Hindawi Publishing Corporation
Boundary Value Problems
Volume 2011, Article ID 268032, 27 pages
doi:10.1155/2011/268032

Research Article

Degenerate Anisotropic Differential Operators and Applications

Ravi Agarwal,[1] Donal O'Regan,[2] and Veli Shakhmurov[3]

[1] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA*
[2] *Department of Mathematics, National University of Ireland, Galway, Ireland*
[3] *Department of Electronics Engineering and Communication, Okan University, Akfirat, Tuzla 34959 Istanbul, Turkey*

Correspondence should be addressed to Veli Shakhmurov, veli.sahmurov@okan.edu.tr

Received 2 December 2010; Accepted 18 January 2011

Academic Editor: Gary Lieberman

The boundary value problems for degenerate anisotropic differential operator equations with variable coefficients are studied. Several conditions for the separability and Fredholmness in Banach-valued L_p spaces are given. Sharp estimates for resolvent, discreetness of spectrum, and completeness of root elements of the corresponding differential operators are obtained. In the last section, some applications of the main results are given.

1. Introduction and Notations

It is well known that many classes of PDEs, pseudo-Des, and integro-DEs can be expressed as differential-operator equations (DOEs). As a result, many authors investigated PDEs as a result of single DOEs. DOEs in H-valued (Hilbert space valued) function spaces have been studied extensively in the literature (see [1–14] and the references therein). Maximal regularity properties for higher-order degenerate anisotropic DOEs with constant coefficients and nondegenerate equations with variable coefficients were studied in [15, 16].

The main aim of the present paper is to discuss the separability properties of BVPs for higher-order degenerate DOEs; that is,

$$\sum_{k=1}^{n} a_k(x) D_k^{[l_k]} u(x) + A(x)u(x) + \sum_{|\alpha:l|<1} A_\alpha(x) D^{[\alpha]} u(x) = f(x), \tag{1.1}$$

where $D_k^{[i]} u(x) = (\gamma_k(x_k)(\partial/\partial x_k))^i u(x)$, γ_k are weighted functions, A and A_α are linear operators in a Banach Space E. The above DOE is a generalized form of an elliptic equation. In fact, the special case $l_k = 2m$, $k = 1, \dots, n$ reduces (1.1) to elliptic form.

Note, the principal part of the corresponding differential operator is nonself-adjoint. Nevertheless, the sharp uniform coercive estimate for the resolvent, Fredholmness, discreetness of the spectrum, and completeness of root elements of this operator are established.

We prove that the corresponding differential operator is separable in L_p; that is, it has a bounded inverse from L_p to the anisotropic weighted space $W_{p,\gamma}^{[l]}$. This fact allows us to derive some significant spectral properties of the differential operator. For the exposition of differential equations with bounded or unbounded operator coefficients in Banach-valued function spaces, we refer the reader to [8, 15–25].

Let $\gamma = \gamma(x)$ be a positive measurable weighted function on the region $\Omega \subset R^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of all strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{p,\gamma} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x)dx\right)^{1/p}, \quad 1 \le p < \infty. \tag{1.2}$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p(\Omega; E)$.

The weight γ we will consider satisfies an A_p condition; that is, $\gamma \in A_p$, $1 < p < \infty$ if there is a positive constant C such that

$$\left(\frac{1}{|Q|}\int_Q \gamma(x)dx\right)\left(\frac{1}{|Q|}\int_Q \gamma^{-1/(p-1)}(x)dx\right)^{p-1} \le C, \tag{1.3}$$

for all cubes $Q \subset R^n$.

The Banach space E is called a UMD space if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}(f(y)/(x-y))dy$ is bounded in $L_p(R, E)$, $p \in (1, \infty)$ (see, e.g., [26]). UMD spaces include, for example, L_p, l_p spaces, and Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda;\ \lambda \in \mathbf{C}, |\arg\lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi < \pi. \tag{1.4}$$

A linear operator A is said to be φ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\left\|(A + \lambda I)^{-1}\right\|_{L(E)} \le M(1 + |\lambda|)^{-1}, \tag{1.5}$$

for all $\lambda \in S_\varphi, \varphi \in [0, \pi)$, I is an identity operator in E, and $B(E)$ is the space of bounded linear operators in E. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by A_λ. It is known [27, Section 1.15.1] that there exists fractional powers A^θ of the sectorial operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{1/p}, \quad 1 \le p < \infty,\ -\infty < \theta < \infty. \tag{1.6}$$

Let E_1 and E_2 be two Banach spaces. Now, $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$ will denote interpolation spaces obtained from $\{E_1, E_2\}$ by the K method [27, Section 1.3.1].

A set $W \subset B(E_1, E_2)$ is called R-bounded (see [3, 25, 26]) if there is a constant $C > 0$ such that for all $T_1, T_2, \ldots, T_m \in W$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in N$

$$\int_0^1 \left\| \sum_{j=1}^{m} r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^{m} r_j(y) u_j \right\|_{E_1} dy, \tag{1.7}$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$.

The smallest C for which the above estimate holds is called an R-bound of the collection W and is denoted by $R(W)$.

Let $S(R^n; E)$ denote the Schwartz class, that is, the space of all E-valued rapidly decreasing smooth functions on R^n. Let F be the Fourier transformation. A function $\Psi \in C(R^n; B(E))$ is called a Fourier multiplier in $L_{p,\gamma}(R^n; E)$ if the map $u \to \Phi u = F^{-1}\Psi(\xi)Fu$, $u \in S(R^n; E)$ is well defined and extends to a bounded linear operator in $L_{p,\gamma}(R^n; E)$. The set of all multipliers in $L_{p,\gamma}(R^n; E)$ will denoted by $M_{p,\gamma}^{p,\gamma}(E)$.

Let

$$V_n = \{\xi : \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n,\ \xi_j \neq 0\},$$
$$U_n = \{\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in N^n : \beta_k \in \{0, 1\}\}. \tag{1.8}$$

Definition 1.1. A Banach space E is said to be a space satisfying a multiplier condition if, for any $\Psi \in C^{(n)}(R^n; B(E))$, the R-boundedness of the set $\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in R^n \setminus 0, \beta \in U_n\}$ implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(R^n; E)$, that is, $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$ for any $p \in (1, \infty)$.

Let $\Psi_h \in M_{p,\gamma}^{p,\gamma}(E)$ be a multiplier function dependent on the parameter $h \in Q$. The uniform R-boundedness of the set $\{\xi^\beta D^\beta \Psi_h(\xi) : \xi \in R^n \setminus 0, \beta \in U\}$; that is,

$$\sup_{h \in Q} R\left(\left\{\xi^\beta D^\beta \Psi_h(\xi) : \xi \in R^n \setminus 0, \beta \in U\right\}\right) \leq K \tag{1.9}$$

implies that Ψ_h is a uniform collection of Fourier multipliers.

Definition 1.2. The φ-positive operator A is said to be R-positive in a Banach space E if there exists $\varphi \in [0, \pi)$ such that the set $\{A(A + \xi I)^{-1} : \xi \in S_\varphi\}$ is R-bounded.

A linear operator $A(x)$ is said to be φ-positive in E uniformly in x if $D(A(x))$ is independent of x, $D(A(x))$ is dense in E and $\|(A(x) + \lambda I)^{-1}\| \leq M/(1 + |\lambda|)$ for any $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$.

The φ-positive operator $A(x)$, $x \in G$ is said to be uniformly R-positive in a Banach space E if there exists $\varphi \in [0, \pi)$ such that the set $\{A(x)(A(x) + \xi I)^{-1} : \xi \in S_\varphi\}$ is uniformly R-bounded; that is,

$$\sup_{x \in G} R\left(\left\{\xi^\beta D^\beta \left[A(x)(A(x) + \xi I)^{-1}\right] : \xi \in R^n \setminus 0, \beta \in U\right\}\right) \leq M. \tag{1.10}$$

Let $\sigma_\infty(E_1, E_2)$ denote the space of all compact operators from E_1 to E_2. For $E_1 = E_2 = E$, it is denoted by $\sigma_\infty(E)$.

For two sequences $\{a_j\}_1^\infty$ and $\{b_j\}_1^\infty$ of positive numbers, the expression $a_j \sim b_j$ means that there exist positive numbers C_1 and C_2 such that

$$C_1 a_j \le b_j \le C_2 a_j. \tag{1.11}$$

Let $\sigma_\infty(E_1, E_2)$ denote the space of all compact operators from E_1 to E_2. For $E_1 = E_2 = E$, it is denoted by $\sigma_\infty(E)$.

Now, $s_j(A)$ denotes the approximation numbers of operator A (see, e.g., [27, Section 1.16.1]). Let

$$\sigma_q(E_1, E_2) = \left\{ A : A \in \sigma_\infty(E_1, E_2), \sum_{j=1}^{\infty} s_j^q(A) < \infty,\ 1 \le q < \infty \right\}. \tag{1.12}$$

Let E_0 and E be two Banach spaces and E_0 continuously and densely embedded into E and $l = (l_1, l_2, \ldots, l_n)$.

We let $W_{p,\gamma}^l(\Omega; E_0, E)$ denote the space of all functions $u \in L_{p,\gamma}(\Omega; E_0)$ possessing generalized derivatives $D_k^{l_k} u = \partial^{l_k} u / \partial x_k^{l_k}$ such that $D_k^{l_k} u \in L_{p,\gamma}(\Omega; E)$ with the norm

$$\|u\|_{W_{p,\gamma}^l(\Omega; E_0, E)} = \|u\|_{L_{p,\gamma}(\Omega; E_0)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{L_{p,\gamma}(\Omega; E)} < \infty. \tag{1.13}$$

Let $D_k^{[i]} u(x) = (\gamma_k(x_k)(\partial/\partial x_k))^i u(x)$. Consider the following weighted spaces of functions:

$$\begin{aligned} W_{p,\gamma}^{[l]}(G; E(A), E) = \Bigg\{ & u : u \in L_p(G; E(A)), D_k^{[l_k]} u \in L_p(G; E), \\ & \|u\|_{W_{p,\gamma}^{[l]}(G; E(A), E)} = \|u\|_{L_p(G; E(A))} + \sum_{k=1}^{n} \left\| D_k^{[l_k]} u \right\|_{L_p(G; E)} \Bigg\}. \end{aligned} \tag{1.14}$$

2. Background

The embedding theorems play a key role in the perturbation theory of DOEs. For estimating lower order derivatives, we use following embedding theorems from [24].

