

Research Article

Minimal Nonnegative Solution of Nonlinear Impulsive Differential Equations on Infinite Interval

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The cone theory and monotone iterative technique are used to investigate the minimal nonnegative solution of nonlocal boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times. All the existing results obtained in previous papers on nonlocal boundary value problems are under the case of the boundary conditions with no impulsive effects or the boundary conditions with impulsive effects on a finite interval with a finite number of impulsive times, so our work is new. Meanwhile, an example is worked out to demonstrate the main results.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. The theory of impulsive differential equations has become an important area of investigation in the recent years and is much richer than the corresponding theory of differential equations. For an introduction of the basic theory of impulsive differential equations in R^n ; see Lakshmikantham et al. [1], Bainov and Simeonov [2], and Samoilenko and Perestyuk [3] and the references therein.

Usually, we only consider the differential equation, integrodifferential equation, functional differential equations, or dynamic equations on time scales on a finite interval with a finite number of impulsive times. To identify a few, we refer the reader to [4–13] and references therein. In particular, we would like to mention some results of Guo and Liu

[5] and Guo [6]. In [5], by using fixed-point index theory for cone mappings, Guo and Liu investigated the existence of multiple positive solutions of a boundary value problem for the following second-order impulsive differential equation:

$$\begin{aligned} -x''(t) &= f(t, x(t)) \quad t \in J, t \neq t_k, k = 1, 2, \dots, m, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ ax(0) - bx'(0) &= \theta, \quad cx(1) + dx'(1) = \theta, \end{aligned} \quad (1.1)$$

where $f \in C(J \times P, P)$, $J = [0, 1]$, P is a cone in the real Banach space E , θ denotes the zero element of E , $f(t, \theta) = \theta$ for $t \in J$, $I_k(\theta) = \theta$, $k = 1, 2, \dots, m$, $0 < t_1 < t_2 < \dots < t_k < \dots < t_m < 1$, $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ and $\delta = ac + ad + bc > 0$.

In [6], by using fixed-point theory, Guo established the existence of solutions of a boundary value problem for the following second-order impulsive differential equation in a Banach space E :

$$\begin{aligned} -x''(t) &= f(t, x, x'(t)) \quad t \in J, t \neq t_k, k = 1, 2, \dots, m, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta x'|_{t=t_k} &= N_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ ax(0) - bx'(0) &= x_0, \quad cx(1) + dx'(1) = x_0^*, \end{aligned} \quad (1.2)$$

where $f \in C(J \times E \times E, E)$, $J = [0, 1]$, $I_k \in C[E, E]$, $N_k \in C[E \times E, E]$, $x_0, x_0^* \in E$, $0 < t_1 < t_2 < \dots < t_k < \dots < t_m < 1$, and $p = ac + ad + bc \neq 0$.

On the other hand, the readers can also find some recent developments and applications of the case that impulse effects on a finite interval with a finite number of impulsive times to a variety of problems from Nieto and Rodríguez-López [14–16], Jankowski [17–19], Lin and Jiang [20], Ma and Sun [21], He and Yu [22], Feng and Xie [23], Yan [24], Benchohra et al. [25], and Benchohra et al. [26].

Recently, in [27], Li and Nieto obtained some new results of the case that impulse effects on an infinite interval with a finite number of impulsive times. By using a fixed-point theorem due to Avery and Peterson [28], Li and Nieto considered the existence of multiple positive solutions of the following impulsive boundary value problem on an infinite interval:

$$\begin{aligned} u''(t) + q(t)f(t, u) &= 0, \quad \forall 0 < t < \infty, t \neq t_k, k = 1, 2, \dots, p \\ \Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \dots, p, \\ u(0) &= \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0, \end{aligned} \quad (1.3)$$

where $f \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$, $I_k \in C([0, +\infty), [0, +\infty))$, $u'(\infty) = \lim_{t \rightarrow +\infty} u'(t)$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < \infty$, $0 < t_1 < t_2 < \dots < t_p < +\infty$, and $q \in C([0, +\infty), [0, +\infty))$.

