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Research Article

On the Behaviour of the Solutions of a Second-Order Difference Equation

Leonid Gutnik and Stevo Stević

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We study the difference equation $x_{n+1} = \alpha - x_n/x_{n-1}$, $n \in \mathbb{N}_0$, where $\alpha \in \mathbb{R}$ and where x_{-1} and x_0 are so chosen that the corresponding solution (x_n) of the equation is defined for every $n \in \mathbb{N}$. We prove that when $\alpha = 3$ the equilibrium $\overline{x} = 2$ of the equation is not stable, which corrects a result due to X. X. Yan, W. T. Li, and Z. Zhao. For the case $\alpha = 1$, we show that there is a strictly monotone solution of the equation, and we also find its asymptotics. An explicit formula for the solutions of the equation are given for the case $\alpha = 0$.

1. Introduction

Recently, there has been a great interest in studying nonlinear and rational difference equations (cf. [1–35] and the references therein).

In [34], the authors study the boundedness, the global asymptotic stability, and the periodicity of positive and negative solutions $(x_n)_{n\in\mathbb{N}_0}$ of the difference equation

$$x_{n+1} = \alpha - \frac{x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and the initial conditions x_{-1}, x_0 are arbitrary real numbers.

First note that (1.1) has the unique equilibrium $\overline{x} = \alpha - 1$.

By the change $x_n = -y_n$ (1.1) is transformed into the equation

$$y_{n+1} = \beta + \frac{y_n}{y_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $\beta = -\alpha$.

When $\beta > 0$ (1.2) was studied in [13] where it was shown that every positive solution of (1.2) converges to the equilibrium $\overline{y} = \beta + 1$. Hence, if $\alpha \in (-\infty, 0)$, then every negative solution of (1.1) converges to the equilibrium $\overline{x} = \alpha - 1$.

The case $\alpha > 0$ was investigated for the first time in [34], where the authors proved the following results (summarized in a theorem).

Theorem 1.1. *Consider (1.1). Then the following statements hold true.*

(a) *$\overline{x}$ is locally asymptotically stable if and only if $\alpha > 3$ or $\alpha < 0$.*
(b) *$\overline{x}$ is a saddle point if and only if $1 < \alpha < 3$.*
(c) *$\overline{x}$ is a repeller if and only if $0 < \alpha < 1$.*
(d) *$\overline{x}$ is stable, but not locally asymptotically stable, if and only if $\alpha = 3$.*
(e) *Equation (1.1) has a periodic solution with minimal period equal to 2 if and only if*

$$\alpha < -1 \quad \text{or} \quad \alpha > 3, \tag{1.3}$$

there are exactly two such solutions and they are defined by the initial conditions

$$x_{-1} = \phi = \frac{\alpha + 1 + \varepsilon\sqrt{(\alpha+1)(\alpha-3)}}{2}, \qquad x_0 = \psi = \frac{\alpha + 1 - \varepsilon\sqrt{(\alpha+1)(\alpha-3)}}{2}, \tag{1.4}$$

where $\varepsilon = 1$ determines one solution and $\varepsilon = -1$ determines another (the values (1.3) are the roots of the quadratic $t^2 - (\alpha+1)t + (\alpha+1)$).
(f) *If condition (1.3) is satisfied, then the two periodic solutions are saddle points of the system $Y_{n+1} = F^2(Y_n)$, where $F^2 = F \circ F$ and $F(u,v) = (v, \alpha - v/u)^T$.*
(g) *If $\alpha > 3$ and $\{x_{-1}, x_0\} \subset [\psi, \phi]$ (where $\varepsilon = 1$ in (1.4)), then all the terms of a positive solution (x_n) of (1.1) lie in the segment and the unique equilibrium $\alpha - 1$ is a global attractor of (1.1) with basin $[\psi, \phi]^2 \setminus \{(\psi,\phi),(\phi,\psi)\}$.*
(h) *If (x_n) is a positive solution of (1.1), which consists of a single semicycle, then this sequence converges monotonically to $\overline{x} = \alpha - 1$.*
(i) *If (x_n) is a positive solution of (1.1), which consists of at least two semicycles, then this sequence is oscillatory. Moreover, with the possible exception of the first semicycle, every semicycle has length one and every term x_n is less than α, and with the possible exception of the first two semicycles, no term x_n is ever equal to $\alpha - 1$.*
(j) *Equation (1.1) has a strictly monotone solution, which converges to $\overline{x} = \alpha - 1$.*
(k) *If $\alpha = 0$, then every nontrivial solution of (1.1) is periodic with prime period six. These solutions are $x_{-1}, x_0, -x_0/x_{-1}, 1/x_{-1}, 1/x_0, -x_{-1}/x_0, \ldots$.*

