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Research Article

Positive Solutions for Third-Order p-Laplacian Functional Dynamic Equations on Time Scales

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The authors study the boundary value problems for a p-Laplacian functional dynamic equation on a time scale, $[\phi_p(x^{\Delta\nabla}(t))]^{\nabla} + a(t)f(x(t), x(\mu(t))) = 0$, $t \in (0, T)$, $x_0(t) = \psi(t)$, $t \in [-r, 0]$, $x^{\Delta}(0) = x^{\Delta\nabla}(0) = 0$, $x(T) + B_0(x^{\Delta}(\eta)) = 0$. By using the twin fixed-point theorem, sufficient conditions are established for the existence of twin positive solutions.

1. Introduction

Let T be a closed nonempty subset of R, and let T have the subspace topology inherited from the Euclidean topology on R. In some of the current literature, T is called a time scale (or measure chain). For notation, we shall use the convention that, for each interval of J of R, J will denote time scales interval, that is, $J := J \cap \mathrm{T}$.

In this paper, let T be a time scale such that $-r$, 0, $T \in \mathrm{T}$. We are concerned with the existence of positive solutions of the p-Laplacian dynamic equation on a time scale

$$\begin{gathered}\left[\phi_p\left(x^{\Delta\nabla}(t)\right)\right]^{\nabla} + a(t)f(x(t), x(\mu(t))) = 0, \quad t \in (0, T), \\ x_0(t) = \psi(t), \quad t \in [-r, 0], \quad x^{\Delta}(0) = x^{\Delta\nabla}(0) = 0, \quad x(T) + B_0\left(x^{\Delta}(\eta)\right) = 0,\end{gathered} \tag{1.1}$$

where $\phi_p(u)$ is the p-Laplacian operator, that is, $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $(\phi_p)^{-1}(u) = \phi_q(u)$, where $1/p + 1/q = 1$; $\eta \in (0, \rho(T))$ and

- (H_1) the function $f : (\mathrm{R}^+)^2 \to \mathrm{R}^+$ is continuous,
- (H_2) the function $a : \mathrm{T} \to \mathrm{R}^+$ is left dense continuous (i.e., $a \in C_{\mathrm{ld}}(\mathrm{T}, \mathrm{R}^+)$) and does not vanish identically on any closed subinterval of $[0, T]$. Here, $C_{\mathrm{ld}}(\mathrm{T}, \mathrm{R}^+)$ denotes the set of all left dense continuous functions from T to R^+,

(H_3) $\varphi : [-r, 0] \to \mathbb{R}^+$ is continuous and $r > 0$,

(H_4) $\mu : [0, T] \to [-r, T]$ is continuous, $\mu(t) \le t$ for all t,

(H_5) $B_0 : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies that there are $\beta \ge \delta \ge 0$ such that

$$\delta s \le B_0(s) \le \beta s, \quad \text{for } s \in \mathbb{R}^+. \tag{1.2}$$

p-Laplacian problems with two-, three-, m-point boundary conditions for ordinary differential equations and finite difference equations have been studied extensively, for example see [1–4] and references therein. However, there are not many concerning the p-Laplacian problems on time scales, especially for p-Laplacian functional dynamic equations on time scales.

The motivations for the present work stems from many recent investigations in [5–8] and references therein. Especially, Kaufmann and Raffoul [8] considered a nonlinear functional dynamic equation on a time scale and obtained sufficient conditions for the existence of positive solutions. In this paper, we apply the twin fixed-point theorem to obtain at least two positive solutions of boundary value problem (BVP for short) (1.1) when growth conditions are imposed on f. Finally, we present two corollaries, which show that under the assumptions that f is superlinear or sublinear, BVP (1.1) has at least two positive solutions.

