Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 34517, 9 pages
doi:10.1155/2007/34517

Research Article

On the Recursive Sequence $x_{n+1} = A + x_n^p / x_{n-1}^p$

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Received 4 July 2006; Revised 7 November 2006; Accepted 9 January 2007

This paper studies the boundedness, global attractivity, and periodicity of the positive solutions of the difference equation $x_{n+1} = A + x_n^p / x_{n-1}^p$, $n \in \mathbb{N}_0$, with $p, A \in (0, \infty)$. The main results give a complete picture regarding the boundedness character of the positive solutions of the equation.

1. Introduction

Recently, there has been great interest in studying nonlinear and rational difference equations (cf. [1–23] and the references therein). One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economics, probability theory, genetics, psychology, sociology, and so forth. Such equations also appear naturally as discrete analogs of differential equations which model various biological and economic systems (see, e.g., [6, 8–11, 16, 22] and the references therein).

In [5], the authors investigated the positive solutions of the difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

in order to prove that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with $A \in (0, \infty)$, converges to a period-two solution. They proved that the positive equilibrium $\overline{x} = A + 1$ is a global attractor of (1.1) relative to the set $(0, \infty)^2$.

In this paper, we investigate the positive solutions of the following extension of (1.1):

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^p}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where A and p are positive numbers.

The linearized equation associated with (1.3) is

$$(A+1)y_{n+1} - py_n + py_{n-1} = 0. \tag{1.4}$$

The characteristic roots of (1.4) are

$$\lambda_{1,2} = \frac{p \pm \sqrt{p^2 - 4p(A+1)}}{2(A+1)}. \tag{1.5}$$

It is easy to see that both roots have modulus strictly less than one, if and only if $p < A+1$.

Based on this fact, one can conjecture that (1.3) is globally stable when $p < A+1$. However, we will prove that this is not true by proving that there are values of p such that $p < A+1$ and that there are unbounded positive solutions of (1.3) in the case.

Our aim here is to give a complete picture regarding the boundedness character of the positive solutions of (1.3). We also present a global stability result for the case $p \in (0,1]$, as well as a result regarding the periodicity of the positive solutions of (1.3). Closely related equations to (1.3) are investigated, for example, in [1–3, 7, 9, 17–21, 23].

2. Boundedness character for (1.3)

In this section, we investigate the boundedness character of the positive solutions of (1.3).

2.1. Case $p \geq 4$.

Here, we investigate (1.3) for the case $p \geq 4$.

Theorem 2.1. *Assume that $p \geq 4$. Then (1.3) has positive unbounded solutions.*

Proof. First, note that for every solution of (1.3), the following inequality holds:

$$x_{n+1} > \left(\frac{x_n}{x_{n-1}}\right)^p, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Let $y_n = \ln x_n$. Taking the logarithm of (2.1), it follows that

$$y_{n+1} - py_n + py_{n-1} > 0, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Note that the roots of the polynomial

$$P(\lambda) = \lambda^2 - p\lambda + p \tag{2.3}$$

are

$$\lambda_{1,2} = \frac{p \pm \sqrt{p^2 - 4p}}{2}. \tag{2.4}$$

It is clear that if $p \geq 4$, then $\lambda_1 \geq 2$. On the other hand, we have that

$$\lambda_2 = \frac{2p}{p + \sqrt{p^2 - 4p}} > 1. \tag{2.5}$$

Hence, if $p \geq 4$, both roots of $P(\lambda)$ are greater than one.

Now note that inequality (2.2) can be written in the following form:

$$y_{n+1} - \lambda_1 y_n - \lambda_2 (y_n - \lambda_1 y_{n-1}) > 0, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Returning to the sequence x_n, we obtain

$$\frac{x_{n+1}}{x_n^{\lambda_1}} > \left(\frac{x_n}{x_{n-1}^{\lambda_1}} \right)^{\lambda_2}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

From (2.7), it follows that

$$\frac{x_n}{x_{n-1}^{\lambda_1}} > \left(\frac{x_0}{x_{-1}^{\lambda_1}} \right)^{\lambda_2^n}. \tag{2.8}$$

Choose x_{-1} and x_0 so that

$$x_0 > 1, \qquad x_0 > x_{-1}^{\lambda_1}. \tag{2.9}$$

From (2.8) and such chosen initial values, it follows that

$$x_n > \left(\frac{x_0}{x_{-1}^{\lambda_1}} \right)^{\lambda_2^n} x_{n-1}^{\lambda_1} > x_{n-1}^{\lambda_1} > \cdots > x_0^{\lambda_1^n}, \tag{2.10}$$

and consequently

$$x_n > x_0^{\lambda_1^n}, \quad n \in \mathbb{N}. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), it follows that

$$\lim_{n \to \infty} x_n = +\infty, \tag{2.12}$$

from which the result follows. □

2.2. Case $p \in (0,4)$. Here, we investigate the boundedness character of the positive solutions of (1.3) for the case $p \in (0,4)$.