Remark 1.2. We would like to point out that statements (e) and (g) in Theorem 1.1 are different from the original ones. Namely, the authors in [34] claim that there is a unique prime period solution of (1.1) which is not quite correct. Also, statements (h)–(j) make sense only if $\alpha > 1$, which was not mentioned in [34].

Equations (1.1) and (1.2) and their extensions have been extensively studied for some time, see, for example, [1–3, 5, 7, 13, 14, 18, 21, 25–27, 33].

2. Case $\alpha = 3$

In this section we consider the case $\alpha = 3$ in detail. The reason for this is the fact that the statement in Theorem 1.1(d) was obtained by the authors of paper [34] by applying the linearized stability theorem which failed in this case. Namely, the characteristic equation associated with (1.1) for the case $\alpha = 3$ is

$$2z_{n+1} + z_n - z_{n-1} = 0, \tag{2.1}$$

and the roots of its characteristic equation

$$2\lambda^2 + \lambda - 1 = 0 \tag{2.2}$$

are

$$\lambda_1 = -1, \qquad \lambda_2 = \frac{1}{2}. \tag{2.3}$$

The following theorem shows that Theorem 1.1(d) is not true.

Theorem 2.1. *The equilibrium $\overline{x} = 2$ of the equation*

$$x_{n+1} = 3 - \frac{x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.4}$$

is unstable.

Proof. Let $\theta_n = x_n - 2$. Then (2.4) takes the form

$$\theta_{n+1} = \frac{\theta_{n-1} - \theta_n}{\theta_{n-1} + 2}, \tag{2.5}$$

and we must prove that $\overline{\theta} = 0$ is an unstable equilibrium of (2.5). We further let $\beta_n = (-1)^n\theta_n$. Then (2.5) turns into the system

$$\beta_{2k+1} = \frac{\beta_{2k-1} + \beta_{2k}}{2 - \beta_{2k-1}}, \qquad \beta_{2k+2} = \frac{\beta_{2k+1} + \beta_{2k}}{2 + \beta_{2k}}, \tag{2.6}$$

and we must prove that $\overline{\beta} = 0$ is an unstable equilibrium of (2.6).

Suppose the inequalities

$$\beta_{2k-1} \in (0,1), \quad \beta_{2k} > 0 \tag{2.7}$$

hold for some $k \in \mathbb{N}_0$. Let $\eta \in (0, \beta_{2k-1})$ be fixed such that

$$1 - \eta < \frac{\beta_{2k}}{\eta}. \tag{2.8}$$

Further let

$$\gamma \in \left(1-\eta, \min\left(1, \frac{\beta_{2k}}{\eta}\right)\right) \tag{2.9}$$

be fixed. Clearly, such η and γ always exist. Then certainly $0 < \eta < \beta_{2k-1} < 1$, $0 < 1-\eta < \gamma$, and $\gamma\eta < \beta_{2k}$. Furthermore,