Given a nonnegative continuous functional γ on a cone P of a real Banach space E, we define for each $d > 0$ the sets

$$\begin{aligned} P(\gamma, d) &= \{x \in P : \gamma(x) < d\}, \\ \partial P(\gamma, d) &= \{x \in P : \gamma(x) = d\}, \\ \overline{P(\gamma, d)} &= \{x \in P : \gamma(x) \le d\}. \end{aligned} \tag{1.3}$$

The following twin fixed-point lemma due to [9] will play an important role in the proof of our results.

Lemma 1.1. *Let E be a real Banach space, P a cone of E, γ and α two nonnegative increasing continuous functionals, θ a nonnegative continuous functional, and $\theta(0) = 0$. Suppose that there are two positive numbers c and M such that*

$$\gamma(x) \le \theta(x) \le \alpha(x), \quad \|x\| \le M\gamma(x), \quad \text{for } x \in \overline{P(\gamma, c)}. \tag{1.4}$$

$F : \overline{P(\gamma, c)} \to P$ is completely continuous. There are positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \le \lambda\theta(x), \quad \forall \lambda \in [0, 1], \; x \in \partial P(\theta, b), \tag{1.5}$$

and

(i) *$\gamma(Fx) > c$ for $x \in \partial P(\gamma, c)$,*

(ii) *$\theta(Fx) < b$ for $x \in \partial P(\theta, b)$,*

(iii) *$\alpha(Fx) > a$ and $P(\alpha, a) \ne \emptyset$ for $x \in \partial P(\alpha, a)$.*

Then, F has at least two fixed points x_1 *and* $x_2 \in \overline{P(\gamma, c)}$ *satisfying*

$$a < \alpha(x_1), \qquad \theta(x_1) < b, \qquad b < \theta(x_2), \qquad \gamma(x_2) < c. \tag{1.6}$$

2. Positive Solutions

We note that $x(t)$ is a solution of (1.1) if and only if

$$x(t) = \begin{cases} \displaystyle\int_0^T (T-s)\phi_q\left(\int_0^s a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\ \displaystyle\quad -B_0\left(\int_0^\eta \phi_q\left(\int_0^s - a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s\right) \\ \displaystyle\quad +\int_0^t (t-s)\phi_q\left(\int_0^s - a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s, & t \in [0,T], \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{2.1}$$

Let $E = C_{\mathrm{ld}}^{\Delta}([0,T], \mathrm{R})$ be endowed with the norm $\|x\| = \max_{t\in[0,T]}|x(t)|$ and $P = \{x \in E : x$ is concave and nonnegative valued on $[0,T]$, and $x^{\Delta}(0) = 0\}$.

Clearly, E is a Banach space with the norm $\|x\|$ and P is a cone in E. For each $x \in E$, extend $x(t)$ to $[-r, T]$ with $x(t) = \psi(t)$ for $t \in [-r, 0]$.

Define $F : P \to E$ as

$$\begin{aligned} Fx(t) = {} & \int_0^T (T-s)\phi_q\left(\int_0^s a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\ & - B_0\left(\int_0^\eta \phi_q\left(\int_0^s -a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s\right) \\ & + \int_0^t (t-s)\phi_q\left(\int_0^s -a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s, \quad t \in [0,T]. \end{aligned} \tag{2.2}$$

We seek a fixed point, x_1, of F in the cone P. Define

$$x(t) = \begin{cases} x_1(t), & t \in [0,T], \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{2.3}$$

Then, $x(t)$ denotes a positive solution of BVP (1.1).

It follows from (2.2) that

Lemma 2.1. *Let F be defined by* (2.2). *If* $x \in P$, *then*

(i) $F(P) \subset P$.

(ii) $F : P \to P$ *is completely continuous.*

(iii) $x(t) \geq ((T-t)/T)\|x\|$, $t \in [0,T]$.

(iv) $x(t)$ *is decreasing on* $[0,T]$.

The proof is similar to the proofs of Lemma 2.3 and Theorem 3.1 in [7], and is omitted.