Theorem A1. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ and suppose that the following conditions are satisfied:*

(1) *E is a Banach space satisfying the multiplier condition with respect to p and γ,*

(2) *A is an R-positive operator in E,*

(3) *$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $l = (l_1, l_2, \ldots, l_n)$ are n-tuples of nonnegative integer such that*

$$\varkappa = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \le 1, \quad 0 \le \mu \le 1 - \varkappa,\ 1 < p < \infty, \tag{2.1}$$

(4) $\Omega \subset R^n$ *is a region such that there exists a bounded linear extension operator from* $W^l_{p,\gamma}(\Omega; E(A), E)$ *to* $W^l_{p,\gamma}(R^n; E(A), E)$.

Then, the embedding $D^\alpha W^l_{p,\gamma}(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; E(A^{1-\varkappa-\mu}))$ *is continuous. Moreover, for all positive number* $h < \infty$ *and* $u \in W^l_{p,\gamma}(\Omega; E(A), E)$, *the following estimate holds*

$$\|D^\alpha u\|_{L_{p,\gamma}(\Omega; E(A^{1-\varkappa-\mu}))} \leq h^\mu \|u\|_{W^l_{p,\gamma}(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_{p,\gamma}(\Omega; E)}. \tag{2.2}$$

Theorem A2. *Suppose that all conditions of Theorem A1 are satisfied. Moreover, let* $\gamma \in A_p$, Ω *be a bounded region and* $A^{-1} \in \sigma_\infty(E)$. *Then, the embedding*

$$W^l_{p,\gamma}(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; E) \tag{2.3}$$

is compact.

Let Sp A *denote the closure of the linear span of the root vectors of the linear operator* A.

From [18, Theorem 3.4.1], we have the following.

Theorem A3. *Assume that*

(1) E *is an UMD space and* A *is an operator in* $\sigma_p(E)$, $p \in (1, \infty)$,

(2) $\mu_1, \mu_2, \ldots, \mu_s$ *are non overlapping, differentiable arcs in the complex plane starting at the origin. Suppose that each of the* s *regions into which the planes are divided by these arcs is contained in an angular sector of opening less then* π/p,

(3) $m > 0$ *is an integer so that the resolvent of* A *satisfies the inequality*

$$\|R(\lambda, A)\| = O\left(|\lambda|^{-1}\right), \tag{2.4}$$

as $\lambda \to 0$ *along any of the arcs* μ.

Then, the subspace Sp A *contains the space* E.

Let

$$G = \{x = (x_1, x_2, \ldots, x_n) : 0 < x_k < b_k\}, \qquad \gamma(x) = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}. \tag{2.5}$$

Let

$$\beta_k = x_k^{\beta_k}, \qquad \nu = \prod_{k=1}^n x_k^{\nu_k}, \qquad \gamma = \prod_{k=1}^n x_k^{\gamma_k}. \tag{2.6}$$

Let $I = I(W^l_{p,\beta,\gamma}(\Omega; E(A), E), L_{p,\gamma}(\Omega; E))$ *denote the embedding operator* $W^l_{p,\beta,\gamma}(\Omega; E(A), E) \to L_{p,\nu}(\Omega; E)$.

From [15, Theorem 2.8], we have the following.

Theorem A4. *Let E_0 and E be two Banach spaces possessing bases. Suppose that*

$$0 \leq \gamma_k < p-1, \quad 0 \leq \beta_k < 1, \quad \nu_k - \gamma_k > p(\beta_k - 1), \quad 1 < p < \infty,$$
$$s_j(I(E_0, E)) \sim j^{-1/k_0}, \quad k_0 > 0, \; j = 1, 2, \ldots, \infty, \qquad \varkappa_0 = \sum_{k=1}^{n} \frac{\gamma_k - \nu_k}{p(l_k - \beta_k)} < 1. \tag{2.7}$$

Then,

$$s_j\left(I\left(W^l_{p,\beta,\gamma}(G; E_0, E), L_{p,\nu}(G; E)\right)\right) \sim j^{-1/(k_0 + \varkappa_0)}. \tag{2.8}$$

3. Statement of the Problem

Consider the BVPs for the degenerate anisotropic DOE

$$\sum_{k=1}^{n} a_k(x) D_k^{[l_k]} u(x) + [A(x) + \lambda] u(x) + \sum_{|\alpha:l|<1} A_\alpha(x) D^{[\alpha]} u(x) = f(x), \tag{3.1}$$

$$\sum_{i=0}^{m_{kj}} \alpha_{kji} D_k^{[i]} u(G_{k0}) = 0, \quad j = 1, 2, \ldots, d_k,$$
$$\sum_{i=0}^{m_{kj}} \beta_{kji} D_k^{[i]} u(G_{kb}) = 0, \quad j = 1, 2, \ldots, l_k - d_k, \; d_k \in (0, l_k), \tag{3.2}$$

where

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad l = (l_1, l_2, \ldots, l_n), \quad |\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k},$$
$$G = \{x = (x_1, x_2, \ldots, x_n), 0 < x_k < b_k, \}, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),$$
$$D^{[\alpha]} = D_1^{[\alpha_1]} D_2^{[\alpha_2]} \cdots D_n^{[\alpha_n]}, \qquad D_k^{[i]} u(x) = \left[x_k^{\gamma_k} (b_k - x_k)^{\nu_k} \frac{\partial}{\partial x_k}\right]^i u(x),$$
$$0 \leq \gamma_k, \quad \nu_k < 1 - \frac{1}{p}, \quad k = 1, 2, \ldots, n, \quad G_{k0} = (x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n), \tag{3.3}$$
$$G_{kb} = (x_1, x_2, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n), \quad 0 \leq m_{kj} \leq l_k - 1,$$
$$x(k) = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad G_k = \prod_{j \neq k} (0, b_j), \quad j, k = 1, 2, \ldots, n,$$

α_{jk}, β_{jk}, λ are complex numbers, a_k are complex-valued functions on G, $A(x)$, and $A_\alpha(x)$ are linear operators in E. Moreover, γ_k and ν_k are such that

$$\int_0^{x_k} x_k^{-\gamma_k} (b_k - x_k)^{-\nu_k} dx_k < \infty, \quad x_k \in [0, b_k], \; k = 1, 2, \ldots, n. \tag{3.4}$$

A function $u \in W_{p,\gamma}^{[l]}(G;E(A),E,L_{kj}) = \{u \in W_{p,\gamma}^{[l]}(G;E(A),E), L_{kj}u = 0\}$ and satisfying (3.1) a.e. on G is said to be solution of the problem (3.1)-(3.2).

We say the problem (3.1)-(3.2) is L_p-separable if for all $f \in L_p(G;E)$, there exists a unique solution $u \in W_{p,\gamma}^{[l]}(G;E(A),E)$ of the problem (3.1)-(3.2) and a positive constant C depending only G, p, γ, l, E, A such that the coercive estimate

$$\sum_{k=1}^{n} \left\| D_k^{[l_k]} u \right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq C\|f\|_{L_p(G;E)} \tag{3.5}$$

holds.

Let Q be a differential operator generated by problem (3.1)-(3.2) with $\lambda = 0$; that is,

$$\begin{gathered} D(Q) = W_{p,\gamma}^{[l]}(G;E(A),E,L_{kj}), \\ Qu = \sum_{k=1}^{n} a_k(x) D_k^{[l_k]} u + A(x)u + \sum_{|\alpha:l|<1} A_\alpha(x) D^{[\alpha]} u. \end{gathered} \tag{3.6}$$

We say the problem (3.1)-(3.2) is Fredholm in $L_p(G;E)$ if $\dim\operatorname{Ker} Q = \dim\operatorname{Ker} Q^* < \infty$, where Q^* is a conjugate of Q.

Remark 3.1. Under the substitutions

$$\tau_k = \int_0^{x_k} x_k^{-\gamma_k} (b_k - x_k)^{-\nu_k} dx_k, \quad k = 1, 2, \ldots, n, \tag{3.7}$$

the spaces $L_p(G;E)$ and $W_{p,\gamma}^{[l]}(G;E(A),E)$ are mapped isomorphically onto the weighted spaces $L_{p,\tilde{\gamma}}(\tilde{G};E)$ and $W_{p,\tilde{\gamma}}^{l}(\tilde{G};E(A),E)$, where

$$\tilde{G} = \prod_{k=1}^{n} \left(0, \tilde{b}_k\right), \quad \tilde{b}_k = \int_0^{b_k} x_k^{-\gamma_k} (b_k - x_k)^{-\nu_k} dx_k. \tag{3.8}$$

Moreover, under the substitution (3.7) the problem (3.1)-(3.2) reduces to the nondegenerate BVP

$$\begin{gathered} \sum_{k=1}^{n} \tilde{a}_k(\tau) D_k^{l_k} u(\tau) + \left(\tilde{A}(\tau) + \lambda\right) u(\tau) + \sum_{|\alpha:l|<1} \tilde{A}_\alpha(\tau) D^\alpha u(\tau) = f(\tau), \\ \sum_{i=0}^{m_{kj}} \alpha_{kji} D_k^i u\left(\tilde{G}_{k0}\right) = 0, \quad j = 1, 2, \ldots, d_k, \ x(k) \in \tilde{G}_k, \\ \sum_{i=0}^{m_{kj}} \beta_{kji} D_k^i u\left(\tilde{G}_{kb}\right) = 0, \quad x(k) \in \tilde{G}_k, \ j = 1, 2, \ldots, l_k - d_k, \ d_k \in (0, l_k), \end{gathered} \tag{3.9}$$

where

$$\tilde{G}_{k0} = (\tau_1, \tau_2, \dots, \tau_{k-1}, 0, \tau_{k+1}, \dots, \tau_n), \qquad \tilde{G}_{kb} = \left(\tau_1, \tau_2, \dots, \tau_{k-1}, \tilde{b}_k, \tau_{k+1}, \dots, \tau_n\right),$$
$$\tilde{a}_k(\tau) = a_k(x_1(\tau), x_2(\tau), \dots, x_n(\tau)), \qquad \tilde{A}(\tau) = A(a_k(x_1(\tau), x_2(\tau), \dots, x_n(\tau))), \tag{3.10}$$
$$\tilde{A}_k(\tau) = A_k(a_k(x_1(\tau), x_2(\tau), \dots, x_n(\tau))), \qquad \tilde{\gamma}(\tau) = \gamma(x_1(\tau), x_2(\tau), \dots, x_n(\tau)).$$