$$\begin{aligned}
\beta_{2k+1} &= \frac{\beta_{2k-1}+\beta_{2k}}{2-\beta_{2k-1}} \geq \frac{\eta+\beta_{2k}}{2-\eta} \geq \frac{\eta+\eta\gamma}{2-\eta} = \eta\frac{1+\gamma}{2-\eta} > \eta, \\
\beta_{2k+2} &= \frac{\beta_{2k+1}+\beta_{2k}}{2+\beta_{2k}} \geq \frac{\eta+\beta_{2k}}{2+\beta_{2k}} \geq \frac{\eta+\eta\gamma}{2+\gamma\eta} = \eta\frac{(1+\gamma)}{2+\gamma\eta} \\
&> \eta\frac{(1+\gamma)}{2+\eta} = \eta\frac{1+\gamma}{2-\eta} \times \frac{2-\eta}{2+\eta}.
\end{aligned} \tag{2.10}$$

Let

$$\eta^* = \eta\frac{1+\gamma}{2-\eta}, \qquad \gamma^* = \frac{2-\eta}{2+\eta}. \tag{2.11}$$

Then

$$\begin{aligned}
1-\eta^*-\gamma^* &= \frac{2\eta}{2+\eta} - \eta^* = \frac{\eta}{4-\eta^2}\left(2(2-\eta)-(2+\eta)(1+\gamma)\right) \\
&= \frac{\eta}{4-\eta^2}\left(2(1-\eta-\gamma)-\eta(1+\gamma)\right) < 0.
\end{aligned} \tag{2.12}$$

Therefore, if

$$\beta_{2k+1} < 1, \tag{2.13}$$

then we can pass from k with η and γ, to $k+1$ with η^* instead of η and γ^* instead of γ. We suppose now that $\overline{\beta} = 0$ is a stable equilibrium of (2.6). Then for $\varepsilon = 1/2$ there exists a $\delta \in (0, 1/2)$ such that, if $|\beta_{-1}| \leq \delta$ and $|\beta_0| \leq \delta$, then

$$|\beta_n| \leq \varepsilon = \frac{1}{2} < 1 \tag{2.14}$$

for all $n \in \mathbb{N}_0 \cup \{-1\}$.

We take now $\beta_{-1} \in (0, \delta)$, $\beta_0 \in (0, \delta)$, and fix $\eta_0 \in (0, \beta_{-1})$ such that

$$1-\eta_0 < \frac{\beta_0}{\eta_0}. \tag{2.15}$$

Further, fix

$$\gamma_0 \in \left(1-\eta_0, \min\left(1, \frac{\beta_0}{\eta_0}\right)\right). \tag{2.16}$$

Then, since (2.14) holds, by induction, it follows that for each $k \in \mathbb{N}_0$, there exist $\eta_k \in (0, \beta_{2k-1})$ and $\gamma_k \in (1-\eta_k, 1)$ such that

$$\eta_k\gamma_k \le \beta_{2k},$$
$$\eta_{k+1} = \eta_k \frac{1+\gamma_k}{2-\eta_k} > \eta_k, \qquad \gamma_{k+1} = \frac{2-\eta_k}{2+\eta_k}. \tag{2.17}$$

Therefore, there exists $c = \lim_{k\to\infty} \eta_k > 0$. In view of (2.17),

$$\gamma_k = (2-\eta_k)\frac{\eta_{k+1}}{\eta_k} - 1, \qquad \gamma_{k+1} = \frac{2-\eta_k}{2+\eta_k}. \tag{2.18}$$

Letting $k \to \infty$ in these two relationships, we obtain

$$1 - c = \lim_{k\to\infty} \gamma_k = \lim_{k\to\infty} \gamma_{k+1} = \frac{2-c}{2+c}. \tag{2.19}$$

Hence, $c^2 + c - 2 = c - 2$, that is, $c = 0$, which is a contradiction. Hence, the result follows. □

3. Case $\alpha = 1$

In this section we address the problem of the existence of solutions of (1.1) converging to zero. For the results devoted to the research area, see the following papers [5, 7, 17, 23, 29, 32, 33] and the references therein.

The following theorem was proved in [33].