THEOREM 2.2. *Assume that $p \in (0,4)$. Then all positive solutions of (1.3) are bounded.*

Proof. From (1.3), we have that

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^p} = A + \left(\frac{A}{x_{n-1}} + \frac{x_{n-1}^{p-1}}{x_{n-2}^p}\right)^p \tag{2.13}$$

$$\begin{aligned}
&= A + \left(\frac{A}{x_{n-1}} + \left(\frac{x_{n-1}}{x_{n-2}^{p/(p-1)}}\right)^{p-1}\right)^p \\
&= A + \left(\frac{A}{x_{n-1}} + \left(\frac{A}{x_{n-2}^{p/(p-1)}} + \frac{x_{n-2}^{p-p/(p-1)}}{x_{n-3}^p}\right)^{p-1}\right)^p \\
&= A + \left(\frac{A}{x_{n-1}} + \left(\frac{A}{x_{n-2}^{p/(p-1)}} + \left(\frac{x_{n-2}}{x_{n-3}^{p/(p-p/(p-1))}}\right)^{p-p/(p-1)}\right)^{p-1}\right)^p = \cdots \\
&= A + \left(\frac{A}{x_{n-1}} + \left(\frac{A}{x_{n-2}^{p/(p-1)}} + \left(\cdots + \left(\frac{A}{x_{n-k}^{p_k}} + \frac{x_{n-k}^{p-p_k}}{x_{n-k-1}^p}\right)^{p-p_{k-1}} \cdots\right)^{p-p/(p-1)}\right)^{p-1}\right)^p,
\end{aligned} \tag{2.14}$$

for every $n \geq k$, where the sequence p_k is defined by

$$p_{k+1} = \frac{p}{p - p_k}, \qquad p_0 = 0. \tag{2.15}$$

If $p = p_{k_1}$, for some $k_1 \in \mathbb{N}$, the recurrence relation in (2.15) defines only $k_1 + 1$ terms, $p_0, p_1, \ldots, p_{k_1}$. In this case, we finish the procedure described in (2.13)-(2.14) after k_1 steps (e.g., if $p = p_1 = 1$, the procedure is finished in (2.13)).

If $p \leq p_1 = 1$, then from (2.13) it follows that

$$x_{n+1} \leq A + \frac{(A+1)^p}{A^p}, \tag{2.16}$$

from which the result follows in this case.

Assume now that $p > 1$. We prove that there is the least $k_0 \in \mathbb{N}$ such that $p_{k_0-1} < p$ and $p_{k_0} \geq p$. Otherwise, $p_k < p$, for every $k \in \mathbb{N}$. Since $0 = p_0 < p_1 = 1$, and the function $f(x) = p/(p-x)$ is strictly increasing on the interval $(0, p)$, the sequence p_k would be strictly increasing. Since p_k is bounded, then it would converge, say, to p^*, and it would be a solution of the equation

$$x^2 - px + p = 0. \tag{2.17}$$

However, since $p \in (0,4)$, the equation does not have real solutions. Hence, there is the least $k_0 \in \mathbb{N}$ such that $p_{k_0-1} < p$ and $p_{k_0} \geq p$, as claimed. From this and (2.14) with $k = k_0$, it follows that

$$\begin{aligned}
x_{n+1} &= A + \left(\frac{A}{x_{n-1}} + \left(\frac{A}{x_{n-2}^{p/(p-1)}} + \left(\cdots + \left(\frac{A}{x_{n-k_0}^{p_{k_0}}} + \frac{1}{x_{n-k_0}^{p_{k_0}-p} x_{n-k_0-1}^p}\right)^{p-p_{k_0-1}} \cdots\right)^{p-p/(p-1)}\right)^{p-1}\right)^p \\
&\leq A + \left(1 + \left(\frac{1}{A^{p/(p-1)-1}} + \left(\cdots + \left(\frac{1}{A^{p_{k_0}-1}} + \frac{1}{A^{p_{k_0}}}\right)^{p-p_{k_0-1}} \cdots\right)^{p-p/(p-1)}\right)^{p-1}\right)^p,
\end{aligned} \tag{2.18}$$

for every $n \geq k_0$. The last expression is an upper bound for the sequence x_n, finishing the proof of the theorem. □

3. Case $p \in (0,1]$

In this section we study the global stability of the positive solutions of (1.3) for the case $p \in (0,1]$. The main result of this section is the following.