By denoting τ, $\tilde{G}$, $\tilde{G}_{k0}$, $\tilde{G}_{kb}$, $\tilde{a}_k(\tau)$, $\tilde{A}(\tau)$, $\tilde{A}_k(y)$, $\tilde{\gamma}_k(\tau)$ again by x, G, G_{k0}, G_{kb}, $a_k(x)$, $A(x)$, $A_k(x)$, γ_k, respectively, we get

$$\sum_{k=1}^{n} a_k(x) D_k^{l_k} u(x) + A_\lambda(x) u(x) + \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u(x) = f(x),$$
$$\sum_{i=0}^{m_{kj}} \alpha_{kji} D_k^i u(G_{kb}) = 0, \quad j = 1, 2, \dots, l_k - d_k, \; x(k) \in (G_k), \tag{3.11}$$
$$\sum_{i=0}^{m_{kj}} \beta_{kji} D_k^i u(G_{kb}) = 0, \quad x(k) \in G_k, \; j = 1, 2, \dots, l_k - d_k, \; d_k \in (0, l_k).$$

4. BVPs for Partial DOE

Let us first consider the BVP for the anisotropic type DOE with constant coefficients

$$(L + \lambda) u = \sum_{k=1}^{n} a_k D_k^{[l_k]} u(x) + (A + \lambda) u(x) = f(x),$$
$$L_{kj} u = f_{kj}, \quad j = 1, 2, \dots, d_k, \qquad L_{kj} u = f_{kj}, \quad j = 1, 2, \dots, l_k - d_k, \tag{4.1}$$

where

$$D_k^{[i]} u(x) = \left[x_k^{\gamma_k} \frac{\partial}{\partial x_k} \right]^i u(x), \tag{4.2}$$

L_{kj} are boundary conditions defined by (3.2), a_k are complex numbers, λ is a complex parameter, and A is a linear operator in a Banach space E. Let $\omega_{k1}, \omega_{k2}, \dots, \omega_{kl_k}$ be the roots of the characteristic equations

$$a_k \omega^{l_k} + 1 = 0, \quad k = 1, 2, \dots, n. \tag{4.3}$$

Now, let

$$F_{kj} = (Y_k, X_k)_{(1-\gamma_k+pm_{kj})/pl_k,p}, \quad X_k = L_p(G_k; E), \ Y_k = W^{[l^{(k)}]}_{p,\gamma^{(k)}}(G_k; E(A), E),$$
$$l^{(k)} = (l_1, l_2, \ldots, l_{k-1}, l_{k+1}, \ldots, l_n), \qquad \gamma^{(k)} = \left(x_1^{\gamma_1}, x_2^{\gamma_2}, \ldots, x_{k-1}^{\gamma_{k-1}}, x_{k+1}^{\gamma_{k+1}}, \ldots, x_n^{l_n}\right), \tag{4.4}$$
$$G_{kx_0} = (x_1, x_2, \ldots, x_{k-1}, x_{k0}, x_{k+1}, \ldots, x_n).$$

By applying the trace theorem [27, Section 1.8.2], we have the following.

Theorem A5. *Let l_k and j be integer numbers, $0 \le j \le l_k - 1$, $\theta_j = (1-\gamma_k + pj + 1)/pl_k$, $x_{k0} \in [0, b_k]$. Then, for any $u \in W^l_{p,\gamma}(G; E_0, E)$, the transformations $u \to D^j_k u(G_{kx_0})$ are bounded linear from $W^l_{p,\gamma}(G; E_0, E)$ onto F_{kj}, and the following inequality holds:*

$$\left\| D^j_k u(G_{kx_0}) \right\|_{F_{kj}} \le C \|u\|_{W^l_{p,\gamma}(G;E_0,E)}. \tag{4.5}$$

Proof. It is clear that

$$W^l_{p,\gamma}(G; E_0, E) = W^{l_k}_{p,\gamma_k}(0, b_k; Y_k, X_k). \tag{4.6}$$

□

Then, by applying the trace theorem [27, Section 1.8.2] to the space $W^{l_k}_{p,\gamma_k}(0, b_k; Y_k, X_k)$, we obtain the assertion.

Condition 1. Assume that the following conditions are satisfied:

(1) E is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$ and the weight function $\gamma = \prod_{k=1}^n x_k^{\gamma_k}$, $0 \le \gamma_k < 1 - 1/p$;

(2) A is an R-positive operator in E for $\varphi \in [0, \pi/2)$;

(3) $a_k \ne 0$, and

$$\left|\arg \omega_{kj} - \pi\right| \le \frac{\pi}{2} - \varphi, \quad j = 1, 2, \ldots, d_k, \qquad \left|\arg \omega_{kj}\right| \le \frac{\pi}{2} - \varphi, \quad j = d_k + 1, \ldots, l_k \tag{4.7}$$

for $0 < d_k < l_k$, $k = 1, 2, \ldots, n$.

Let B denote the operator in $L_p(G; E)$ generated by BVP (4.1). In [15, Theorem 5.1] the following result is proved.

Theorem A6. *Let Condition 1 be satisfied. Then,*

(a) *the problem* (4.1) *for $f \in L_p(G; E)$ and $|\arg \lambda| \le \varphi$ with sufficiently large $|\lambda|$ has a unique solution u that belongs to $W^{[l]}_p(G; E(A), E)$ and the following coercive uniform estimate holds:*

$$\sum_{k=1}^n \sum_{i=0}^{l_k} |\lambda|^{1-i/l_k} \left\| D^{[i]}_k u \right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \le M \|f\|_{L_p(G;E)}, \tag{4.8}$$

(b) *the operator B is R-positive in $L_p(G; E)$.*

From Theorems A5 and A6 we have.

Theorem A7. *Suppose that Condition 1 is satisfied. Then, for sufficiently large* $|\lambda|$ *with* $|\arg\lambda| \leq \varphi$ *the problem* (4.1) *has a unique solution* $u \in W_{p,\gamma}^{[l]}(G; E(A), E)$ *for all* $f \in L_p(G; E)$ *and* $f_{kj} \in F_{kj}$*. Moreover, the following uniform coercive estimate holds:*

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^{[i]}u\right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq M\|f\|_{L_p(G;E)} + \sum_{k=1}^{n}\sum_{j=1}^{l_k}\|f_{kj}\|_{F_{kj}}. \tag{4.9}$$

Consider BVP (3.11)*. Let* $\omega_{k1}(x), \omega_{k2}(x), \ldots, \omega_{kl_k}(x)$ *be roots of the characteristic equations*

$$a_k(x)\omega^{l_k} + 1 = 0, \quad k = 1, 2, \ldots, n. \tag{4.10}$$

Condition 2. Suppose the following conditions are satisfied:

(1) $a_k \neq 0$ and

$$\begin{gathered}\left|\arg\omega_{kj} - \pi\right| \leq \frac{\pi}{2} - \varphi, \quad j = 1, 2, \ldots, d_k, \\ \left|\arg\omega_{kj}\right| \leq \frac{\pi}{2} - \varphi, \quad j = d_k + 1, \ldots, l_k,\end{gathered} \tag{4.11}$$

for

$$0 < d_k < l_k, \quad k = 1, 2, \ldots, n, \ \varphi \in \left[0, \frac{\pi}{2}\right), \tag{4.12}$$

(2) E is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$ and the weighted function $\gamma = \prod_{k=1}^{n} x_k^{\gamma_k}(b_k - x_k)^{\gamma_k}$, $0 \leq \gamma_k < 1 - 1/p$.

Remark 4.1. Let $l = 2m_k$ and $a_k = (-1)^{m_k} b_k(x)$, where b_k are real-valued positive functions. Then, Condition 2 is satisfied for $\varphi \in [0, \pi/2)$.

Consider the inhomogenous BVP (3.1)-(3.2); that is,

$$(L + \lambda)u = f, \qquad L_{kj}u = f_{kj}. \tag{4.13}$$

Lemma 4.2. *Assume that Condition 2 is satisfied and the following hold:*

(1) $A(x)$ *is a uniformly R-positive operator in E for* $\varphi \in [0, \pi/2)$*, and* $a_k(x)$ *are continuous functions on* $\overline{G}$*,* $\lambda \in S_\varphi$*,*

(2) $A(x)A^{-1}(\overline{x}) \in C(\overline{G}; B(E))$ *and* $A_\infty A^{(1-|\alpha:l|-\mu)} \in L_\infty(G; B(E))$ *for* $0 < \mu < 1 - |\alpha : l|$*.*

Then, for all $\lambda \in S(\varphi)$ *and for sufficiently large* $|\lambda|$ *the following coercive uniform estimate holds:*

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^{i}u\right\|_{L_{p,\gamma}(G;E)} + \|Au\|_{L_{p,\gamma}(G;E)} \leq C\|f\|_{L_{p,\gamma}(G;E)} + \sum_{k=1}^{n}\sum_{j=1}^{l_k}\|f_{kj}\|_{F_{kj}}, \tag{4.14}$$

for the solution of problem (4.13).