THEOREM 3.1. *Let $f \in C(I^2, \mathbb{R})$ for some interval I, $f(x,y)$ is decreasing in x on I for a fixed y and increasing in y on I for a fixed x and let f have a unique equilibrium $\overline{x} \in I$. Then the following inequality*

$$(f(x,x) - x)(x - \overline{x}) < 0 \quad \text{for some } x \in I \setminus \{\overline{x}\} \tag{3.1}$$

is a necessary and sufficient condition for the existence of a strictly monotone solution (x_n) of the equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0 \tag{3.2}$$

such that x_n converges to $\overline{x}$.

Using Theorem 3.1, the authors of [34] prove that (1.1) has a strictly monotone solution, which converges to $\overline{x} = \alpha - 1$, if $\alpha \ge 1$. However, they apply the theorem to the open interval $(0, \infty)$ so that when $\alpha = 1$, the equilibrium point $\overline{x} = 0$ does not belong to the interval, which is essential for applying the theorem. Hence, the problem of the existence of monotone solutions for the case $\alpha = 1$ was not solved in [34].

Here, we give an answer to the problem of the existence of monotone solutions of (1.1) for the case $\alpha = 1$.

A general result which can help in proving the existence of monotone solutions (even in the nonautonomous case) was developed in [29], based on Berg's nice ideas in [7] which use asymptotics. We have proved the following inclusion theorem.

Theorem 3.2. *Let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous and nondecreasing function in each argument, and let (y_n) and (z_n) be sequences such that $y_n < z_n$ for $n \ge n_0$ and*

$$y_{n-k} \le f(y_{n-k+1},\dots,y_{n+1}), \qquad f(z_{n-k+1},\dots,z_{n+1}) \le z_{n-k}, \quad \text{for } n > n_0+k-1. \tag{3.3}$$

Then, the difference equation

$$x_{n-k} = f(x_{n-k+1},\dots,x_{n+1}) \tag{3.4}$$

has a solution such that

$$y_n \le x_n \le z_n \quad \text{for } n \ge n_0. \tag{3.5}$$

Remark 3.3. Theorem 3.2 can be improved if we assume that f is strictly increasing. Namely, it can be proved that in this case, the solution x_n is uniquely defined by its initial values. Also, in the formulation of Theorem 3.2, we can replace $\mathbb{R}$ by an interval $I \subset \mathbb{R}$.

Asymptotics for solutions of difference equations have been investigated by Berg and the second author of this paper for some time, see, for example, [6–10, 12, 19–22, 25, 28, 29] and the reference therein. Some methods for construction of the bounds (y_n) and (z_n) can be found in [6–8, 10].

Note that if (x_n) is a positive solution of the equation

$$x_{n+1} = 1 - \frac{x_n}{x_{n-1}}, \tag{3.6}$$

then $0 < x_n < 1$, $n \in \mathbb{N}$, and from this and (3.6) it follows that $x_n < x_{n-1}$. If $\lim_{n\to\infty} x_n = \overline{x}$, then from (3.6) we obtain $\overline{x} = 0$. Hence, the following statement is true.

Theorem 3.4. *Every positive solution of (3.6) decreasingly converges to zero.*

Now we turn to the problem of the existence of solutions of (3.6) converging to zero. Since the linearized equation of (3.6) is

$$x_{n+1} - x_n = 0, \tag{3.7}$$

we expect that there is a solution which has the following asymptotics

$$x_n = \frac{a}{n} + \frac{b\ln n + c}{n^2} + \frac{d\ln^2 n + e\ln n}{n^3} + O\left(\frac{1}{n^3}\right). \tag{3.8}$$

For some explanations how to guess the asymptotics, see [29] (see also [5, 19, 20, 22]).

Since we ask for solutions which are defined for all $n \in \mathbb{N}$, then we can write (3.6) in the following form:

$$H(x_{n-1},x_n,x_{n+1}) := x_{n-1} - x_n - x_{n+1}x_{n-1} = 0. \tag{3.9}$$

If we replace the asymptotics (3.8) into (3.9), then equating the coefficients nearby

$\ln^l n/n^m$ in the obtained equality to zero, we find that

$$a = b = d = 1, \qquad e = 2c - 1, \qquad f = c^2 - c + \frac{3}{2}, \tag{3.10}$$

where c is an arbitrary real number and f the coefficient of $1/n^3$.