At the same time, we also notice that there has been increasing interest in studying nonlinear differential equation and impulsive integrodifferential equation on an infinite interval with an infinite number of impulsive times; to identify a few, we refer the reader to Guo and Liu [29], Guo [30–32], and Li and Shen [33]. It is here worth mentioning the works by Guo [31]. In [31], Guo investigated the minimal nonnegative solution of the following initial value problem for a second order nonlinear impulsive integrodifferential equation of Volterra type on an infinite interval with an infinite number of impulsive times in a Banach space E :

$$\begin{aligned}x'' &= f(t, x, Tx), \quad \forall t \geq 0, t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \\ \Delta x'|_{t=t_k} &= N_k(x(t_k)) \quad (k = 1, 2, \dots), \\ x(0) &= x_0, \quad x'(0) = x_0^*,\end{aligned}\tag{1.4}$$

where $f \in C(J \times P \times P, E)$, $I_k, N_k \in C[P, P]$, $J = [0, \infty)$, $x_0, x_0^* \in P$, $0 < t_1 < \dots < t_k < \dots < \dots$, $t_k \rightarrow \infty$, as $k \rightarrow \infty$, P is a cone of E .

However, the corresponding theory for nonlocal boundary value problems for impulsive differential equations on an infinite interval with an infinite number of impulsive times is not investigated till now. Now, in this paper, we will use the cone theory and monotone iterative technique to investigate the existence of minimal nonnegative solution for a class of second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times.

Consider the following boundary value problem for second-order nonlinear impulsive differential equation:

$$\begin{aligned}-x''(t) &= f(t, x(t), x'(t)) \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, \dots, \\ \Delta x'|_{t=t_k} &= \bar{I}_k(x(t_k)), \quad k = 1, 2, \dots, \\ x(0) &= \int_0^\infty g(t)x(t)dt, \quad x'(\infty) = 0,\end{aligned}\tag{1.5}$$

where $J = [0, \infty)$, $f \in C(J \times R^+ \times R^+, R^+)$, $R^+ = [0, +\infty)$, $0 < t_1 < t_2 < \dots < t_k < \dots$, $t_k \rightarrow \infty$, $I_k \in C[R^+, R^+]$, $\bar{I}_k \in C[R^+, R^+]$, $g(t) \in C(R^+, R^+)$, with $\int_0^\infty g(t)dt < 1$. $x'(\infty) = \lim_{t \rightarrow \infty} x'(t)$. $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, that is,

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-),\tag{1.6}$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t = t_k$, respectively. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

Let

$$\begin{aligned}
 PC[J, R] &= \{x : x \text{ is a map from } J \text{ into } R \text{ such that } x(t) \text{ is continuous at } t \neq t_k, \\
 &\quad \text{left continuous at } t = t_k \text{ and } x(t_k^+) \text{ exist for } k = 1, 2, \dots\}, \\
 PC^1[J, R] &= \{x \in PC[J, R] : x'(t) \text{ exists and is continuous at } t \neq t_k, \\
 &\quad \text{left continuous at } t = t_k \text{ and } x'(t_k^+) \text{ exist for } k = 1, 2, \dots\}.
 \end{aligned}
 \tag{1.7}$$

Let $E = \{x \in PC^1[J, R] : \sup_{t \in J} (|x(t)|/(1+t)) < \infty, \sup_{t \in J} |x'(t)| < \infty\}$ with the norm $\|x\| = \max\{\|x\|_1, \|x'\|_\infty\}$, where

$$\|x\|_1 = \sup_{t \in J} \frac{|x(t)|}{1+t}, \quad \|x'\|_\infty = \sup_{t \in J} |x'(t)|.
 \tag{1.8}$$

Define a cone $P \subset E$ by

$$P = \{x \in E : x(t) \geq 0, x'(t) \geq 0\}.
 \tag{1.9}$$

Let $J' = J \setminus \{t_1, t_2, \dots, t_k, \dots\}$, $J_0 = [0, t_1]$, and $J_i = (t_i, t_{i+1}]$ ($i = 1, 2, 3, \dots$). $x \in E \cap C^2[J', R]$ is called a nonnegative solution of (1.5), if $x(t) \geq 0, x'(t) \geq 0$ and $x(t)$ satisfies (1.5).