Fix $l \in \mathrm{T}$ such that $0 < l < \eta < T$, and set

$$Y_1 := \{t \in [0,T] : \mu(t) < 0\}, \qquad Y_2 := \{t \in [0,T] : \mu(t) \geq 0\}, \qquad Y_3 := Y_1 \cap [0,l]. \tag{2.4}$$

Throughout this paper, we assume $Y_3 \neq \emptyset$ and $\int_{Y_3} \phi_q(\int_0^s a(r)\nabla r)\nabla s > 0$.

Now, we define the nonnegative, increasing, continuous functionals γ, θ, and α on P by

$$\begin{aligned} \gamma(x) &= \max_{t\in[l,\eta]} x(t) = x(l), \\ \theta(x) &= \min_{t\in[0,l]} x(t) = x(l), \\ \alpha(x) &= \max_{t\in[\eta,T]} x(t) = x(\eta). \end{aligned} \tag{2.5}$$

We have

$$\begin{gathered} \gamma(x) = \theta(x) \leq \alpha(x), \quad x \in P, \\ \theta(x) = \gamma(x) = x(l) \geq \frac{T-l}{T}\|x\|, \quad \alpha(x) = x(\eta) \geq \frac{T-\eta}{T}\|x\|, \quad \text{for each } x \in P. \end{gathered} \tag{2.6}$$

Then,

$$\|x\| \leq \frac{T}{T-l}\gamma(x), \quad \|x\| \leq \frac{T}{T-\eta}\alpha(x), \quad \text{for each } x \in P. \tag{2.7}$$

We also see that

$$\theta(\lambda x) = \lambda\theta(x), \quad \forall \lambda \in [0,1], \; x \in \partial P(\theta, b). \tag{2.8}$$

For the notational convenience, we denote σ_1, σ_2 and ρ_1, ρ_2 by

$$\sigma = \beta \int_{Y_3} \phi_q\left(\int_0^s a(r)\nabla r\right)\nabla s; \qquad \rho = T(2T+\delta)\phi_q\left(\int_0^T a(r)\nabla r\right). \tag{2.9}$$

Theorem 2.2. *Suppose that there are positive numbers $a < b < c$ such that*

$$0 < a < \frac{\sigma}{\rho} b < \frac{(T-l)\sigma}{T\rho} c. \tag{2.10}$$

Assume f satisfies the following conditions:

(A) $f(x,\psi(s)) > \phi_p(c/\sigma)$ *for* $c \le x \le (T/(T-l))c$, *uniformly in* $s \in [-r,0]$,

(B) $f(x,\psi(s)) < \phi_p(b/\rho)$ *for* $0 \le x \le (T/(T-l))b$, *uniformly in* $s \in [-r,0]$,

$$f(x_1,x_2) < \phi_p\left(\frac{b}{\rho}\right), \quad \text{for } 0 \le x_i \le \frac{T}{T-l}b,\ i=1,2, \tag{2.11}$$

(C) $f(x,\psi(s)) > \phi_p(a/\sigma)$ *for* $a \le x \le (T/(T-\eta))a$, *uniformly in* $s \in [-r,0]$.

Then, BVP (1.1) *has at least two positive solutions of the form*

$$x(t) = \begin{cases} \psi(t), & t \in [-r,0], \\ x_i(t), & t \in [0,T],\ i=1,2, \end{cases} \tag{2.12}$$

where $a < \max_{t\in[\eta,T]} x_1(t)$, $\min_{t\in[0,l]} x_1(t) < b$ *and* $b < \min_{t\in[0,l]} x_2(t)$, $\max_{t\in[l,\eta]} x_2(t) < c$.

Proof. By the definition of operator F and its properties, it suffices to show that the conditions of Lemma 1.1 hold with respect to F.

First, we verify that $x \in \partial P(\gamma,c)$ implies $\gamma(Fx) > c$.