This motivated us to choose the following expression:

$$\varphi_n = \frac{1}{n} + \frac{\ln n}{n^2} + p\frac{(\ln n)^2}{n^3}. \tag{3.11}$$

Now we are in a position to formulate and prove the main result of this section.

THEOREM 3.5. *There exists an absolute integer constant $n_0 > 0$, such that for any $a \in [n_0, +\infty) \cap \mathbb{Z}$ there is a solution x_n, of (3.6), which has the following form:*

$$x_n = \frac{1}{n+a} + \frac{\ln(n+a)}{(n+a)^2} + O\left(\frac{(\ln n)^2}{n^3}\right), \tag{3.12}$$

where $n \in \mathbb{N}_0$.

Proof. Write (3.6) in the following form:

$$G(x_{n-1}, x_n, x_{n+1}) = x_{n-1} - \frac{x_n}{1 - x_{n+1}} = 0. \tag{3.13}$$

Since

$$H(x_{n-1}, x_n, x_{n+1}) = (1 - x_{n+1})G(x_{n-1}, x_n, x_{n+1}), \tag{3.14}$$

we have that (3.9), and (3.13) have identical sets of solutions x_n satisfying the condition $x_n \in (0,1)$ for $n \in \mathbb{N}_0 \cup \{-1\}$ and these sets are equal to the set of all positive solutions of (3.6).

Note that the function

$$f(y,z) = \frac{y}{1-z} \tag{3.15}$$

is increasing in y and z if they belong to the interval (0,1).

Now we show that

$$G(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) \sim (p-1)\frac{(\ln n)^2}{n^4}. \tag{3.16}$$

Let $\eta^2 = 1$. Then

$$\ln(n - \eta) = \ln n - q_0(\eta, n) + O\left(\frac{1}{n^5}\right), \tag{3.17}$$

where

$$q_0(\eta,n)=\frac{\eta}{n}+\frac{1}{2n^2}+\frac{\eta}{3n^3}+\frac{1}{4n^4};\tag{3.18}$$

$$\frac{1}{n-\eta}=q_1(\eta,n)+O\left(\frac{1}{n^5}\right),\tag{3.19}$$

where

$$q_1(\eta,n)=\frac{1}{n}+\frac{\eta}{n^2}+\frac{1}{n^3}+\frac{\eta}{n^4};\tag{3.20}$$

$$\frac{1}{(n-\eta)^2}=q_2(\eta,n)+O\left(\frac{1}{n^5}\right),\tag{3.21}$$

where

$$q_2(\eta,n)=\frac{1}{n^2}+\frac{2\eta}{n^3}+\frac{3}{n^4};\tag{3.22}$$

$$\frac{1}{(n-\eta)^3}=q_3(\eta,n)+O\left(\frac{1}{n^5}\right),\tag{3.23}$$

where

$$q_3(\eta,n)=\frac{1}{n^3}+\frac{3\eta}{n^4}.\tag{3.24}$$

Employing (3.18), (3.22), and (3.24), we have that

$$q_0(\eta,n)q_2(\eta,n)=\left(\frac{\eta}{n}+\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)\right)\left(\frac{1}{n^2}+\frac{2\eta}{n^3}+O\left(\frac{1}{n^4}\right)\right)=q_4(\eta,n)+O\left(\frac{1}{n^5}\right),\tag{3.25}$$

where

$$q_4(\eta,n)=\frac{\eta}{n^3}+\frac{5}{2n^4};\tag{3.26}$$

$$\left(q_0(\eta,n)\right)^2=\left(\frac{\eta}{n}+\frac{1}{2n^2}+\frac{\eta}{3n^3}+\frac{1}{4n^4}\right)^2=q_5(\eta,n)+O\left(\frac{1}{n^5}\right),\tag{3.27}$$

