{A^n}{x_0}, \dots\right). \tag{3.14}$$

If $A > 1$, then $x_2 = Ax_0$ and

$$x_3 = \max\left\{\frac{1}{x_2}, Ax_1\right\} = \max\left\{\frac{1}{Ax_0}, \frac{A}{x_0}\right\} = \frac{A}{x_0} > 0. \tag{3.15}$$

By induction we obtain $x_{2n-1} = A^{n-1}/x_0$, $x_{2n} = A^n x_0$, for all $n \geq 1$, that is,

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \dots, A^n x_0, \frac{A^n}{x_0}, \dots\right). \tag{3.16}$$

(c)-(d) If $x_{-1} > 0$ and $x_0 < 0$, then $x_1 = Ax_{-1} > 0$, $x_2 = 1/(Ax_{-1}) > 0$, and

$$x_3 = \max\left\{\frac{1}{x_2}, Ax_1\right\} = Ax_{-1} \max\{1, A\}. \tag{3.17}$$

Clearly $x_n > 0$, for all $n \geq 1$.

If $A \in (0,1]$, then $x_3 = Ax_{-1}$ and $x_4 = 1/(Ax_{-1})$. By induction we obtain that $x_{2n-1} = Ax_{-1}$, $x_{2n} = 1/(Ax_{-1})$, for all $n \geq 1$, that is,

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \dots, Ax_{-1}, \frac{1}{Ax_{-1}}, \dots\right). \tag{3.18}$$

If $A > 1$, then $x_3 = A^2 x_{-1}$ and $x_4 = 1/x_{-1}$. By induction we obtain $x_{2n} = A^{n-2}/x_{-1}$, $x_{2n-1} = A^n x_{-1}$, for all $n \geq 2$. Hence

$$(x_n) = \left(x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, A^2 x_{-1}, \frac{1}{x_{-1}}, \dots, \frac{A^{n-2}}{x_{-1}}, A^{n+1} x_{-1}, \dots\right). \tag{3.19}$$

(e)–(h) If $x_{-1}, x_0 > 0$, then $x_1 = \max\{1/x_0, Ax_{-1}\}$. If $x_1 = 1/x_0$ then $x_2 = \max\{x_0, Ax_0\}$.

Hence, if $A \in (0,1]$, then $x_3 = 1/x_0$. Using induction we obtain $x_{2n-1} = 1/x_0$ and $x_{2n} = x_0$, for all $n \geq 1$, that is,

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \dots, x_0, \frac{1}{x_0}, \dots\right). \tag{3.20}$$

If $A \in (1, \infty)$, we get $x_2 = Ax_0$, $x_3 = A/x_0$, and by induction it follows that $x_{2n} = A^n x_0$ and $x_{2n-1} = A^{n-1}/x_0$, for all $n \geq 1$, that is,

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \dots, A^n x_0, \frac{A^n}{x_0}, \dots\right). \tag{3.21}$$

If $x_1 = Ax_{-1} \geq 1/x_0$ and $A \in [1, \infty)$, then $x_2 = \max\{1/(Ax_{-1}), Ax_0\} = Ax_0$ and $x_3 = \max\{1/(Ax_0), A^2 x_{-1}\} = A^2 x_{-1}$. By induction we get $x_{2n-1} = A^n x_{-1}$, $x_{2n} = A^n x_0$, for all $n \geq 1$. Thus

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2 x_{-1}, A^2 x_0, \dots, A^n x_{-1}, A^n x_0, \dots). \tag{3.22}$$

The case when $x_1 = Ax_{-1} \geq 1/x_0$ and $A \in (0,1)$ is more complicated. Because $x_n > 0$, for all $n \geq -1$, we can multiply (1.1) by x_n and use the change of variable $y_n = x_n x_{n-1}$ to obtain (3.1). Since all the conditions of Lemma 3.1 are satisfied, we obtain that in this case the sequence (y_n) is eventually constant. This means that each solution (x_n) of (1.1) in the case is eventually two-periodic.

(i)–(l) If $x_{-1}, x_0 < 0$, then $x_n < 0$, for all $n \geq -1$. If $x_1 = \max\{1/x_0, Ax_{-1}\} = 1/x_0$ and $A \in (0,1]$, we have $x_2 = \max\{x_0, Ax_0\} = Ax_0$ and $x_3 = \max\{1/(Ax_0), A/x_0\} = A/x_0$. By induction we have $x_{2n} = A^n x_0$, $x_{2n-1} = A^{n-1}/x_0$, for all $n \geq 1$, that is,

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots\right). \tag{3.23}$$

If $x_{-1}, x_0 < 0$, $x_1 = 1/x_0$, and $A > 1$, then $x_2 = x_0$ and $x_3 = \max\{1/x_0, A/x_0\} = 1/x_0$. By induction we have that $x_{2n} = x_0$, $x_{2n-1} = 1/x_0$, for all $n \geq 1$. Hence

$$(x_n) = \left(x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots\right). \tag{3.24}$$

If $x_{-1}, x_0 < 0$, $x_1 = Ax_{-1}$, and $A \in (0,1]$, then $x_2 = \max\{1/(Ax_{-1}), Ax_0\} = Ax_0$ and $x_3 = \max\{1/(Ax_0), A^2 x_{-1}\} = A^2 x_{-1}$. Using induction we get $x_{2n} = A^n x_0$, $x_{2n-1} = A^n x_{-1}$, for all $n \geq 1$. Thus

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2 x_{-1}, \ldots, A^n x_0, A^{n+1} x_{-1}, \ldots). \tag{3.25}$$

Finally, if $x_{-1}, x_0 < 0$, $x_1 = Ax_{-1}$, and $A > 1$, then since $x_n < 0$, for all $n \geq -1$, multiplying (1.1) by x_n and using the change of variable $z_n = x_n x_{n-1}$, we obtain that the sequence (z_n) satisfies (3.2) and $z_n > 0$, for all $n \geq 0$. Since $A > 1$, by Lemma 3.2, we obtain that (z_n) is eventually constant which implies that the sequence (x_n) is eventually two-periodic, as desired. □

Note. A slightly different version of the paper (mostly without recent references), circulated among some experts in the research field since 2005, was accepted for publication in the International Journal of Pure and Applied Mathematics in April 2005. However, it was withdrawn since it had not been published in a reasonable long period of time.

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