Since $\gamma(x) = x(l) = c$, one gets $x(t) \ge c$ for $t \in [0,l]$. Recalling that (2.7), we know $c \le x \le (T/(T-l))c$ for $t \in [0,l]$. Then, we get

$$\begin{aligned}
\gamma(Fx) &= \int_0^T (T-s)\phi_q\left(\int_0^s a(r)f(x(r),x(\mu(r)))\nabla r\right)\nabla s \\
&\quad - B_0\left(\int_0^\eta \phi_q\left(\int_0^s -a(r)f(x(r),x(\mu(r)))\nabla r\right)\nabla s\right) \\
&\quad + \int_0^l (l-s)\phi_q\left(\int_0^s -a(r)f(x(r),x(\mu(r)))\nabla r\right)\nabla s \\
&\ge -B_0\left(\int_0^\eta \phi_q\left(\int_0^s -a(r)f(x(r),x(\mu(r)))\nabla r\right)\nabla s\right) \\
&\ge \beta\int_0^l \phi_q\left(\int_0^s a(r)f(x(r),x(\mu(r)))\nabla r\right)\nabla s \\
&\ge \beta\int_{Y_3} \phi_q\left(\int_0^s a(r)f(x(r),\psi(\mu(r)))\nabla r\right)\nabla s \\
&> \beta\int_{Y_3} \phi_q\left(\int_0^s a(r)\nabla r\right)\nabla s\frac{c}{\sigma} = c.
\end{aligned} \tag{2.13}$$

Secondly, we prove that $x \in \partial P(\theta,b)$ implies $\theta(Fx) < b$.

Since $\theta(x) = b$ implies $x(l) = b$, it holds that $b \le x(t) \le \|x\| \le (T/(T-l))\theta(x) = (T/(T-l))b$ for $t \in [0,l]$, and for all $x \in \partial P(\theta,b)$ implies

$$0 \le x(t) \le b, \quad \text{for } t \in [l,T]. \tag{2.14}$$

Then,

$$0 \le x(t) \le \frac{T}{T-l} b, \quad t \in [0,T]. \tag{2.15}$$

So, we have

$$\begin{aligned}
\theta(Fx) &= \int_0^T (T-s)\phi_q\left(\int_0^s a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\
&\quad - B_0\left(\int_0^\eta \phi_q\left(\int_0^s -a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s\right) \\
&\quad + \int_0^l (l-s)\phi_q\left(\int_0^s -a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\
&< \int_0^T T\phi_q\left(\int_0^T a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s + \delta\int_0^T \phi_q\left(\int_0^T a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\
&\quad + \int_0^T T\phi_q\left(\int_0^T a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\
&= T(2T+\delta)\phi_q\left[\int_{Y_1} a(r) f\left(x(r), \psi\left(\mu(r)\right)\right)\nabla r + \int_{Y_2} a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right] \\
&< \frac{b}{\rho} T(2T+\delta)\phi_q\left(\int_0^T a(r)\nabla r\right) = b.
\end{aligned} \tag{2.16}$$

Finally, we show that

$$P(\alpha, a) \ne \emptyset, \quad \alpha(Fx) > a, \quad \forall x \in \partial P(\alpha, a). \tag{2.17}$$

It is obvious that $P(\alpha, a) \ne \emptyset$. On the other hand, $\alpha(x) = x(\eta) = a$ and (2.7) imply

$$a \le x \le \frac{T}{T-\eta} a, \quad \text{for } t \in [0,\eta]. \tag{2.18}$$

Thus,

$$\begin{aligned}
\alpha(Fx) &= \int_0^T (T-s)\phi_q\left(\int_0^s a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s \\
&\quad - B_0\left(\int_0^\eta \phi_q\left(\int_0^s -a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s\right) \\
&\quad + \int_0^\eta (\eta-s)\phi_q\left(\int_0^s -a(r) f\left(x(r), x\left(\mu(r)\right)\right)\nabla r\right)\nabla s
\end{aligned}$$

$$
\begin{aligned}
&\geq -B_0\left(\int_0^{\eta} \phi_q\left(\int_0^s -a(r) f(x(r), x(\mu(r)))\nabla r\right)\nabla s\right) \\
&\geq \beta \int_0^l \phi_q\left(\int_0^s a(r) f(x(r), x(\mu(r)))\nabla r\right)\nabla s \\
&\geq \beta \int_{Y_3} \phi_q\left(\int_0^s a(r) f(x(r), \varphi(\mu(r)))\nabla r\right)\nabla s \\
&> \beta \int_{Y_3} \phi_q\left(\int_0^s a(r)\nabla r\right)\nabla s \frac{a}{\sigma} = a.
\end{aligned}
\tag{2.19}
$$