where

$$\begin{gathered}q_5(\eta,n)=\frac{1}{n^2}+\frac{\eta}{n^3}+\frac{11}{12n^4};\\ q_0(\eta,n)q_3(\eta,n)=\left(\frac{\eta}{n}+O\left(\frac{1}{n^2}\right)\right)\left(\frac{1}{n^3}+\frac{3\eta}{n^4}\right)=q_6(\eta,n)+O\left(\frac{1}{n^5}\right),\end{gathered}\tag{3.28}$$

where

$$q_6(\eta,n)=\frac{\eta}{n^4}.\tag{3.29}$$

By using (3.17), (3.21), (3.26), and (3.27), we have that

$$\begin{aligned}\frac{\ln(n-\eta)}{(n-\eta)^2} &= \left(\ln n - q_0(\eta,n) + O\left(\frac{1}{n^5}\right)\right)\left(q_2(\eta,n) + O\left(\frac{1}{n^5}\right)\right)\\ &= (\ln n)q_2(\eta,n) - q_0(\eta,n)q_2(\eta,n) + O\left(\frac{\ln n}{n^5}\right)\\ &= (\ln n)q_2(\eta,n) - q_4(\eta,n) + O\left(\frac{\ln n}{n^5}\right),\end{aligned} \tag{3.30}$$

$$\begin{aligned}\left(\ln(n-\eta)\right)^2 &= \left(\ln n - q_0(\eta,n) + O\left(\frac{1}{n^5}\right)\right)^2\\ &= (\ln n)^2 - 2(\ln n)q_0(\eta,n) + \left(q_0(\eta,n)\right)^2 + O\left(\frac{\ln n}{n^5}\right)\\ &= (\ln n)^2 - 2(\ln n)q_0(\eta,n) + q_5(\eta,n) + O\left(\frac{\ln n}{n^5}\right).\end{aligned} \tag{3.31}$$

Employing (3.31), (3.23), (3.24), (3.28), (3.29), and (3.27), we have that

$$\begin{aligned}\frac{\left(\ln(n-\eta)\right)^2}{(n-\eta)^3} &= \left((\ln n)^2 - 2(\ln n)q_0(\eta,n) + O\left(\frac{1}{n^2}\right)\right)\left(q_3(\eta,n) + O\left(\frac{1}{n^5}\right)\right)\\ &= (\ln n)^2 q_3(\eta,n) - 2(\ln n)q_0(\eta,n)q_3(\eta,n) + O\left(\frac{(\ln n)^2}{n^5}\right)\\ &= (\ln n)^2 q_3(\eta,n) - 2(\ln n)q_6(\eta,n) + O\left(\frac{(\ln n)^2}{n^5}\right).\end{aligned} \tag{3.32}$$

Since we study φ with fixed p and increasing n, from (3.17), (3.30), (3.32), it follows that

$$\begin{aligned}\varphi_{n-\eta} = \frac{1}{n-\eta} + \frac{\ln(n-\eta)}{(n-\eta)^2} + p\frac{\left(\ln(n-\eta)\right)^2}{(n-\eta)^3} &= (\ln n)q_2(\eta,n) + q_7(\eta,n)\\ &+ p(\ln n)^2 q_3(\eta,n) - 2p(\ln n)q_6(\eta,n) + O\left(\frac{(\ln n)^2}{n^5}\right),\end{aligned} \tag{3.33}$$

where, from (3.20), (3.26),

$$\begin{aligned}q_7(\eta,n) = q_1(\eta,n) - q_4(\eta,n) &= \left(\frac{1}{n} + \frac{\eta}{n^2} + \frac{1}{n^3} + \frac{\eta}{n^4}\right) - \left(\frac{\eta}{n^3} + \frac{5}{2n^4}\right)\\ &= \frac{1}{n} + \frac{\eta}{n^2} + \frac{1-\eta}{n^3} + \frac{\eta - 5/2}{n^4}.\end{aligned} \tag{3.34}$$