Proof. Let $G_1, G_2, \dots, G_N$ be regions covering G and let $\varphi_1, \varphi_2, \dots, \varphi_N$ be a corresponding partition of unity; that is, $\varphi_j \in C_0^\infty$, $\sigma_j = \operatorname{supp}\varphi_j \subset G_j$ and $\sum_{j=1}^N \varphi_j(x) = 1$. Now, for $u \in W_{p,\gamma}^l(G; E(A), E)$ and $u_j(x) = u(x)\varphi_j(x)$, we get

$$(L+\lambda)u_j = \sum_{k=1}^n a_k(x) D_k^{l_k} u_j(x) + A_\lambda(x) u_j(x) = f_j(x), \quad L_{ki} u_j = \Phi_{ki}, \tag{4.15}$$

where

$$\begin{gathered} f_j = f\varphi_j + \sum_{k=1}^n a_k \sum_{|\alpha:l|<1} A_\alpha(x) \prod_{k=1}^n \sum_{i=0}^{\alpha_k - 1} C_{\alpha i k} D_k^{\alpha_k - i} \varphi_j D_k^i u - \sum_{|\alpha:l|<1} \varphi_j A_\alpha(x) D^\alpha u(x), \\ \Phi_{ki} = \varphi_j L_{ki} u + B_{ki}(\varphi_j) L'_{ki} u, \end{gathered} \tag{4.16}$$

here, L'_{ki} and B_{ki} are boundary operators which orders less than $m_{ki} - 1$. Freezing the coefficients of (4.15), we have

$$\begin{gathered} \sum_{k=1}^n a_k(x_{0j}) D_k^{l_k} u_j(x) + A_\lambda(x_{0j}) u_j(x) = F_j, \\ L_{ki} u_j = \Phi_{ki}, \quad i = 1, 2, \dots, l_k, \ k = 1, 2, \dots, n, \end{gathered} \tag{4.17}$$

where

$$F_j = f_j + \left[A(x_{0j}) - A(x)\right] u_j + \sum_{k=1}^n \left[a_k(x) - a(x_{0j})\right] D_k^{l_k} u_j(x). \tag{4.18}$$

It is clear that $\gamma(x) \sim \prod_{k=1}^n x_k^{\gamma_k}$ on neighborhoods of $G_j \cap G_{k0}$ and

$$\gamma(x) \sim \prod_{k=1}^n (b_k - x_k)^{\nu_k}, \tag{4.19}$$

on neighborhoods of $G_j \cap G_{kb}$ and $\gamma(x) \sim C_j$ on other parts of the domains G_j, where C_j are positive constants. Hence, the problems (4.17) are generated locally only on parts of the boundary. Then, by Theorem A7 problem (4.17) has a unique solution u_j and for $|\arg\lambda| \le \varphi$ the following coercive estimate holds:

$$\sum_{k=1}^n \sum_{i=0}^{l_k} |\lambda|^{1-i/l_k} \left\| D_k^i u_j \right\|_{G_j,p,\gamma} + \left\| A u_j \right\|_{G_j,p,\gamma} \le C \left[\left\| F_j \right\|_{G_j,p,\gamma} + \sum_{k=1}^n \sum_{i=1}^{l_k} \left\| \Phi_{kj} \right\|_{F_{ki}} \right]. \tag{4.20}$$

From the representation of F_j, Φ_{ki} and in view of the boundedness of the coefficients, we get

$$\|F_j\|_{G_j,p,\gamma} \leq \|f_j\|_{G_j,p,\gamma} + \|[A(x_{0j}) - A(x)]u_j\|_{G_j,p,\gamma} + \sum_{k=1}^{n}\left\|[a_k(x) - a(x_{0j})]D_k^{l_k}u_j(x)\right\|_{G_j,p,\gamma},$$
$$\|\Phi_{kj}\|_{F_{ki}} \leq \|\varphi_j L_{ki}u\|_{F_{ki}} + \|B_j(\varphi_j)L'_{ki}u\|_{F_{ki}} \leq M\Big(\|L_{ki}u\|_{F_{ki}} + \|L'_{ki}u\|_{F_{ki}}\Big). \tag{4.21}$$

Now, applying Theorem A1 and by using the smoothness of the coefficients of (4.16), (4.18) and choosing the diameters of σ_j so small, we see there is an $\varepsilon > 0$ and $C(\varepsilon)$ such that

$$\begin{aligned}\|F_j\|_{G_j,p,\gamma} &\leq \|f_j\|_{G_j,p,\gamma} + \varepsilon\|A(x_{0j})u_j\|_{G_j,p,\gamma} + \varepsilon\sum_{k=1}^{n}\left\|D_k^{l_k}u_j(x)\right\|_{G_j,p,\gamma}\\ &\leq \|f\varphi_j\|_{G_j,p,\gamma} + M\sum_{|\alpha:l|<1}\|A_\alpha(x)D^\alpha u_j(x)\|_{G_j,p,\gamma} + \varepsilon\|u_j\|_{W^l_{p,\gamma}(G_j;E(A),E)}\\ &\leq \|f\varphi_j\|_{G_j,p,\gamma} + \varepsilon\|u_j\|_{W^l_{p,\gamma}(G_j;E(A),E)} + C(\varepsilon)\|u_j\|_{G_j,p,\gamma}.\end{aligned} \tag{4.22}$$

Then, using Theorem A5 and using the smoothness of the coefficients of (4.16), (4.18), we get

$$\|\Phi_{ki}\|_{F_{kj}} \leq M\Big[\|L_{ki}u\|_{F_{ki}} + \|L'_{ki}u\|_{F_{ki}}\Big] \leq M\left[\|L_{ki}u\|_{F_{ki}} + \|u_j\|_{W^{l_k-1}_{p,\gamma}(0,b_{kj};Y_k,X_k)}\right]. \tag{4.23}$$

Now, using Theorem A1, we get that there is an $\varepsilon > 0$ and $C(\varepsilon)$ such that

$$\begin{aligned}\|u_j\|_{W^{l_k-1}_{p,\gamma_k}(0,b_{kj};Y_k,X_k)} &\leq \varepsilon\|u_j\|_{W^{l_k}_{p,\gamma_k}(0,b_{kj};Y_k,X_k)} + C(\varepsilon)\|u_j\|_{L_{p,\gamma_k}}\\ &\leq \varepsilon\|u_j\|_{W^l_{p,\gamma}(G_j;E(A),E)} + C(\varepsilon)\|u_j\|_{G_j,p,\gamma},\end{aligned} \tag{4.24}$$

where

$$(0,b_{kj}) = (0,b_k)\cap G_j. \tag{4.25}$$

Using the above estimates, we get

$$\|\Phi_{kj}\|_{F_{ki}} \leq M\|L_{ki}u\|_{F_{ki}} + \varepsilon\|u_j\|_{W^l_{p,\gamma}(G_j;E(A),E)} + C(\varepsilon)\|u_j\|_{G_j,p,\gamma}. \tag{4.26}$$

Consequently, from (4.22)–(4.26), we have

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i u_j\right\|_{G_j,p,\gamma} + \|Au_j\|_{G_j,p,\gamma}\\ &\qquad\leq C\|f\|_{G_j,p,\gamma} + \varepsilon\|u_j\|_{W^2_{p,\gamma}} + M(\varepsilon)\|u_j\|_{G_j,p,\gamma} + C\sum_{k=1}^{n}\sum_{i=1}^{l_k}\|f_{ki}\|_{F_{ki}}.\end{aligned} \tag{4.27}$$

Choosing $\varepsilon < 1$ from the above inequality, we obtain

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i u_j\right\|_{G_j,p,\gamma}+\left\|Au_j\right\|_{G_j,p,\gamma}\le C\left[\|f\|_{G_j,p,\gamma}+\left\|u_j\right\|_{G_j,p,\gamma}+\sum_{k=1}^{n}\sum_{i=1}^{l_k}\|f_{ki}\|_{F_{ki}}\right]. \quad (4.28)$$

Then, by using the equality $u(x)=\sum_{j=1}^{N}u_j(x)$ and the above estimates, we get (4.14). □

Condition 3. Suppose that part (1.1) of Condition 1 is satisfied and that E is a Banach space satisfying the multiplier condition with respect to $p\in(1,\infty)$ and the weighted function $\gamma=\prod_{k=1}^{n}x_k^{\gamma_k}(b_k-x_k)^{\nu_k}$, $0\le\gamma_k,\nu_k<1-1/p$.

Consider the problem (3.11). Reasoning as in the proof of Lemma 4.2, we obtain.

Proposition 4.3. *Assume Condition 3 hold and suppose that*

(1) *$A(x)$ is a uniformly R-positive operator in E for $\varphi\in[0,\pi/2)$, and that $a_k(x)$ are continuous functions on $\overline{G}$, $\lambda\in S_\varphi$,*

(2) *$A(x)A^{-1}(\overline{x})\in C(\overline{G};B(E))$ and $A_\infty A^{(1-|\alpha:l|-\mu)}\in L_\infty(G;B(E))$ for $0<\mu<1-|\alpha:l|$.*

Then, for all $\lambda\in S(\varphi)$ and for sufficiently large $|\lambda|$, the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i u\right\|_{L_{p,\gamma}(G;E)}+\|Au\|_{L_{p,\gamma}(G;E)}\le C\|f\|_{L_{p,\gamma}(G;E)}, \quad (4.29)$$

for the solution of problem (3.11).