If $I_k = 0, \bar{I}_k = 0, k = 1, 2, \dots, g(t) = 0$, then boundary value problem (1.5) reduces to the following two point boundary value problem:

$$\begin{aligned}
 -x''(t) &= f(t, x(t), x'(t)) \quad t \in J, \\
 x(0) &= 0, \quad x'(\infty) = 0,
 \end{aligned}
 \tag{1.10}$$

which has been intensively studied; see Ma [34], Agarwal and O'Regan [35], Constantin [36], Liu [37, 38], and Yan and Liu [39] for some references along this line.

The organization of this paper is as follows. In Section 2, we provide some necessary background. In Section 3, the main result of problem (1.5) will be stated and proved. In Section 4, we give an example to illustrate how the main results can be used in practice.

2. Preliminaries

To establish the existence of minimal nonnegative solution in E of problem (1.5), let us list the following assumptions, which will stand throughout this paper.

(H₁) Suppose that $f \in C[J \times R^+ \times R^+, R^+]$, $I_k \in C[R^+, R^+]$, $\bar{I}_k \in C[R^+, R^+]$, and there exist $p, q, r \in C(J, R^+)$ and nonnegative constants c_k, d_k, e_k, f_k such that

$$\begin{aligned} f(t, u, v) &\leq p(t)u + q(t)v + r(t), \quad \forall t \in J, \text{ and } \forall u, v \in R^+, \\ I_k(u) &\leq c_k u + d_k, \quad \forall u \in R^+ \quad (k = 1, 2, 3, \dots), \\ \bar{I}_k(u) &\leq e_k u + f_k, \quad \forall u \in R^+ \quad (k = 1, 2, 3, \dots), \\ p^* &= \int_0^\infty p(t)(t+1)dt < \infty, \quad q^* = \int_0^\infty q(t)dt < \infty, \\ r^* &= \int_0^\infty r(t)dt < \infty, \quad c^* = \sum_{k=1}^\infty (t_k + 1)c_k < \infty, \\ d^* &= \sum_{k=1}^\infty d_k < \infty, \quad e^* = \sum_{k=1}^\infty (t_k + 1)e_k < \infty, \quad f^* = \sum_{k=1}^\infty f_k < \infty. \end{aligned} \quad (2.1)$$

(H₂) $f(t, u_1, v_1) \leq f(t, u_2, v_2)$, $I_k(u_1) \leq I_k(u_2)$, $\bar{I}_k(u_1) \leq \bar{I}_k(u_2)$, for $t \in J, u_1 \leq u_2, v_1 \leq v_2$ ($k = 1, 2, 3, \dots$).

Lemma 2.1. Suppose that (H₁) holds. Then for all $x \in P$, $\int_0^\infty f(t, x(t), x'(t))dt$, $\sum_{k=1}^\infty I_k(x(t_k))$, and $\sum_{k=1}^\infty \bar{I}_k(x(t_k))$ are convergent.

Proof. By (H₁), we have

$$\begin{aligned} f(t, x(t), x'(t)) &\leq p(t)(t+1)\frac{x(t)}{t+1} + q(t)x'(t) + r(t), \\ I_k(x(t_k)) &\leq c_k(t_k+1)\frac{x(t_k)}{t_k+1} + d_k, \\ \bar{I}_k(x(t_k)) &\leq e_k(t_k+1)\frac{x(t_k)}{t_k+1} + f_k. \end{aligned} \quad (2.2)$$