Clearly, if $(\eta_1)^2 = (\eta_2)^2 = 1$, then

$$q_k(\eta_1,n)q_3(\eta_2,n) = O\left(\frac{1}{n^5}\right) \tag{3.35}$$

for $k = 2,3,5$,

$$q_k(\eta_1,n)q_6(\eta_2,n) = O\left(\frac{1}{n^5}\right) \tag{3.36}$$

for $k = 0,1,2,3,4,5,6,7$. Let

$$q_9(\eta,n) = q_2(\eta,n) - \frac{1}{n^2} = \frac{2\eta}{n^3} + \frac{3}{n^4}, \tag{3.37}$$

$$q_{10}(\eta,n) = q_7(\eta,n) - \frac{1}{n} = \frac{\eta}{n^2} + \frac{1-\eta}{n^3} + \frac{\eta - 5/2}{n^4}, \tag{3.38}$$

$$q_{11}(\eta,n) = q_3(\eta,n) - \frac{1}{n^3} = \frac{3\eta}{n^4}, \tag{3.39}$$

$$q_{12}(n) = q_2(\eta,n)q_2(-\eta,n) = \frac{1}{n^4} + O\left(\frac{1}{n^5}\right), \tag{3.40}$$

$$q_{13}(n) = \sum_{k=0}^{1} q_2((-1)^k,n)q_7(-(-1)^k,n) \tag{3.41}$$

$$q_{14}(n) = q_7(\eta,n)q_7(-\eta,n), \tag{3.42}$$

$$q_{15}(n) = \sum_{k=0}^{1} q_3((-1)^k,n)q_7(-(-1)^k,n). \tag{3.43}$$

From (3.22) and (3.34), we obtain

$$q_2(\eta,n)q_7(-\eta,n) = \left(\frac{1}{n^2} + \frac{2\eta}{n^3} + \frac{3}{n^4}\right)\left(\frac{1}{n} - \frac{\eta}{n^2} + O\left(\frac{1}{n^3}\right)\right) = \frac{1}{n^3} + \frac{\eta}{n^4} + O\left(\frac{1}{n^5}\right). \tag{3.44}$$

Therefore, from (3.41),

$$q_{13}(n) = \frac{2}{n^3} + O\left(\frac{1}{n^5}\right). \tag{3.45}$$

From (3.34), we obtain

$$q_{14}(n) = \left(\frac{1}{n} + \frac{1}{n^3} - \frac{5}{2n^4}\right)^2 - \left(\frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4}\right)^2 = \frac{1}{n^2} + \frac{1}{n^4} + O\left(\frac{1}{n^5}\right) \tag{3.46}$$

and by (3.24) and (3.34)

$$q_3(\eta,n)q_7(-\eta,n) = \left(\frac{1}{n^3} + \frac{3\eta}{n^4}\right)\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = \frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{3.47}$$

Therefore, from (3.43)

$$q_{15}(n) = \frac{2}{n^4} + O\left(\frac{1}{n^5}\right). \tag{3.48}$$

In view of (3.11) and (3.33),

$$\varphi_{n-\eta} - \varphi_n = u(\eta,n) + pv(\eta,n) + O\left(\frac{(\ln n)^2}{n^5}\right), \tag{3.49}$$

where, (see (3.37)–(3.39)),

$$\begin{aligned} u(\eta,n) &= (\ln n)q_9(\eta,n) + q_{10}(\eta,n), \\ v(\eta,n) &= (\ln n)^2 q_{11}(\eta,n) - 2(\ln n)q_6(\eta,n). \end{aligned} \tag{3.50}$$