Let O denote the operator generated by problem (3.11) for $\lambda=0$; that is,

$$D(O)=W_{p,\gamma}^{l}(G;E(A),E,L_{kj}),$$
$$Ou=\sum_{k=1}^{n}a_k(x)D_k^{l_k}u+A(x)u+\sum_{|\alpha:l|<1}A_\alpha(x)D^\alpha u. \quad (4.30)$$

Theorem 4.4. *Assume that Condition 3 is satisfied and that the following hold:*

(1) *$A(x)$ is a uniformly R-positive operator in E, and $a_k(x)$ are continuous functions on $\overline{G}$,*

(2) *$A(x)A^{-1}(\overline{x})\in C(\overline{G};B(E))$, and $A_\alpha A^{(1-|\alpha:l|-\mu)}\in L_\infty(G;B(E))$ for $0<\mu<1-|\alpha:l|$.*

Then, problem (3.11) *has a unique solution $u\in W_{p,\gamma}^{l}(G;E(A),E)$ for $f\in L_{p,\gamma}(G;E)$ and $\lambda\in S_\varphi$ with large enough $|\lambda|$. Moreover, the following coercive uniform estimate holds:*

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i u\right\|_{L_{p,\gamma}(G;E)}+\|Au\|_{L_{p,\gamma}(G;E)}\le C\|f\|_{L_{p,\gamma}(G;E)}. \quad (4.31)$$

Proof. By Proposition 4.3 for $u \in W^l_{p,\gamma}(G; E(A), E)$, we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i u\right\|_{p,\gamma} + \|Au\|_{p,\gamma} \le C\left[\|(L+\lambda)u\|_{p,\gamma} + \|u\|_{p,\gamma}\right]. \tag{4.32}$$

It is clear that

$$\|u\|_{p,\gamma} = \frac{1}{|\lambda|}\|(L+\lambda)u - Lu\|_{p,\gamma} \le \frac{1}{|\lambda|}\left[\|(L+\lambda)u\|_{p,\gamma} + \|Lu\|_{p,\gamma}\right]. \tag{4.33}$$

Hence, by using the definition of $W^l_{p,\gamma}(G; E(A), E)$ and applying Theorem A1, we obtain

$$\|u\|_{p,\gamma} \le \frac{1}{|\lambda|}\left[\|(L+\lambda)u\|_{p,\gamma} + \|u\|_{W^l_{p,\gamma}(G;E(A),E)}\right]. \tag{4.34}$$

From the above estimate, we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i u\right\|_{p,\gamma} + \|Au\|_{p,\gamma} \le C\|(L+\lambda)u\|_{p,\gamma}. \tag{4.35}$$

The estimate (4.35) implies that problem (3.11) has a unique solution and that the operator $O+\lambda$ has a bounded inverse in its rank space. We need to show that this rank space coincides with the space $L_{p,\gamma}(G;E)$; that is, we have to show that for all $f \in L_{p,\gamma}(G;E)$, there is a unique solution of the problem (3.11). We consider the smooth functions $g_j = g_j(x)$ with respect to a partition of unity $\varphi_j = \varphi_j(y)$ on the region G that equals one on supp φ_j, where supp $g_j \subset G_j$ and $|g_j(x)| < 1$. Let us construct for all j the functions u_j that are defined on the regions $\Omega_j = G \cap G_j$ and satisfying problem (3.11). The problem (3.11) can be expressed as

$$\begin{gathered}\sum_{k=1}^{n} a_k(x_{0j}) D_k^{l_k} u_j(x) + A_\lambda(x_{0j}) u_j(x) \\ = g_j\left\{ f + \left[A(x_{0j}) - A(x)\right]u_j + \sum_{k=1}^{n}\left[a_k(x) - a_k(x_{0j})\right]D_k^{l_k}u_j - \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u_j \right\}, \\ L_{ki}u_j = 0,\ \ j = 1,2,\ldots,N.\end{gathered} \tag{4.36}$$

Consider operators $O_{j\lambda}$ in $L_{p,\gamma}(G_j;E)$ that are generated by the BVPs (4.17); that is,

$$\begin{gathered} D(O_{j\lambda}) = W^l_{p,\gamma}(G_j; E(A), E, L_{ki}), \quad i = 1,2,\ldots,l_k,\ k = 1,2,\ldots,n, \\ O_{j\lambda}u = \sum_{k=1}^{n} a_k(x_{0j}) D_k^{l_k} u_j(x) + A_\lambda(x_{0j}) u_j(x), \quad j = 1,\ldots,N. \end{gathered} \tag{4.37}$$

By virtue of Theorem A6, the operators $O_{j\lambda}$ have inverses $O_{j\lambda}^{-1}$ for $|\arg\lambda| \le \varphi$ and for sufficiently large $|\lambda|$. Moreover, the operators $O_{j\lambda}^{-1}$ are bounded from $L_{p,\gamma}(G_j;E)$ to $W_{p,\gamma}^{l}(G_j;E(A),E)$, and for all $f \in L_{p,\gamma}(G_j;E)$, we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^i O_{j\lambda}^{-1} f\right\|_{L_{p,\gamma}(G_j;E)} + \left\|A O_{j\lambda}^{-1} f\right\|_{L_{p,\gamma}(G_j;E)} \le C\|f\|_{L_{p,\gamma}(G_j;E)}. \tag{4.38}$$

Extending u_j to zero outside of $\operatorname{supp}\varphi_j$ in the equalities (4.36), and using the substitutions $u_j = O_{j\lambda}^{-1}\upsilon_j$, we obtain the operator equations

$$\upsilon_j = K_{j\lambda}\upsilon_j + g_j f, \quad j = 1,2,\ldots,N, \tag{4.39}$$

where $K_{j\lambda}$ are bounded linear operators in $L_p(G_j;E)$ defined by

$$K_{j\lambda} = g_j\left\{ f + \left[A(x_{0j}) - A(x)\right]O_{j\lambda}^{-1} + \sum_{k=1}^{n}\left[a_k(x) - a_k(x_{0j})\right]D_k^{l_k}O_{j\lambda}^{-1} - \sum_{|\alpha:l|<1} A_\alpha(x)D^\alpha O_{j\lambda}^{-1}\right\}. \tag{4.40}$$

In fact, because of the smoothness of the coefficients of the expression $K_{j\lambda}$ and from the estimate (4.38), for $|\arg\lambda| \le \varphi$ with sufficiently large $|\lambda|$, there is a sufficiently small $\varepsilon > 0$ such that

$$\begin{gathered}\left\|\left[A(x_{0j}) - A(x)\right]O_{j\lambda}^{-1}\upsilon_j\right\|_{L_{p,\gamma}(G_j;E)} \le \varepsilon\|\upsilon_j\|_{L_{p,\gamma}(G_j;E)}, \\ \sum_{k=1}^{n}\left\|\left[a_k(x) - a_k(x_{0j})\right]D_k^{l_k}O_{j\lambda}^{-1}\upsilon_j\right\|_{L_{p,\gamma}(G_j;E)} \le \varepsilon\|\upsilon_j\|_{L_{p,\gamma}(G_j;E)}.\end{gathered} \tag{4.41}$$

Moreover, from assumption (2.2) of Theorem 4.4 and Theorem A1 for $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that

$$\sum_{|\alpha:l|<1}\left\|A_\alpha(x)D^\alpha O_{j\lambda}^{-1}\upsilon_j\right\|_{L_{p,\gamma}(G_j;E)} \le \varepsilon\|\upsilon_j\|_{W_{p,\gamma}^{l}(G_j;E(A),E)} + C(\varepsilon)\|\upsilon_j\|_{L_{p,\gamma}(G_j;E)}. \tag{4.42}$$

Hence, for $|\arg\lambda| \le \varphi$ with sufficiently large $|\lambda|$, there is a $\delta \in (0,1)$ such that $\|K_{j\lambda}\| < \delta$. Consequently, (4.39) for all j have a unique solution $\upsilon_j = [I - K_{j\lambda}]^{-1}g_j f$. Moreover,

$$\|\upsilon_j\|_{L_{p,\gamma}(G_j;E)} = \left\|\left[I - K_{j\lambda}\right]^{-1}g_j f\right\|_{L_{p,\gamma}(G_j;E)} \le \|f\|_{L_{p,\gamma}(G_j;E)}. \tag{4.43}$$

Thus, $[I - K_{j\lambda}]^{-1}g_j$ are bounded linear operators from $L_{p,\gamma}(G;E)$ to $L_{p,\gamma}(G_j;E)$. Thus, the functions

$$u_j = U_{j\lambda}f = O_{j\lambda}^{-1}\left[I - K_{j\lambda}\right]^{-1}g_j f \tag{4.44}$$

are solutions of (4.38). Consider the following linear operator $(U+\lambda)$ in $L_p(G;E)$ defined by

$$D(U+\lambda) = W^l_{p,\gamma}(G;E(A),E,L_{kj}), \quad j=1,2,\ldots,l_k,\ k=1,2,\ldots,n,$$
$$(U+\lambda)f = \sum_{j=1}^{N}\varphi_j(y)U_{j\lambda}f = O^{-1}_{j\lambda}\left[I-K_{j\lambda}\right]^{-1}g_j f. \tag{4.45}$$

It is clear from the constructions U_j and from the estimate (4.39) that the operators $U_{j\lambda}$ are bounded linear from $L_{p,\gamma}(G;E)$ to $W^l_{p,\gamma}(G_j;E(A),E)$, and for $|\arg\lambda|\le\varphi$ with sufficiently large $|\lambda|$, we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D^i_k U_{j\lambda}f\right\|_p + \left\|AU_{j\lambda}f\right\|_p \le C\|f\|_p. \tag{4.46}$$

Therefore, $(U+\lambda)$ is a bounded linear operator in $L_{p,\gamma}(G;E)$. Since the operators $U_{j\lambda}$ coincide with the inverse of the operator O_λ in $L_{p,\gamma}(G_j;E)$, then acting on O_λ to $u=\sum_{j=1}^{N}\varphi_j U_{j\lambda}f$ gives

$$O_\lambda u = \sum_{j=1}^{N}\varphi_j O_\lambda(U_{j\lambda}f) + \Phi_\lambda f = f + \sum_{j=1}^{N}\Phi_{j\lambda}f, \tag{4.47}$$

where $\Phi_{j\lambda}$ are bounded linear operators defined by

$$\Phi_{j\lambda}f = \left\{\sum_{k=1}^{n}a_k\sum_{\nu=1}^{l_k}C_{k\nu}D^\nu_k\varphi_j D^{l_k-\nu}_k(U_{j\lambda}f) + \sum_{|\alpha:l|<1}A_\alpha\prod_{k=1}^{n}\sum_{\nu_k=1}^{\alpha_k}C_{\alpha,\nu_k}D^{\nu_k}_k\varphi_j D^{\alpha_k-\nu_k}_k(U_{j\lambda}f)\right\}. \tag{4.48}$$

Indeed, from Theorem A1 and estimate (4.46) and from the expression $\Phi_{j\lambda}$, we obtain that the operators $\Phi_{j\lambda}$ are bounded linear from $L_p(G;E)$ to $L_p(G;E)$, and for $|\arg\lambda|\le\varphi$ with sufficiently large $|\lambda|$, there is an $\varepsilon\in(0,1)$ such that $\|\Phi_{j\lambda}\|<\varepsilon$. Therefore, there exists a bounded linear invertible operator $(I+\sum_{j=1}^{N}\Phi_{j\lambda})^{-1}$; that is, we infer for all $f\in L_{p,\gamma}(G;E)$ that the BVP (3.11) has a unique solution

$$u(x) = O^{-1}_\lambda f = \sum_{j=1}^{N}\varphi_j O^{-1}_{j\lambda}\left[I-K_{j\lambda}\right]^{-1}g_j\left(I+\sum_{j=1}^{N}\Phi_{j\lambda}\right)^{-1}f. \tag{4.49}$$

□

Result 1. Theorem 4.4 implies that the resolvent $(O+\lambda)^{-1}$ satisfies the following anisotropic type sharp estimate:

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D^i_k(O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(G;E))} + \left\|A(O+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(G;E))} \le C, \tag{4.50}$$

for $|\arg\lambda|\le\varphi$, $\varphi\in[0,\pi/2)$.