Theorem 3.1. *Assume that $p \in (0,1]$. Then the positive equilibrium $\overline{x} = A+1$ of (1.3) is globally asymptotically stable.*

Before proving Theorem 3.1, we need an auxiliary result which is incorporated in the following lemma.

Lemma 3.2. *Let (x_n) be a nontrivial positive solution of (1.3). Then the following statements are true.*

(a) *(x_n) oscillates about the equilibrium $\overline{x} = A+1$ with semicycles of length two or three and the extreme in a semicycle occurs in the first or the second term.*

(b) *For $n > 2$, one has that*

$$A < x_n < A + \frac{(A+1)^p}{A^p}. \tag{3.1}$$

Proof. First, we show that every positive semicycle, except possibly the first, has two or three terms. To this end, assume that $x_{N-1} < \overline{x}$ and $x_N \geq \overline{x}$, for some $N \in \mathbb{N}$. Then, we have

$$x_{N+1} = A + \frac{x_N^p}{x_{N-1}^p} > A + 1 = \overline{x}. \tag{3.2}$$

If $x_{N+1} > x_N$, then we have

$$x_{N+2} = A + \frac{x_{N+1}^p}{x_N^p} > A + 1 = \overline{x}. \tag{3.3}$$

On the other hand, since $p \in (0,1]$, we have that

$$x_{N+2} = A + \frac{x_{N+1}^p}{x_N^p} \leq A + \frac{x_{N+1}^p}{\overline{x}^p} \leq A + \frac{x_{N+1}}{A+1} < x_{N+1}, \tag{3.4}$$

so that $\overline{x} < x_{N+2} < x_{N+1}$.

From this, we have that

$$x_{N+3} = A + \frac{x_{N+2}^p}{x_{N+1}^p} < A + 1 = \overline{x}, \tag{3.5}$$

from which it follows that (a) holds true.

(b) From the proof of Theorem 2.2 and since $p \in (0,1]$, we have that

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^p} = A + \left(\frac{A}{x_{n-1}} + \frac{1}{x_{n-1}^{1-p}x_{n-2}^p}\right)^p < A + \frac{(A+1)^p}{A^p}, \tag{3.6}$$

as desired. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By the linearized stability theorem, it follows that $\overline{x} = A+1$ is locally asymptotically stable if $p \in (0,1)$. Now, define the sequences as follows:

$$\begin{aligned} L_1 &= A, \qquad U_1 = A + \frac{(A+1)^p}{A^p}, \\ L_{n+1} &= A + \frac{(A+1)^p}{U_n^p}, \qquad U_{n+1} = A + \frac{(A+1)^p}{L_{n+1}^p}. \end{aligned} \tag{3.7}$$

As in Lemma 3.2, it can be shown that the sequences (U_n) and (L_n) are upper and lower bounds for the semicycles of the solutions (x_n) of (1.3). On the other hand, it is easy to see that

$$L_1 < L_2 < \cdots < L_n < L_{n+1} < \cdots < \overline{x} < \cdots < U_{n+1} < U_n < \cdots < U_2 < U_1, \tag{3.8}$$

and that they are solutions of the difference equation

$$w_{n+1} = A + \left[\frac{A+1}{A + ((A+1)/w_n)^p}\right]^p. \tag{3.9}$$

Now we prove that every convergent solution of (3.9) converges to $\overline{x} = A+1$. In order to prove this, it is enough to prove that the equation

$$f(x) = \frac{1}{p}\ln(x-A) + \ln\left(Ax^p + (A+1)^p\right) - p\ln x - \ln(A+1) \tag{3.10}$$

has a unique solution $x = A+1$ on the interval $(A, +\infty)$.