By Lemma 1.1, F has at least two different fixed points x_1 and x_2 satisfying

$$
a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c. \tag{2.20}
$$

Let

$$
x(t) = \begin{cases} \varphi(t), & t \in [-r, 0], \\ x_i(t), & t \in [0, T],\ i = 1, 2, \end{cases} \tag{2.21}
$$

which are twin positive solutions of BVP (1.1). The proof is complete. □

In analogy to Theorem 2.2, we have the following result.

Theorem 2.3. *Suppose that there are positive numbers $a < b < c$ such that*

$$
0 < a < \frac{T-\eta}{T} b < \frac{(T-\eta)\sigma}{T\rho} c. \tag{2.22}
$$

Assume f satisfies the following conditions:

(A′) $f(x, \varphi(s)) < \phi_p(c/\rho)$ *for* $0 \leq x \leq (T/(T-l))c$, *uniformly in* $s \in [-r, 0]$,

$$
f(x_1, x_2) < \phi_p\left(\frac{c}{\rho}\right), \quad \text{for } 0 \leq x_i \leq \frac{T}{T-l} c,\ i = 1, 2, \tag{2.23}
$$

(B′) $f(x, \varphi(s)) > \phi_p(b/\sigma)$ *for* $b \leq x \leq (T/(T-l))b$, *uniformly in* $s \in [-r, 0]$,

(C′) $f(x, \varphi(s)) < \phi_p(a/\rho)$ *for* $0 \leq x \leq (T/(T-\eta))a$, *uniformly in* $s \in [-r, 0]$,

$$
f(x_1, x_2) < \phi_p\left(\frac{a}{\rho}\right), \quad \text{for } 0 \leq x_i \leq \frac{T}{T-\eta} a,\ i = 1, 2. \tag{2.24}
$$

Then, BVP (1.1) *has at least two positive solutions of the form*

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ x_i(t), & t \in [0,T],\ i = 1,2. \end{cases} \tag{2.25}$$

Now, we give theorems, which may be considered as the corollaries of Theorems 2.2 and 2.3.

Let

$$f_0 = \lim_{x\to 0^+} \frac{f(x,\varphi(s))}{x^{p-1}}, \qquad f_\infty = \lim_{x\to\infty} \frac{f(x,\varphi(s))}{x^{p-1}}, \qquad f_{00} = \lim_{x_1\to 0^+; x_2\to 0^+} \frac{f(x_1,x_2)}{\max\left\{x_1^{p-1}, x_2^{p-1}\right\}}, \tag{2.26}$$

and choose k_1, k_2, k_3 such that

$$k_1\sigma > 1, \qquad k_2\sigma > 1, \qquad 0 < k_3\rho < \frac{T-\eta}{T}. \tag{2.27}$$

From above, we deduce that $0 < k_3\rho < l/T$.

Theorem 2.4. *If the following conditions are satisfied:*

(D) $f_0 > {k_1}^{p-1}$, $f_\infty > {k_2}^{p-1}$, *uniformly in* $s \in [-r,0]$,

(E) *there exists a* $p_1 > 0$ *such that for all* $0 \le x \le (T/(T-l))p_1$, *one has*

$$\begin{gathered} f(x,\varphi(s)) < \left(\frac{p_1}{\rho}\right)^{p-1}, \quad \text{uniformly in } s \in [-r,0], \\ f(x_1,x_2) < \left(\frac{p_1}{\rho}\right)^{p-1}, \quad \text{for } 0 \le x_i \le \frac{T}{T-l}p_1,\ i = 1,2. \end{gathered} \tag{2.28}$$