Let Q denote the operator generated by BVP (3.1)-(3.2). From Theorem 4.4 and Remark 3.1, we get the following.

Result 2. Assume all the conditions of Theorem 4.4 hold. Then,

(a) the problem (3.1)-(3.2) for $f \in L_p(G;E)$, $|\arg\lambda| \leq \varphi$ and for sufficiently large $|\lambda|$ has a unique solution $u \in W_{p,\gamma}^{[l]}(G;E(A),E)$, and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\left\|D_k^{[i]}u\right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq M\|f\|_{L_p(G;E)}, \tag{4.51}$$

(b) if $A^{-1} \in \sigma_\infty(E)$, then the operator O is Fredholm from $W_{p,\gamma}^{[l]}(G;E(A),E)$ into $L_p(G;E)$.

Example 4.5. Now, let us consider a special case of (3.1)-(3.2). Let $E = \mathbf{C}$, $l_1 = 2$ and $l_2 = 4$, $n = 2$, $G = (0,1)\times(0,1)$ and $A = q$; that is, consider the problem

$$\begin{gathered} Lu = a_1 D_x^{[2]}u + a_2 D_y^{[4]}u + bD_x^{[1]}D_y^{[1]}u + a_0 u = f, \\ \sum_{i=0}^{m_{1j}}\alpha_{1i}D_x^{[i]}u(0,y) = 0, \quad \sum_{i=0}^{m_{1j}}\beta_{1i}u^{[i]}(1,y) = 0, \quad m_{1j} \in \{0,1\}, \\ \sum_{i=0}^{m_{2j}}\alpha_{2i}D_y^{[i]}u(x,0) = 0, \quad \sum_{i=0}^{m_{2j}}\beta_{2i}u^{[i]}(x,1) = 0, \quad 0 \leq m_{2j} \leq 3, \end{gathered} \tag{4.52}$$

where

$$\begin{gathered} D_x^{[i]} = \left[x^{\alpha_1}(1-x)^{\alpha_2}\frac{\partial}{\partial x}\right]^i, \qquad D_y^{[i]} = \left[y^{\beta_1}(1-y)^{\beta_2}\frac{\partial}{\partial y}\right]^i, \\ u = u(x,y), \qquad a_k \in C\left(\overline{G}\right), \qquad a_1 < 0, \qquad a_2 > 0. \end{gathered} \tag{4.53}$$

Theorem 4.4 implies that for each $f \in L_p(G)$, problem (4.52) has a unique solution $u \in W_p^{[l]}(G)$ satisfying the following coercive estimate:

$$\left\|D_x^{[2]}u\right\|_{L_p(G)} + \left\|D_y^{[4]}u\right\|_{L_p(G)} + \|u\|_{L_p(G)} \leq C\|f\|_{L_p(G)}. \tag{4.54}$$

Example 4.6. Let $l_k = 2m_k$ and $a_k = b_k(x)(-1)^{m_k}$, where b_k are positive continuous function on G, $E = \mathbf{C}^\nu$ and $A(x)$ is a diagonal matrix-function with continuous components $d_m(x) > 0$.

Then, we obtain the separability of the following BVPs for the system of anisotropic PDEs with varying coefficients:

$$\begin{gathered}\sum_{k=1}^{n}(-1)^{m_k}b_k(x)D_k^{2m_k}u_m(x)+d_m(x)u_m(x)=f_m(x),\\ \sum_{i=0}^{m_{kj}}\alpha_{kji}D_k^{i}u_m(G_{k0})=0,\qquad \sum_{i=0}^{m_{kj}}\beta_{kji}D_k^{i}u_m(G_{kb})=0,\\ j=1,2,\ldots,m_k,\quad m=1,2,\ldots,\nu,\end{gathered}\tag{4.55}$$

in the vector-valued space $L_{p,\gamma}(G;\mathbf{C}^{\nu})$.

5. The Spectral Properties of Anisotropic Differential Operators

Consider the following degenerated BVP:

$$\begin{gathered}\sum_{k=1}^{n}a_k(x)D_k^{[l_k]}u(x)+A(x)u(x)+\sum_{|\alpha:l|<1}A_\alpha(x)D^{[\alpha]}u(x)=f(x),\\ \sum_{i=0}^{m_{kj}}\alpha_{kji}D_k^{[i]}u(G_{k0})=0,\quad j=1,2,\ldots,d_k,\\ \sum_{i=0}^{m_{kj}}\beta_{kji}D_k^{[i]}u(G_{kb})=0,\quad j=1,2,\ldots,l_k-d_k,\ d_k\in(0,l_k),\end{gathered}\tag{5.1}$$

where

$$\begin{gathered}G=\{x=(x_1,x_2,\ldots,x_n),0<x_k<b_k\},\quad \alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n),\\ D^{[\alpha]}=D_1^{[\alpha_1]}D_2^{[\alpha_2]}\cdots D_n^{[\alpha_n]},\qquad D_k^{[i]}u(x)=\left[x_k^{\gamma_k}\frac{\partial}{\partial x_k}\right]^{i}u(x),\end{gathered}\tag{5.2}$$

Consider the operator Q generated by problem (5.1).

Theorem 5.1. *Let all the conditions of Theorem 4.4 hold for $\nu_k=0$ and $A^{-1}\in\sigma_\infty(E)$. Then, the operator Q is Fredholm from $W_{p,\gamma}^{[l]}(G;E(A),E)$ into $L_p(G;E)$.*

Proof. Theorem 4.4 implies that the operator $Q+\lambda$ for sufficiently large $|\lambda|$ has a bounded inverse $(O+\lambda)^{-1}$ from $L_p(G;E)$ to $W_p^{[l]}(G;E(A),E)$; that is, the operator $Q+\lambda$ is Fredholm from $W_p^{[l]}(G;E(A),E)$ into $L_p(G;E)$. Then, from Theorem A2 and the perturbation theory of linear operators, we obtain that the operator Q is Fredholm from $W_{p,\gamma}^{[l]}(G;E(A),E)$ into $L_p(G;E)$. □

Theorem 5.2. *Suppose that all the conditions of Theorem 5.1 are satisfied with $\nu_k = 0$. Assume that E is a Banach space with a basis and*

$$\varkappa = \sum_{k=1}^{n} \frac{1}{l_k} < 1, \qquad s_j(I(E(A),E)) \sim j^{-1/\nu}, \quad j = 1,2,\ldots,\infty,\ \nu > 0. \tag{5.3}$$

Then,

(a) *for a sufficiently large positive d*

$$s_j\Big((Q+d)^{-1}(L_p(G;E))\Big) \sim j^{-1/(\nu+\varkappa)}, \tag{5.4}$$

(b) *the system of root functions of the differential operator Q is complete in $L_p(G;E)$.*

Proof. Let $I(E_0,E)$ denote the embedding operator from E_0 to E. From Result 2, there exists a resolvent operator $(Q+d)^{-1}$ which is bounded from $L_p(G;E)$ to $W_{p,\gamma}^{[l]}(G;E(A),E)$. Moreover, from Theorem A4 and Remark 3.1, we get that the embedding operator

$$I\Big(W_{p,\gamma}^{[l]}(G;E(A),E), L_p(G;E)\Big) \tag{5.5}$$

is compact and

$$s_j\Big(I\Big(W_{p,\gamma}^{[l]}(G;E(A),E), L_p(G;E)\Big)\Big) \sim j^{-1/(\nu+\varkappa)}. \tag{5.6}$$

It is clear that

$$\begin{aligned}(Q+d)^{-1}(L_p(G;E)) = (Q+d)^{-1}\Big(L_p(G;E), W_{p,\gamma}^{[l]}(G;E(A),E)\Big)\\ \times I\Big(W_{p,\gamma}^{[l]}(G;E(A),E), L_p(G;E)\Big).\end{aligned} \tag{5.7}$$

Hence, from relations (5.6) and (5.7), we obtain (5.4). Now, Result 1 implies that the operator $Q+d$ is positive in $L_p(G;E)$ and

$$(Q+d)^{-1} \in \sigma_q(L_p(G;E)), \quad q > \frac{1}{\nu+\varkappa}. \tag{5.8}$$

Then, from (4.52) and (5.6), we obtain assertion (b). □

Consider now the operator O in $L_{p,\gamma}(G;E)$ generated by the nondegenerate BVP obtained from (5.1) under the mapping (3.7); that is,

$$D(O) = W^{l}_{p,\gamma}(G;E(A),E,L_{kj}),$$
$$Ou = \sum_{k=1}^{n} a_k(x) D_k^{l_k} u + A(x)u(x) + \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u. \tag{5.9}$$

From Theorem 5.2 and Remark 3.1, we get the following.

Result 3. Let all the conditions of Theorem 5.1 hold. Then, the operator O is Fredholm from $W^{l}_{p,\gamma}(G;E(A),E)$ into $L_{p,\gamma}(G;E)$.

Result 4. Then,

(a) for a sufficiently large positive d

$$s_j\Big((O+d)^{-1}(L_p(G;E))\Big) \sim j^{-1/(\nu+\varkappa)}, \tag{5.10}$$

(b) the system of root functions of the differential operator O is complete in $L_{p,\gamma}(G;E)$.