Thus,

$$\begin{aligned} \int_0^\infty f(s, x(s), x'(s))ds &\leq p^*\|x\|_1 + q^*\|x'\|_\infty + r^* < \infty, \\ \sum_{k=1}^\infty I_k(x(t_k)) &\leq c^*\|x\|_1 + d^* < \infty, \\ \sum_{k=1}^\infty \bar{I}_k(x(t_k)) &\leq e^*\|x\|_1 + f^* < \infty. \end{aligned} \quad (2.3)$$

The proof is complete. □

Lemma 2.2. *Suppose that (H_1) holds. If $0 \leq \int_0^\infty g(t)dt < 1$, then $x \in E \cap C^2[J', R]$ is a solution of problem (1.5) if and only if $x \in E$ is a solution of the following impulsive integral equation:*

$$\begin{aligned} x(t) = & \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)) \\ & + \frac{1}{1 - \int_0^\infty g(t)dt} \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) \right. \\ & \left. + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)) \right] dt, \quad \forall t \in J, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} G(t,s) &= \begin{cases} t, & 0 \leq t \leq s < +\infty, \\ s, & 0 \leq s \leq t < +\infty, \end{cases} \\ G'_s(t,s) &= \begin{cases} 0, & 0 \leq t \leq s < +\infty, \\ 1, & 0 \leq s \leq t < +\infty. \end{cases} \end{aligned} \quad (2.5)$$

Proof. First, suppose that $x \in E \cap C^2[J', R]$ is a solution of problem (1.5). It is easy to see by integration of (1.5) that

$$-x'(t) + x'(0) = \int_0^t f(s,x(s),x'(s))ds + \sum_{t_k < t} \bar{I}_k(x(t_k)). \quad (2.6)$$

Taking limit for $t \rightarrow \infty$, by Lemma 2.1 and the boundary conditions, we have

$$x'(0) = \int_0^\infty f(s,x(s),x'(s))ds + \sum_{k=1}^\infty \bar{I}_k(x(t_k)). \quad (2.7)$$

Thus,

$$x'(t) = \int_0^\infty f(s,x(s),x'(s))ds + \sum_{k=1}^\infty \bar{I}_k(x(t_k)) - \int_0^t f(s,x(s),x'(s))ds - \sum_{t_k < t} \bar{I}_k(x(t_k)). \quad (2.8)$$

Integrating (2.8), we can get

$$\begin{aligned} x(t) &= x(0) + \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)) \\ &= \int_0^\infty g(t)x(t)dt + \int_0^\infty G(t,s)f(s,x(s),x'(s))ds \\ &\quad + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)). \end{aligned} \quad (2.9)$$

It follows that

$$\begin{aligned} \int_0^\infty g(t)dt &= \frac{1}{1 - \int_0^\infty g(t)x(t)dt} \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \\ &\quad \left. + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)) \right] dt. \end{aligned} \quad (2.10)$$

So we have (2.4).

Conversely, suppose that $x \in E$ is a solution of (2.4). Evidently,

$$\Delta x|_{t=t_k} = I_k(x(t_k)), \quad (k = 1, 2, \dots). \quad (2.11)$$

Direct differentiation of (2.4) implies, for $t \neq t_k$,

$$\begin{aligned} x'(t) &= \int_t^\infty f(s,x(s),x'(s))ds + \sum_{t_k \geq t} \bar{I}_k(x(t_k)), \\ \Delta x'|_{t=t_k} &= \bar{I}_k(x(t_k)), \quad (k = 1, 2, \dots), \\ x''(t) &= -f(t,x(t),x'(t)). \end{aligned} \quad (2.12)$$

So $x \in C^2[J', R]$. It is easy to verify that $x(0) = \int_0^\infty g(t)x(t)dt$, $x'(\infty) = 0$. The proof of Lemma 2.2 is complete. \square

Define an operator $T : E \rightarrow E$,

$$\begin{aligned} (Tx)(t) &= \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)) \\ &\quad + \frac{1}{1 - \int_0^\infty g(t)dt} \int_0^\infty g(t) \left[\int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)\bar{I}_k(x(t_k)) \right. \\ &\quad \left. + \sum_{k=1}^\infty G'_s(t,t_k)I_k(x(t_k)) \right] dt, \quad \forall t \in J. \end{aligned} \quad (2.13)$$