Since

$$f'(x) = \frac{Ax^{p+1} + (A+1)^p x(1-p^2) + p^2A(A+1)^p}{px(x-A)\left(Ax^p + (A+1)^p\right)} > 0, \tag{3.11}$$

when $p \in (0,1]$,

$$\lim_{x \to A+0} f(x) = -\infty, \qquad \lim_{x \to +\infty} f(x) = +\infty, \tag{3.12}$$

it follows that $x = A+1$ is indeed a unique solution of (3.10) on the interval $(A, +\infty)$, finishing the proof of the theorem. □

Remark 3.3. Let $g(x)$ be the numerator of $f'(x)$, that is,

$$g(x) = Ax^{p+1} - (A+1)^p(p^2-1)x + p^2A(A+1)^p \tag{3.13}$$

and let $p > 1$.

We have

$$g'(x) = (p+1)\left(Ax^p - (p-1)(A+1)^p\right), \tag{3.14}$$

so that the function $g(x)$ attains its minimum at the point $m = ((p-1)(A+1)^p/A)^{1/p}$, and

$$g(m) = p(A+1)^p(Ap - m(p-1)) = p(A+1)^p\left(Ap - \left(\frac{(p-1)(A+1)^p}{A}\right)^{1/p}(p-1)\right). \tag{3.15}$$

Now we study when this minimum is nonnegative, which is equivalent to

$$A^{p+1}p^p \geq (p-1)^{p+1}(A+1)^p. \tag{3.16}$$

Note that when $p = A+1$, the above inequality becomes equality.

Let

$$h(x) = (p+1)\ln x + p\ln p - (p+1)\ln(p-1) - p\ln(x+1), \tag{3.17}$$

then

$$h(p-1) = 0, \qquad h(+\infty) = +\infty. \tag{3.18}$$

On the other hand,

$$h'(x) = \frac{(p+1)(x+1) - px}{x(x+1)} > 0 \tag{3.19}$$

when $x > 0$, which implies that the function h is increasing on the interval $(0, +\infty)$. From this and (3.18), we obtain that $h(x) \geq 0$, if and only if $x \geq p-1$, and consequently inequality (3.16) holds for $A \geq p-1$, from which it follows that $g(x) \geq 0$ for $A \geq p-1$, showing that the function $f(x)$ has a unique zero, as desired.

This consideration and Theorem 2.2 motivate us to believe that the following conjecture is true.

Conjecture 3.4. If $1 < p < \min\{4, A+1\}$, then $\overline{x} = A+1$ is a global attractor of (1.3) relative to the set $(0,\infty)^2$.

4. Prime two-periodic solutions of (1.3)

In this section, we investigate the existence of prime two-periodic solutions of (1.3).

THEOREM 4.1. *There are no positive prime two-periodic solutions of (1.3).*

Proof. Assume that

$$\ldots, x, y, x, y, \ldots, \tag{4.1}$$

where $x, y \in (A, \infty)$, is a prime two-periodic solution of (1.3). Then it must be

$$x = A + \left(\frac{y}{x}\right)^p, \qquad y = A + \left(\frac{x}{y}\right)^p, \tag{4.2}$$

which implies that

$$y = A + \frac{1}{x - A}. \tag{4.3}$$

Substituting (4.3) into (4.2) and after some simple calculation, it follows that

$$(x-A)^{p+1} x^p = \left(A(x-A)+1\right)^p. \tag{4.4}$$

Taking the logarithm on both sides of (4.4), we obtain

$$f(x) = (p+1)\ln(x-A) + p\ln x - p\ln\left(A(x-A)+1\right) = 0. \tag{4.5}$$

It is clear that $x = A + 1$ is an obvious solution of (4.5). Now we prove that this is a unique solution of the equation. Since

$$f'(x) = \frac{(x-A)\left(Ax + p\left(A(x-A)+1\right)\right) + (p+1)x}{x(x-A)\left(A(x-A)+1\right)}, \tag{4.6}$$

we have that $f'(x) > 0$, for $x \in (A, \infty)$, which implies that the function f is strictly monotonous on the interval (A, ∞). Hence, $A + 1$ is the unique solution of (4.5), and consequently $(A+1, A+1)$ is a unique solution of system (4.2), finishing the proof of the theorem. □

Research project. Investigate whether or not (1.3) has positive prime periodic solutions of period greater than two.

Acknowledgment

The author would like to express his sincere thanks to the referees for valuable comments.

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