6. BVPs for Degenerate Quasielliptic PDE

In this section, maximal regularity properties of degenerate anisotropic differential equations are studied. Maximal regularity properties for PDEs have been studied, for example, in [3] for smooth domains and in [28] for nonsmooth domains.

Consider the BVP

$$Lu = \sum_{k=1}^{n} a_k(x) D_k^{[l_k]} u(x,y) + \sum_{|\beta|\le 2m} a_\beta(y) D_y^\beta u(x,y) + \sum_{|\alpha:l|<1} v_\alpha(x,y) D_y^{[\alpha]} u(x,y)$$
$$= f(x,y), \quad x\in G,\ y\in\Omega,$$
$$\sum_{i=0}^{m_{kj}} \alpha_{kji} D_k^{[i]} u(G_{k0},y) = 0, \quad x(k)\in G_k,\ y\in\Omega,\ j=1,2,\ldots,d_k,$$
$$\sum_{i=0}^{m_{kj}} \beta_{kji} D_k^{[i]} u(G_{kb},y) = 0, \quad x(k)\in G_k,\ y\in\Omega, \tag{6.1}$$
$$j=1,2,\ldots,l_k-d_k, \quad d_k\in(0,l_k),$$
$$B_j u = \sum_{|\beta|=m_j} b_{j\beta}(y) D_y^\beta u(x,y)\Big|_{y\in\partial\Omega} = 0, \quad x\in G,\ j=1,2,\ldots,m,$$

where $D_j = -i(\partial/\partial y_j)$, α_{kji}, β_{kji} are complex number, $y = (y_1, \dots, y_\mu) \in \Omega \subset R^\mu$ and

$$\begin{gathered} G = \{x = (x_1, x_2, \dots, x_n), 0 < x_k < b_k\}, \qquad D_k^{[i]} = \left(x_k^{\gamma_k}(b_k - x_k)^{\nu_k}\frac{\partial}{\partial x_k}\right)^i, \\ G_{k0} = (x_1, x_2, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n), \\ G_{kb} = (x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n), \\ 0 \le m_{kj} \le l_k - 1, \qquad |\alpha_{kj}| + |\beta_{kj}| > 0, \quad j = 1, 2, \dots, l_k. \\ x(k) = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n), \qquad G_k = \prod_{j \ne k}(0, b_j), \\ j, k = 1, 2, \dots, n. \end{gathered} \tag{6.2}$$

Let $\tilde{\Omega} = G \times \Omega$, $\mathbf{p} = (p_1, p)$. Now, $L_{\mathbf{p}}(\tilde{\Omega})$ will denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm (see, e.g., [29, Section 1, page 6]), that is, the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L_{\mathbf{p}}(\tilde{\Omega})} = \left(\int_G \left(\int_\Omega |f(x,y)|^{p_1} dx\right)^{p/p_1} dy\right)^{1/p} < \infty. \tag{6.3}$$

Analogously, $W_{\mathbf{p}}^m(\tilde{\Omega})$ denotes the Sobolev space with corresponding mixed norm.

Let $\omega_{kj} = \omega_{kj}(x)$, $j = 1, 2, \dots, l_k$, $k = 1, 2, \dots, n$ denote the roots of the equations

$$a_k(x)\omega^{l_k} + 1 = 0. \tag{6.4}$$

Let Q denote the operator generated by BVP (6.1). Let

$$F = B\left(L_{\mathbf{p}}\left(\tilde{\Omega}\right)\right). \tag{6.5}$$

Theorem 6.1. *Let the following conditions be satisfied:*

(1) $a_\alpha \in C(\overline{\Omega})$ *for each* $|\alpha| = 2m$ *and* $a_\alpha \in [L_\infty + L_{r_k}](\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \ge p_1$, $p_1 \in (1, \infty)$ *and* $2m - k > l/r_k$, $\nu_\alpha \in L_\infty$,

(2) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ *for each* $j, \beta, m_j < 2m$, $\gamma = \prod_{k=1}^n x_k^{\gamma_k}(b_k - x_k)^{\nu_k}$, $0 \le \gamma_k$, $\nu_k < 1 - 1/p$, $p \in (1, \infty)$,

(3) *for* $y \in \overline{\Omega}$, $\xi \in R^\mu$, $\eta \in S(\varphi_1)$, $\varphi_1 \in [\,0, \pi/2)$, $|\xi| + |\eta| \ne 0$ *let*

$$\eta + \sum_{|\alpha|=2m} a_\alpha(y)\xi^\alpha \ne 0, \tag{6.6}$$

(4) *for each* $y_0 \in \partial\Omega$, *the local BVPs in local coordinates corresponding to* y_0

$$\eta + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) = 0,$$
$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta \vartheta(y) = h_j, \quad j = 1, 2, \ldots, m \tag{6.7}$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, \ldots, h_m) \in R^m$ *and for* $\xi' \in R^{\mu-1}$ *with* $|\xi'| + |\eta| \neq 0$,

(5) $a_k \in C(\overline{G})$, $a_k(x) \neq 0$ *and*

$$\left|\arg \omega_{kj} - \pi\right| \leq \frac{\pi}{2} - \varphi, \quad j = 1, 2, \ldots, d_k,$$
$$\left|\arg \omega_{kj}\right| \leq \frac{\pi}{2} - \varphi, \quad \varphi \in \left[0, \frac{\pi}{2}\right), \tag{6.8}$$
$$j = d_k + 1, \ldots, l_k, \quad 0 < d_k < l_k, \quad k = 1, 2, \ldots, n, \quad v_\alpha \in L_\infty(G), \quad x \in G.$$

Then,

(a) *the following coercive estimate*

$$\sum_{k=1}^{n} \left\| D_k^{[l_k]} u \right\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \sum_{|\beta|=2m} \left\| D_y^\beta u \right\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \|u\|_{L_{\mathbf{p}}(\tilde{\Omega})} \leq C \|f\|_{L_{\mathbf{p}}(\tilde{\Omega})} \tag{6.9}$$

holds for the solution $u \in W_{\mathbf{p},\gamma}^{[l],2m}(\tilde{\Omega})$ *of problem* (6.1),

(b) *for* $\lambda \in S(\varphi)$ *and for sufficiently large* $|\lambda|$, *there exists a resolvent* $(Q + \lambda)^{-1}$ *and*

$$\sum_{k=1}^{n} \sum_{i=0}^{l_k} |\lambda|^{1-i/l_k} \left\| D_k^i (Q + \lambda)^{-1} \right\|_F + \left\| A(Q + \lambda)^{-1} \right\|_F \leq M, \tag{6.10}$$

(c) *the problem* (6.1) *for* $\nu_k = 0$ *is Fredholm in* $L_{\mathbf{p}}(\tilde{\Omega})$,

(d) *the relation with* $\nu_k = 0$

$$s_j\left((Q + \lambda)^{-1}\left(L_{p,q}\left(\tilde{\Omega}\right)\right)\right) \sim j^{-1/(l_0 + \varkappa_0)}, \quad l_0 = \sum_{k=1}^{n} \frac{1}{l_k}, \ \varkappa_0 = \frac{\mu}{2m} \tag{6.11}$$

holds,

(e) *for* $\nu_k = 0$ *the system of root functions of the BVP* (6.1) *is complete in* $L_{\mathbf{p}}(\tilde{\Omega})$.

Proof. Let $E = L_{p_1}(\Omega)$. Then, from [3, Theorem 3.6], part (1.1) of Condition 1 is satisfied. Consider the operator A which is defined by

$$D(A) = W_{p_1}^{2m}(\Omega; B_j u = 0), \qquad Au = \sum_{|\beta| \le 2m} a_\beta(y) D^\beta u(y). \tag{6.12}$$

For $x \in \Omega$, we also consider operators

$$A_\alpha(x)u = v_\alpha(x, y) D^\alpha u(y), \quad |\alpha : l| < 1. \tag{6.13}$$

The problem (6.1) can be rewritten as the form of (3.1)-(3.2), where $u(x) = u(x, \cdot)$ and $f(x) = f(x, \cdot)$ are functions with values in $E = L_{p_1}(\Omega)$. From [3, Theorem 8.2] problem

$$\begin{gathered} \eta u(y) + \sum_{|\beta| \le 2m} a_\beta(y) D^\beta u(y) = f(y), \\ B_j u = \sum_{|\beta| \le m_j} b_{j\beta}(y) D^\beta u(y) = 0, \quad j = 1, 2, \ldots, m \end{gathered} \tag{6.14}$$

has a unique solution for $f \in L_{p_1}(\Omega)$ and $\arg \eta \in S(\varphi_1)$, $|\eta| \to \infty$. Moreover, the operator A, generated by (5.8) is R-positive in L_{p_1}; that is, part (2.2) of Condition 1 holds. From (2.2), (3.7), and by [29, Section 18], we have

$$\sum_{|\alpha : l| < 1} \|A_\alpha(x)u\|_{L_{p_1}} \le C \sum_{|\alpha : l| < 1} \|D^\alpha u\|_{L_{p_1}} \le \varepsilon \|u\|_{B_{p_1,1}^{2m(1-|\alpha : l|)}} + \|u\|_{L_{p_1}}, \tag{6.15}$$

that is, all the conditions of Theorem 5.2 and Result 4 are fulfilled. As a result, we obtain assertion (a) and (b) of the theorem. Also, it is known (e.g., [27, Theorem 3.2.5, Section 4.10]) that the embedding $W_{p_1}^{2m}(\Omega) \subset L_{p_1}(\Omega)$ is compact and

$$s_j\Big(I\Big(W_{p_1}^{2m}(G), L_{p_1}(G)\Big)\Big) \sim j^{-1/\varkappa_0}. \tag{6.16}$$