In view of (3.33), (3.34), (3.29), (3.22), (3.24), (3.35), (3.36), and (3.40)–(3.46),

$$\varphi_{n-1}\varphi_{n+1} = A(n) + q_{15}(n)(\ln n)^2 p + O\left(\frac{(\ln n)^2}{n^5}\right), \tag{3.51}$$

where

$$A(n) = (\ln n)^2 q_{12}(n) + (\ln n)q_{13}(n) + q_{14}(n). \tag{3.52}$$

Since

$$\begin{gathered} q_{10}(1,n) - q_{14}(n) = -\frac{5/2}{n^4} + O\left(\frac{1}{n^5}\right), \qquad q_9(1,n) - q_{13}(n) = \frac{3}{n^4} + O\left(\frac{1}{n^5}\right), \\ q_{11}(1,n) - q_{15}(n) = \frac{1}{n^4} + O\left(\frac{1}{n^5}\right), \end{gathered} \tag{3.53}$$

it follows from (3.14), (3.49), (3.51), (3.40), (3.29) that

$$\begin{aligned} H(\varphi_{n-1},\varphi_n,\varphi_{n+1}) &= (p-1)\frac{(\ln n)^2}{n^4} + (3-2p)\frac{\ln n}{n^4} - \frac{5}{2n^4} + O\left(\frac{(\ln n)^2}{n^5}\right) \\ &= (p-1)\frac{(\ln n)^2}{n^4} + O\left(\frac{\ln n}{n^4}\right). \end{aligned} \tag{3.54}$$

From (3.54) and the definition of H, (3.16) follows.

With the notation

$$\begin{aligned} y_n &= \frac{1}{n} + \frac{\ln n}{n^2} + p_1\frac{\ln^2 n}{n^3}, \\ z_n &= \frac{1}{n} + \frac{\ln n}{n^2} + p_2\frac{\ln^2 n}{n^3}, \end{aligned} \tag{3.55}$$

where $p_1 < 1 < p_2$, we obtain

$$G(y_{n-1},y_n,y_{n+1}) \sim (p_1-1)\frac{\ln^2 n}{n^4} < 0, \qquad G(z_{n-1},z_n,z_{n+1}) \sim (p_2-1)\frac{\ln^2 n}{n^4} > 0. \tag{3.56}$$

These relations show that the inequalities (3.3) are satisfied for sufficiently large n, say $n \ge n_0$, with f defined by (3.15) and G is given by (3.13). In view of Theorem 3.2 (with

$k = 1$), it follows that there is a solution of (3.6) with the asymptotics

$$x_n = \frac{1}{n} + \frac{\ln n}{n^2} + O\left(\frac{(\ln n)^2}{n^3}\right), \tag{3.57}$$

from which the result follows. □

As a consequence of Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *There is a decreasing positive solution of (3.6).*

4. A remark on the case $\alpha = 0$

As we have already mentioned if $\alpha = 0$, then all well-defined solutions of (1.1) are

$$x_{-1}, x_0, -\frac{x_0}{x_{-1}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, -\frac{x_{-1}}{x_0}, \ldots, \tag{4.1}$$

and they are periodic with period six.

Using the change $x_n = -y_n$, (1.1) is transformed into (1.2) with $\beta = 0$ for which it is well-known that all well-defined solutions are periodic with period six. It is interesting that the general real solution of (1.1) can be written in the following form:

$$x_n = e^{i\pi(F_n+1)} y_n, \tag{4.2}$$

where $y_n = |x_n|$ is a positive solution of (1.2) with $\beta = 0$ and

$$F_n = \lambda\chi(n) + \mu\chi(n+1), \tag{4.3}$$

where

$$\lambda^2 = \lambda, \qquad \mu^2 = \mu, \qquad \chi(3k) = 0, \qquad \chi(3k+1) = \chi(3k+2) = 1, \tag{4.4}$$

for $k \in \mathbb{Z}$ (i.e F_n is an nth element of a Fibonacci sequence mod 2).

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Leonid Gutnik: Moscow State Institute of Electronics and Mathematics, Moscow, Russia
Email address: gutnik@gutnik.mccme.ru

Stevo Stević: Mathematical Institute of the Serbian Academy of Science, Knez Mihailova 35/I, 11000 Beograd, Serbia
Email addresses: sstevic@ptt.yu; sstevo@matf.bg.ac.yu