Lemma 2.3. Assume that (H_1) and (H_2) hold. Then operator T maps P into P , and

$$\|Tx\| \leq \beta + \alpha\|x\|, \quad \forall x \in P, \quad (2.14)$$

where

$$\alpha = \frac{2 - \int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} (p^* + q^* + c^* + e^*), \quad \beta = \frac{2 - \int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} (r^* + f^* + d^*). \quad (2.15)$$

Moreover, for $x, y \in P$ with $x(t) \leq y(t), x'(t) \leq y'(t)$, for all $t \in J'$, one has

$$(Tx)(t) \leq (Ty)(t), \quad (Tx)'(t) \leq (Ty)'(t), \quad \forall t \in J'. \quad (2.16)$$

Proof. Let $x \in P$. From the definition of T and (H_1) , we can obtain that T is an operator from P into P , and

$$\begin{aligned} & \frac{|(Tx)(t)|}{1+t} \\ & \leq \int_0^\infty |f(s, x(s), x'(s))| ds + \sum_{k=1}^\infty |\bar{I}_k(x(t_k))| \\ & \quad + \sum_{k=1}^\infty |I_k(x(t_k))| + \frac{1}{1 - \int_0^\infty g(t)dt} \left(\int_0^\infty |f(s, x(s), x'(s))| ds + \sum_{k=1}^\infty |\bar{I}_k(x(t_k))| + \sum_{k=1}^\infty |I_k(x(t_k))| \right) \\ & \leq \frac{2 - \int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} (p^* + q^* + c^* + e^*) \|x\| + \frac{2 - \int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} (r^* + f^* + d^*) \\ & = \alpha \|x\| + \beta, \quad \forall t \in J. \end{aligned} \quad (2.17)$$

Direct differentiation of (2.13) implies, for $t \neq t_k$,

$$(Tx)'(t) = \int_t^\infty f(s, x(s), x'(s)) ds + \sum_{t_k \geq t} \bar{I}_k(x(t_k)). \quad (2.18)$$

Thus we have $|(Tx)'(t)| \leq \int_0^\infty |f(s, x(s), x'(s))| ds + \sum_{k=1}^\infty |\bar{I}_k(x(t_k))| \leq \alpha \|x\| + \beta$, for all $t \in J'$. It follows that (2.14) is satisfied. Equation (2.16) is easily obtained by (H_2) . \square

3. Main Result

In this section, we establish the existence of a minimal nonnegative solution for problem (1.5).

Theorem 3.1. *Let conditions (H_1) - (H_2) be satisfied. Suppose further that*

$$\alpha = \frac{2 - \int_0^\infty g(t) dt}{1 - \int_0^\infty g(t) dt} (p^* + q^* + c^* + e^*) < 1. \quad (3.1)$$

Then problem (1.5) has the minimal nonnegative solution \bar{x} with $\|\bar{x}\| \leq \beta / (1 - \alpha)$, where β is defined as in Lemma 2.3. Here, the meaning of minimal nonnegative solution is that if x is an arbitrary nonnegative solution of (1.5), then $x(t) \geq \bar{x}(t)$, $x'(t) \geq \bar{x}'(t)$, for all $t \in J'$. Moreover, if we let $x_0(t) = 0$, $x_n(t) = (Tx_{n-1})(t)$, for all $t \in J$ ($n = 1, 2, \dots$), then $x_n \in P$ with

$$\begin{aligned} 0 = x_0(t) &\leq x_1(t) \leq \dots \leq x_n(t) \leq \dots \leq \bar{x}(t), \quad \forall t \in J, \\ 0 = x'_0(t) &\leq x'_1(t) \leq \dots \leq x'_n(t) \leq \dots \leq \bar{x}'(t), \quad \forall t \in J', \end{aligned} \quad (3.2)$$

and $\{x_n(t)\}$ and $\{x'_n(t)\}$ converge uniformly to $\bar{x}(t)$ and $\bar{x}'(t)$ on J_i , ($i = 0, 1, 2, \dots$), respectively.