Then, BVP (1.1) *has at least two positive solutions of the form*

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ x_i(t), & t \in [0,T],\ i = 1,2. \end{cases} \tag{2.29}$$

Proof. First, choose $b = p_1$, one gets

$$\begin{gathered} f(x,\varphi(s)) < \phi_p\left(\frac{b}{\rho}\right), \quad \text{for } 0 \le x \le \frac{T}{T-l}b, \text{ uniformly in } s \in [-r,0], \\ f(x_1,x_2) < \phi_p\left(\frac{b}{\rho}\right), \quad \text{for } 0 \le x_i \le \frac{T}{T-l}b,\ i = 1,2. \end{gathered} \tag{2.30}$$

Secondly, since $f_0 > k_1^{p-1}$, there is $R_1 > 0$ sufficiently small such that

$$f(x,\varphi(s)) > (k_1 x)^{p-1}, \quad \text{for } 0 \le x \le R_1. \tag{2.31}$$

Without loss of generality, suppose $R_1 \le ((T-\eta)\sigma/T\rho)b$. Choose $a > 0$ so that $a < ((T-\eta)/T)R_1$. For $a \le x \le (T/(T-\eta))a$, we have $x \le R_1$ and $a < (\sigma/\rho)b$. Thus,

$$f(x,\varphi(s)) > (k_1 x)^{p-1} \ge (k_1 a)^{p-1} > \phi_p\left(\frac{a}{\sigma}\right), \quad \text{for } a \le x \le \frac{T}{T-\eta}a. \tag{2.32}$$

Thirdly, since $f_\infty > k_2{}^{p-1}$, there is $R_2 > 0$ sufficiently large such that

$$f(x,\varphi(s)) > (k_2 x)^{p-1}, \quad \text{for } x \ge R_2. \tag{2.33}$$

Without loss of generality, suppose $R_2 > (T/(T-l))b$. Choose $c \ge R_2$. Then,

$$f(x,\varphi(s)) > (k_2 x)^{p-1} \ge (k_2 c)^{p-1} > \phi_p\left(\frac{c}{\sigma}\right), \quad \text{for } c \le x \le \frac{T}{T-l}c. \tag{2.34}$$

We get now $0 < a < (\sigma/\rho)b < ((T-l)\sigma/T\rho)c$, and then the conditions in Theorem 2.2 are all satisfied. By Theorem 2.2, BVP (1.1) has at least two positive solutions. The proof is complete. □

Theorem 2.5. *If the following conditions are satisfied:*

(F) $f_0 < k_3{}^{p-1}$, *uniformly in* $s \in [-r,0]$; $f_{00} < k_3{}^{p-1}$,

(G) *there exists a* $p_2 > 0$ *such that for all* $0 \le x \le (T/(T-l))p_2$, *one has*

$$f(x,\varphi(s)) > \left(\frac{p_2}{\sigma}\right)^{p-1}, \quad \text{uniformly in } s \in [-r,0]. \tag{2.35}$$

Then, BVP (1.1) *has at least two positive solutions of the form*

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ x_i(t), & t \in [0,T],\ i = 1,2. \end{cases} \tag{2.36}$$

The proof is similar to that of Theorem 2.4 and we omitted it.

The following Corollaries are obvious.

Corollary 2.6. *If the following conditions are satisfied:*

(D′) $f_0 = \infty$, $f_\infty = \infty$, *uniformly in* $s \in [-r,0]$,

(E) *there exists a* $p_1 > 0$ *such that for all* $0 \le x \le (T/(T-l))p_1$, *one has*

$$f(x,\varphi(s)) < \left(\frac{p_1}{\rho}\right)^{p-1}, \quad \text{uniformly in } s \in [-r,0],$$
$$f(x_1,x_2) < \left(\frac{p_1}{\rho}\right)^{p-1}, \quad \text{for } 0 \le x_i \le \frac{T}{T-l}p_1, \ i = 1,2. \tag{2.37}$$