Then, Results 3 and 4 imply assertions (c), (d), (e). □

7. Boundary Value Problems for Infinite Systems of Degenerate PDE

Consider the infinity systems of BVP for the degenerate anisotropic PDE

$$\begin{gathered}
\sum_{k=1}^{n} a_k(x) D_k^{[l_k]} u_m(x) + \sum_{j=1}^{\infty} (d_j(x) + \lambda) u_m(x) + \sum_{|\alpha:l|<1} \sum_{j=1}^{\infty} d_{\alpha jm}(x) D^{[\alpha]} u_m(x) \\
= f_m(x), \quad x \in G, \ m = 1,2,\dots,\infty, \\
\sum_{i=0}^{m_{kj}} \alpha_{kji} D_k^{[i]} u(G_{k0}) = 0, \quad x(k) \in G_k, \ j = 1,2,\dots,d_k, \\
\sum_{i=0}^{m_{kj}} \beta_{kji} D_k^{[i]} u(G_{kb}) = 0, \quad x(k) \in G_k, \ j = 1,2,\dots,l_k - d_k,
\end{gathered} \tag{7.1}$$

where $d_k \in (0, l_k)$, a_k are complex-valued functions, α_{kji}, β_{kji} are complex numbers. Let

$$\begin{gathered}
G = \{x = (x_1, x_2, \dots, x_n), 0 < x_k < b_k\}, \qquad D_k^{[i]} = \left(x_k^{\gamma_k} (b_k - x_k)^{\nu_k} \frac{\partial}{\partial x_k} \right)^i, \\
G_{k0} = (x_1, x_2, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n), \qquad |\alpha_{kj}| + |\beta_{kj}| > 0, \\
G_{kb} = (x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n), \quad 0 \le m_{kj} \le l_k - 1, \\
x(k) = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n), \qquad G_k = \prod_{j \ne k} (0, b_j), \quad j,k = 1,2,\dots,n, \\
D(x) = \{d_m(x)\}, \quad d_m > 0, \quad u = \{u_m\}, \quad du = \{d_m u_m\}, \quad m = 1,2,\cdots\infty, \\
l_q(D) == \left\{ u : u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q \right)^{1/q} < \infty \right\}, \\
j = 1,2,\dots,l_k, \ k = 1,2,\dots,n.
\end{gathered} \tag{7.2}$$

Let V denote the operator in $L_p(G; l_q)$ generated by problem (7.1). Let

$$B = B\big(L_p(G; l_q)\big). \tag{7.3}$$

Theorem 7.1. *Let* $\gamma = \prod_{k=1}^{n} x_k^{\gamma_k}(b_k - x_k)^{\nu_k}$, $0 \le \gamma_k, \nu_k < 1 - 1/p$, $p \in (1,\infty)$, $a_k \in C(\overline{G})$, $a_k(x) \neq 0$, *and* $|\arg \omega_{kj} - \pi| \le \pi/2 - \varphi$, $|\arg \omega_{kj}| \le \pi/2 - \varphi$, $j = 1,2,\ldots,l_k$, $\varphi \in \varphi \in [0,\pi/2)$, $x \in G$, $d_m \in C(\overline{G})$, $d_{\alpha m} \in L_\infty(G)$ *such that*

$$\max_{\alpha} \sup_{m} \sum_{j=1}^{\infty} d_{\alpha jm}(x) d_j^{-(1-|\alpha:l|-\mu)} < M, \quad \forall x \in G,\ 0 < \mu < 1 - |\alpha : l|. \tag{7.4}$$

Then,

(a) *for all* $f(x) = \{f_m(x)\}_1^\infty \in L_p(G;l_q)$, *for* $|\arg \lambda| \le \varphi$ *and sufficiently large* $|\lambda|$, *the problem* (7.1) *has a unique solution* $u = \{u_m(x)\}_1^\infty$ *that belongs to the space* $W_{p,\gamma}^{[l]}(G, l_q(D), l_q)$ *and the following coercive estimate holds:*

$$\begin{aligned} &\sum_{k=1}^{n} \left[\int_G \left(\sum_{m=1}^{\infty} \left| D_k^{[l_k]} u_m(x) \right|^q \right)^{p/q} dx \right]^{1/p} + \left[\int_G \left(\sum_{m=1}^{\infty} |d_m u_m(x)|^q \right)^{p/q} dx \right]^{1/p} \\ &\quad \le C \left[\int_G \left(\sum_{m=1}^{\infty} |f_m(x)|^q \right)^{p/q} dx \right]^{1/p}, \end{aligned} \tag{7.5}$$

(b) *there exists a resolvent* $(V+\lambda)^{-1}$ *of the operator* V *and*

$$\sum_{k=1}^{n} \sum_{j=0}^{l_k} |\lambda|^{1-j/l_k} \left\| D_k^{[j]} (V+\lambda)^{-1} \right\|_B + \left\| A(V+\lambda)^{-1} \right\|_B \le M, \tag{7.6}$$

(c) *for* $\nu_k = 0$, *the system of root functions of the BVP* (7.1) *is complete in* $L_p(G;l_q)$.

Proof. Let $E = l_q$, A and $A_\alpha(x)$ be infinite matrices such that

$$A = [d_m \delta_{mj}], \quad A_\alpha(x) = [d_{\alpha jm}(x)], \quad m,j = 1,2,\ldots,\infty. \tag{7.7}$$

It is clear that the operator A is R-positive in l_q. The problem (7.1) can be rewritten in the form (1.1). From Theorem 4.4, we obtain that problem (7.1) has a unique solution $u \in W_p^l(G; l_q(D), l_q)$ for all $f \in L_p(G;l_q)$ and

$$\sum_{k=1}^{n} \sum_{i=0}^{l_k} |\lambda|^{1-i/l_k} \left\| D_k^{[i]} u \right\|_{L_p(G;l_q)} + \|Au\|_{L_p(G;l_q)} \le M \|f\|_{L_p(G;l_q)}. \tag{7.8}$$

From the above estimate, we obtain assertions (a) and (b). The assertion (c) is obtained from Result 4. □

References

[1] M. S. Agranovich, "Spectral problems in Lipschitz domains for strongly elliptic systems in the Banach spaces H_p^σ and B^σ," *Functional Analysis and Its Applications*, vol. 42, no. 4, pp. 249–267, 2008.

[2] A. Ashyralyev, "On well-posedness of the nonlocal boundary value problems for elliptic equations," *Numerical Functional Analysis and Optimization*, vol. 24, no. 1-2, pp. 1–15, 2003.

[3] R. Denk, M. Hieber, and J. Prüss, R-*Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, Mem. Amer. Math. Soc., American Mathematical Society, 2003.

[4] A. Favini, V. Shakhmurov, and Y. Yakubov, "Regular boundary value problems for complete second order elliptic differential-operator equations in UMD Banach spaces," *Semigroup Forum*, vol. 79, no. 1, pp. 22–54, 2009.

[5] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Differential-Operator Equations*, "Naukova Dumka", Kiev, Ukraine, 1984.

[6] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, The Clarendon Press/Oxford University Press, New York, NY, USA, 1985.

[7] J.-L. Lions and E. Magenes, *Nonhomogenous Boundary Value Problems*, Mir, Moscow, Russia, 1971.

[8] A. Ya. Shklyar, *Complete Second Order Linear Differential Equations in Hilbert Spaces*, vol. 92 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1997.

[9] V. B. Shakhmurov, "Nonlinear abstract boundary-value problems in vector-valued function spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 3, pp. 745–762, 2007.

[10] V. B. Shakhmurov, "Theorems on the embedding of abstract function spaces and their applications," *Matematicheskiĭ Sbornik. Novaya Seriya*, vol. 134(176), no. 2, pp. 260–273, 1987.

[11] V. B. Shakhmurov, "Embedding theorems and their applications to degenerate equations," *Differentsial'nye Uravneniya*, vol. 24, no. 4, pp. 672–682, 1988.

[12] S. Yakubov, *Completeness of Root Functions of Regular Differential Operators*, vol. 71 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, UK, 1994.

[13] S. Yakubov, "A nonlocal boundary value problem for elliptic differential-operator equations and applications," *Integral Equations and Operator Theory*, vol. 35, no. 4, pp. 485–506, 1999.

[14] S. Yakubov and Y. Yakubov, *Differential-Operator Equations: Ordinary and Partial Differential Equations*, vol. 103 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2000.

[15] R. P. Agarwal, R. Bohner, and V. B. Shakhmurov, "Maximal regular boundary value problems in Banach-valued weighted spaces," *Boundary Value Problems*, vol. 2005, no. 1, pp. 9–42, 2005.

[16] V. B. Shakhmurov, "Separable anisotropic differential operators and applications," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1182–1201, 2007.

[17] H. Amann, "Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications," *Mathematische Nachrichten*, vol. 186, pp. 5–56, 1997.

[18] H. Amann, *Linear and Quasi-Linear Equations*, Birkhauser, 1995.

[19] S. G. Krein, *Linear Differential Equations in Banach Space*, American Mathematical Society, Providence, RI, USA, 1971.

[20] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications, 16, Birkhäuser, Basel, Switzerland, 1995.

[21] V. P. Orlov, "Regular degenerate differential operators of arbitrary order with unbounded operators coefficients," *Proceedings of Voronej State University*, vol. 2, pp. 33–41, 1974.

[22] P. E. Sobolevskiĭ, "Inequalities coerciveness for abstract parabolic equations," *Doklady Akademii Nauk SSSR*, vol. 57, pp. 27–40, 1964.

[23] V. B. Shakhmurov, "Coercive boundary value problems for regular degenerate differential-operator equations," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 605–620, 2004.

[24] V. B. Shakhmurov, "Degenerate differential operators with parameters," *Abstract and Applied Analysis*, vol. 2007, Article ID 51410, 27 pages, 2007.

[25] Lutz Weis, "Operator-valued Fourier multiplier theorems and maximal L_p-regularity," *Mathematische Annalen*, vol. 319, no. 4, pp. 735–758, 2001.

[26] D. L. Burkholder, "A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions," in *Proceedings of the Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. I, II*, Wadsworth Math. Ser., pp. 270–286, Chicago, Ill, USA, 1981.

[27] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, vol. 18 of *North-Holland Mathematical Library*, North-Holland, Amsterdam, The Netherlands, 1978.

[28] P. Grisvard, *Elliptic Problems in Non Smooth Domains*, Pitman, 1985.

[29] O. V. Besov, V. P. Ilin, and S. M. Nikolskii, *Integral Representations of Functions and Embedding Theorems*, Wiley, New York, NY, USA, 1978.