Proof. By Lemma 2.3 and the definition of operator T , we have $x_n \in P$, and

$$\|x_n\| \leq \beta + \alpha \|x_{n-1}\| \quad (n = 1, 2, 3, \dots), \quad (3.3)$$

$$0 = x_0(t) \leq x_1(t) \leq \dots \leq x_n(t) \leq \dots, \quad \forall t \in J, \quad (3.4)$$

$$0 = x'_0(t) \leq x'_1(t) \leq \dots \leq x'_n(t) \leq \dots, \quad \forall t \in J'. \quad (3.5)$$

By (3.3), we have

$$\|x_n\| \leq \beta + \alpha\beta + \alpha^2\beta + \dots + \alpha^{n-1}\beta = \frac{\beta(1 - \alpha^n)}{1 - \alpha} \leq \frac{\beta}{1 - \alpha}, \quad (n = 1, 2, \dots). \quad (3.6)$$

From (3.4), (3.5), and (3.6), we know that $\lim_{n \rightarrow \infty} x_n(t)$ and $\lim_{n \rightarrow \infty} x'_n(t)$ exist. Suppose that

$$\lim_{n \rightarrow \infty} x_n(t) = \bar{x}(t), \quad \lim_{n \rightarrow \infty} x'_n(t) = y(t), \quad \forall t \in J'. \quad (3.7)$$

By the definition of $x_n(t)$, we have

$$x'_n(t) = \int_t^\infty f(s, x_{n-1}(s), x'_{n-1}(s)) ds + \sum_{t_k \geq t} \bar{I}_k(x_n(t_k)), \quad \forall t \in J', \quad (n = 1, 2, \dots), \quad (3.8)$$

$$x''_n(t) = -f(t, x_{n-1}(t), x'_{n-1}(t)), \quad \forall t \in J', \quad (n = 1, 2, \dots). \quad (3.9)$$

From (3.6), we obtain

$$\frac{|x_n(t)|}{t+1} \leq \frac{\beta}{1-\alpha}, \quad |x'_n(t)| \leq \frac{\beta}{1-\alpha}, \quad \forall t \in J', \quad (n = 1, 2, \dots). \quad (3.10)$$

It follows that $\{x_n(t)\}$ is equicontinuous on every J_i ($i = 0, 1, 2, \dots$). Combining this with Ascoli-Arzela theorem and diagonal process, there exists a subsequence which converges uniformly to $\bar{x}(t)$ on J_i ($i = 0, 1, 2, \dots$). Which together with (3.4) imply that $\{x_n(t)\}$ converges uniformly to $\bar{x}(t)$ on J_i ($i = 0, 1, 2, \dots$), and $\bar{x} \in PC[J, R]$, $\|\bar{x}\|_1 \leq \beta/(1-\alpha)$. On the other hand, by (H_1) , (3.6), and (3.9), we have

$$\begin{aligned} |x''_n(t)| &\leq p(t)(t+1)\|x_{n-1}\|_1 + q(t)\|x'_{n-1}\|_\infty + r(t) \\ &\leq p(t)(t+1)\frac{\beta}{1-\alpha} + q(t)\frac{\beta}{1-\alpha} + r(t) = s(t) \in C(J, R^+), \quad \forall t \in J' \quad (n = 1, 2, \dots). \end{aligned} \quad (3.11)$$

Since $s(t)$ is bounded on $[0, M]$ (M is a finite positive number), $\{x'_n(t)\}$ is equicontinuous on every J_i , $i = 1, 2, \dots$. Combining this with Ascoli-Arzela theorem and diagonal process, there exists a subsequence which converges uniformly to $y(t)$ on J_i ($i = 0, 1, 2, \dots$), which together with (3.5) imply that $\{x'_n(t)\}$ converges uniformly to $y(t)$ on J_i ($i = 0, 1, 2, \dots$), and $y \in PC[J, R]$, $\|y\|_\infty \leq \beta/(1-\alpha)$. From above, we know that $x'(t)$ exists and $x'(t) = y(t)$, for all $t \in J'$. It follows that $\bar{x} \in P$ and

$$\|\bar{x}\| \leq \frac{\beta}{1-\alpha}. \quad (3.12)$$

Now we prove that $\bar{x}(t) = (T\bar{x})(t)$.