Then, BVP (1.1) *has at least two positive solutions of the form*

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ x_i(t), & t \in [0,T], \ i = 1,2. \end{cases} \tag{2.38}$$

Corollary 2.7. *If the following conditions are satisfied:*

(F′) $f_0 = 0$, *uniformly in* $s \in [-r,0]$, $f_{00} = 0$;

(G) *there exists a* $p_2 > 0$ *such that for all* $0 \le x \le (T/(T-l))p_2$, *one has*

$$f(x,\varphi(s)) > \left(\frac{p_2}{\sigma}\right)^{p-1}, \quad \text{uniformly in } s \in [-r,0]. \tag{2.39}$$

Then, BVP (1.1) *has at least two positive solutions of the form*

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ x_i(t), & t \in [0,T], \ i = 1,2. \end{cases} \tag{2.40}$$

3. Example

Example 3.1. Let $\mathrm{T} = [-1/2,0] \cup \{1/2^n : n \in \mathrm{N}_0\}$, $a(t) \equiv 1$, $r = 1/2$, $\eta = 1/2$, $p = 3$, $B_0(x) = x$.

We consider the following boundary value problem:

$$\left(\left|x^{\Delta\nabla}(t)\right| x^{\Delta\nabla}(t)\right)^{\nabla} + \frac{10^4 x^3(t)}{x^3(t) + x^3(t-1/2) + 1} = 0, \quad t \in (0,1),$$
$$x_0(t) = \varphi(t) \equiv 0, \quad t \in \left[-\frac{1}{2}, 0\right], \qquad x^{\Delta}(0) = x^{\Delta\nabla}(0) = 0, \qquad x(1) + x^{\Delta}\left(\frac{1}{2}\right) = 0, \tag{3.1}$$

where $\mu : [0,1] \to [-1/2,1]$ and $\mu(t) = t - 1/2$; $f(x,\varphi(s)) = 6x^3/(x^3+1)$, $f(x_1,x_2) = 6x_1^3/(x_1^3 + x_2^3 + 1)$.

Choosing $a = 1/2 \times 10^{10}$, $b = 1$, $c = 10^3$, $l = 1/4$, direct calculation shows that

$$Y_1 = \left[0, \frac{1}{2}\right), \qquad Y_2 = \left[\frac{1}{2}, 1\right], \qquad Y_3 = \left[0, \frac{1}{4}\right], \qquad \sigma = \frac{4+\sqrt{2}}{224}, \qquad \rho = 3. \tag{3.2}$$

Consequently, $0 < a < ((T-\eta)/T)b < ((T-\eta)\sigma/T\rho)c$ and f satisfies

(A′) $f(x,\psi(s)) < \phi_p(c/\rho) = 10^6/9$ for $0 \le x \le 4\times 10^3/3$, uniformly in $s \in [-1/2, 0]$,

$$f(x_1,x_2) < \phi_p\left(\frac{c}{\rho}\right) = \frac{10^6}{9}, \quad \text{for } 0 \le x_i \le \frac{4\times 10^3}{3},\ i = 1,2, \tag{3.3}$$

(B′) $f(x,\psi(s)) > \phi_p(b/\sigma) = 1/\sigma^2$ for $1 \le x \le 4/3$, uniformly in $s \in [-1/2, 0]$,

(C′) $f(x,\psi(s)) < \phi_p(a/\rho) = 1/36\times 10^{20}$ for $0 \le x \le 1/10^{10}$, uniformly in $s \in [-1/2, 0]$,

$$f(x_1,x_2) < \phi_p\left(\frac{a}{\rho}\right) = \frac{1}{36\times 10^{20}}, \quad \text{for } 0 \le x_i \le \frac{1}{10^{10}},\ i = 1,2. \tag{3.4}$$

Then all conditions of Theorem 2.3 hold. Thus, with Theorem 2.3, the BVP (3.1) has at least two positive solutions.

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