By the continuity of f and the uniform convergence of $x_n(t)$, $x'_n(t)$, we know that

$$f(s, x_n(s), x'_n(s)) \longrightarrow f(s, \bar{x}(s), \bar{x}'(s)), \quad n \longrightarrow \infty, \quad \forall t \in J'. \quad (3.13)$$

On the other hand, by (H_1) and (3.6) and (3.12), we have

$$\begin{aligned} &\left| f(s, x_n(s), x'_n(s)) - f(s, \bar{x}(s), \bar{x}'(s)) \right| \\ &\leq 2(p(s)(s+1) + q(s))\frac{\beta}{1-\alpha} + 2r(s) = z(s) \in L[J, R^+] \quad (n = 1, 2, \dots). \end{aligned} \quad (3.14)$$

Combining this with the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_t^\infty f(s, x_n(s), x'_n(s)) ds &= \int_t^\infty f(s, \bar{x}(s), \bar{x}'(s)) ds, \quad \forall t \in J, \\ \lim_{n \rightarrow \infty} \int_0^t f(s, x_n(s), x'_n(s)) ds &= \int_0^t f(s, \bar{x}(s), \bar{x}'(s)) ds, \quad \forall t \in J.\end{aligned}\tag{3.15}$$

Moreover, we can see that

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{0 < t_k < t} I_k(x_n(t_k)) &= \sum_{0 < t_k < t} I_k(\bar{x}(t_k)), \\ \lim_{n \rightarrow \infty} \sum_{0 < t_k < t} \bar{I}_k(x_n(t_k)) &= \sum_{0 < t_k < t} \bar{I}_k(\bar{x}(t_k)), \\ \lim_{n \rightarrow \infty} \sum_{t_k \geq t} I_k(x_n(t_k)) &= \sum_{t_k \geq t} I_k(\bar{x}(t_k)), \\ \lim_{n \rightarrow \infty} \sum_{t_k \geq t} \bar{I}_k(x_n(t_k)) &= \sum_{t_k \geq t} \bar{I}_k(\bar{x}(t_k)).\end{aligned}\tag{3.16}$$

Now taking limits from two sides of $x_n(t) = (Tx_{n-1})(t)$ and using (3.15)–(3.16), we have

$$\begin{aligned}\bar{x}(t) &= \int_0^\infty G(t, s) f(s, \bar{x}(s), \bar{x}'(s)) ds + \sum_{k=1}^\infty G(t, t_k) \bar{I}_k(\bar{x}(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k) I_k(\bar{x}(t_k)) \\ &\quad + \frac{1}{1 - \int_0^\infty g(t) dt} \int_0^\infty g(t) \left[\int_0^\infty G(t, s) f(s, \bar{x}(s), \bar{x}'(s)) ds \right. \\ &\quad \left. + \sum_{k=1}^\infty G(t, t_k) \bar{I}_k(\bar{x}(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k) I_k(\bar{x}(t_k)) \right] dt, \quad \forall t \in J.\end{aligned}\tag{3.17}$$

By Lemma 2.2, $\bar{x}(t)$ is a nonnegative solution of (1.5).

Suppose that $x \in P \cap C^2[J', R]$ is an arbitrary nonnegative solution of (1.5). Then $x(t) = (Tx)(t)$. It is clear that $x(t) \geq 0$, $x'(t) \geq 0$, $\forall t \in J'$. Suppose that $x(t) \geq x_{n-1}(t)$, $x'(t) \geq x'_{n-1}(t)$, for all $t \in J'$. By (2.16) we have $(Tx)(t) \geq (Tx_{n-1})(t)$, $(Tx)'(t) \geq (Tx_{n-1})'(t)$, for all $t \in J'$. This means that $x(t) \geq x_n(t)$, $x'(t) \geq x'_n(t)$, for all $t \in J'$ ($n = 1, 2, \dots$). Taking limit, we have $x(t) \geq \bar{x}(t)$, $x'(t) \geq \bar{x}'(t)$, for all $t \in J'$. The proof of Theorem 3.1 is complete. \square

4. Example

To illustrate how our main results can be used in practice, we present an example.

Example 4.1. Consider the following boundary value problem of second-order impulsive differential equation on infinite interval

$$\begin{aligned}
 -x'' &= \frac{1}{100}(t+1)^{-9/4}(1+x+x')^{2/3}, \quad t \in J, t \neq k, \\
 \Delta x|_{t=k} &= \frac{1}{8^k}(x(k)+1)^{1/2} \quad (k=1,2,\dots), \\
 \Delta x|_{t=k} &= \frac{1}{10^k}(x(k)+1)^{1/2} \quad (k=1,2,\dots), \\
 x(0) &= \int_0^\infty \frac{1}{2}e^{-t}x(t)dt, \quad x'(\infty) = 0,
 \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
 f(t, x, y) &= \frac{1}{100}(t+1)^{-9/4}(1+x+y)^{2/3}, \\
 I_k(x(t_k)) &= \frac{1}{8^k}(x(k)+1)^{1/2}, \quad k=1,2,\dots, \\
 \bar{I}_k(x(t_k)) &= \frac{1}{10^k}(x(k)+1)^{1/2}, \quad k=1,2,\dots, \\
 g(t) &= \frac{1}{2}e^{-t}.
 \end{aligned} \tag{4.2}$$

Evidently, $x(t) \equiv 0$ is not the solution of (4.1).

Conclusion

Problem (4.1) has minimal positive solution.

Proof. It is clear that $\int_0^\infty (1/2)e^{-t}dt < 1$ and (H_2) is satisfied.

By the inequality $(1+x)^\gamma \leq 1+\gamma x$, for all $0 \leq x < \infty, 0 < \gamma < 1$, we see that

$$\begin{aligned}
 f(t, x, y) &\leq \frac{1}{100}(t+1)^{-9/4}\left(1+\frac{2}{3}x+\frac{2}{3}y\right), \\
 I_k(x) &\leq \frac{1}{8^k}\left(1+\frac{1}{2}x\right), \quad \bar{I}_k(x) \leq \frac{1}{10^k}\left(1+\frac{1}{2}x\right), \quad k=1,2,\dots
 \end{aligned} \tag{4.3}$$

Let

$$\begin{aligned} p(t) &= \frac{1}{150}(t+1)^{-9/4}, & q(t) &= \frac{1}{150}(t+1)^{-9/4}, & r(t) &= \frac{1}{100}(t+1)^{-9/4}, \\ c_k &= \frac{1}{2 \times 8^k}, & d_k &= \frac{1}{8^k}, & e_k &= \frac{1}{2 \times 10^k}, & f_k &= \frac{1}{10^k}. \end{aligned} \quad (4.4)$$

Then, we easily obtain that

$$\begin{aligned} \int_0^\infty p(t)(1+t)dt &= \frac{4}{150}, & \int_0^\infty q(t)dt &= \frac{4}{750}, & \int_0^\infty r(t)dt &= \frac{2}{250}, \\ \sum_{k=1}^\infty (t_k+1)c_k &= \frac{15}{98}, & \sum_{k=1}^\infty d_k &= \frac{1}{7}, & \sum_{k=1}^\infty (t_k+1)e_k &= \frac{19}{162}, & \sum_{k=1}^\infty f_k &= \frac{1}{9}. \end{aligned} \quad (4.5)$$

Thus, (H_1) is satisfied and $\alpha = 150001/165375 < 1$. By Theorem 3.1, it follows that problem (4.1) has a minimal positive solution. \square

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