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*Research Article*

# Second-Order Boundary Value Problem with Integral Boundary Conditions

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The nonlinear alternative of the Leray Schauder type and the Banach contraction principle are used to investigate the existence of solutions for second-order differential equations with integral boundary conditions. The compactness of solutions set is also investigated.

## 1. Introduction

This paper is concerned with the existence of solutions for the second-order boundary value problem

$$\begin{gathered} -y''(t) = f\big(t, y(t)\big), \quad \text{a.e. } t \in (0,1), \\ y(0) = 0, \qquad y(1) = \int_0^1 g(s)y(s)ds, \end{gathered} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function and $g : [0,1] \to \mathbb{R}$ is an integrable function.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [1–9] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for example [10–14]. The goal of this paper is to give existence and uniqueness results for the problem (1.1).

Our approach here is based on the Banach contraction principle and the Leray-Schauder alternative [15].

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $AC^1((0,1),\mathbb{R})$ be the space of differentiable functions $y : (0,1) \to \mathbb{R}$, whose first derivative, $y'$, is absolutely continuous.

We take $C([0,1],\mathbb{R})$ to be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le 1\}, \tag{2.1}$$

and we let $L^1([0,1],\mathbb{R})$ denote the Banach space of functions $y : [0,1] \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^1 |y(t)|dt. \tag{2.2}$$

*Definition 2.1.* A map $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be $L^1$-Carathéodory if

(i) $t \mapsto f(t,u)$ is measurable for each $u \in \mathbb{R}$,

(ii) $u \mapsto f(t,u)$ is continuous for almost each $t \in [0,1]$,

(iii) for every $r > 0$ there exists $h_r \in L^1([0,1],\mathbb{R})$ such that

$$|f(t,u)| \le h_r(t) \quad \text{for a.e. } t \in [0,1] \text{ and all } |u| \le r. \tag{2.3}$$

## 3. Existence and Uniqueness Results

*Definition 3.1.* A function $y \in AC^1((0,1),\mathbb{R})$ is said to be a solution of (1.1) if $y$ satisfies (1.1).

In what follows one assumes that $g_* = \int_0^1 sg(s)ds \neq 1$. One needs the following auxiliary result.

**Lemma 3.2.** *. Let $\sigma : L^1([0,1],\mathbb{R})$. Then the function defined by*

$$y(t) = \int_0^1 H(t,s)\sigma(s)ds \tag{3.1}$$

*is the unique solution of the boundary value problem*

$$\begin{gathered} -y''(t) = \sigma(t), \quad \text{a.e. } t \in (0,1), \\ y(0) = 0, \qquad y(1) = \int_0^1 g(s)y(s)ds, \end{gathered} \tag{3.2}$$

*where*

$$H(t,s)=G(t,s)+\frac{t}{1-\int_0^1 sg(s)ds}\int_0^1 G(r,s)g(r)dr,$$
$$G(t,s)=\begin{cases} s(1-t) & \text{if } 0\le s\le t\le 1,\\ t(1-s) & \text{if } 0\le t\le s\le 1.\end{cases} \tag{3.3}$$

*Proof.* Let $y$ be a solution of the problem (3.2). Then integratingly, we obtain

$$y(t)=y(0)+ty'(0)-\int_0^t (t-s)\sigma(s)ds,$$
$$y(1)=y'(0)-\int_0^1 (1-s)\sigma(s)ds. \tag{3.4}$$

Hence

$$y(t)=\int_0^1 tg(s)y(s)ds+\int_0^1 t(1-s)\sigma(s)ds-\int_0^t (t-s)\sigma(s)ds, \tag{3.5}$$

$$y(t)=\int_0^1 tg(s)y(s)ds+\int_0^1 G(t,s)\sigma(s)ds, \tag{3.6}$$

where

$$G(t,s)=\begin{cases} s(1-t) & \text{if } 0\le s\le t\le 1,\\ t(1-s) & \text{if } 0\le t\le s\le 1.\end{cases} \tag{3.7}$$

Now, multiply (3.6) by $g$ and integrate over $(0,1)$, to get

$$\int_0^1 g(s)y(s)ds=\int_0^1 g(s)\left[s\int_0^1 g(r)y(r)dr+\int_0^1 G(s,r)\sigma(r)dr\right]ds$$
$$=\int_0^1 sg(s)\left[\int_0^1 g(s)y(s)ds\right]+\int_0^1 g(s)\left[\int_0^1 G(s,r)\sigma(r)dr\right]ds. \tag{3.8}$$

Thus,

$$\int_0^1 g(s)y(s)ds=\frac{\int_0^1 g(s)\left[\int_0^1 G(s,r)\sigma(r)dr\right]ds}{1-\int_0^1 sg(s)ds}. \tag{3.9}$$

Substituting in (3.6) we have

$$y(t)=\int_0^1 G(t,s)\sigma(s)ds+\frac{t\int_0^1 g(s)\left[\int_0^1 G(s,r)\sigma(r)dr\right]ds}{1-\int_0^1 sg(s)ds}. \tag{3.10}$$

Therefore

$$y(t)=\int_0^1 H(t,s)\sigma(s)ds. \tag{3.11}$$

Set $g^*=|1-g_*|$. Note that

$$|G(t,s)|\leq\frac{1}{4}\quad\text{for }(t,s)\in[0,1]\times[0,1]. \tag{3.12}$$

Our first result reads □

**Theorem 3.3.** *Assume that $f$ is an $L^1$-Carathéodory function and the following hypothesis*

(A1) *There exists $l\in L^1([0,1],\mathbb{R}_+)$ such that*

$$\left|f(t,x)-f(t,\overline{x})\right|\leq l(t)|x-\overline{x}|\quad\forall\ x,\overline{x}\in\mathbb{R},\ t\in[0,1] \tag{3.13}$$

*holds. If*

$$\|l\|_{L^1}+\frac{\|g\|_{L^1}\|l\|_{L^1}}{g^*}<4, \tag{3.14}$$

*then the BVP* (1.1) *has a unique solution.*

*Proof.* Transform problem (1.1) into a fixed-point problem. Consider the operator $N$ : $C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ defined by

$$N(y)(t)=\int_0^1 H(t,s)f(s,y(s))ds,\quad t\in[0,1]. \tag{3.15}$$

We will show that $N$ is a contraction. Indeed, consider $y,\overline{y}\in C([0,1],\mathbb{R})$. Then we have for each $t\in[0,1]$

$$\begin{aligned}\left|N(y)(t)-N(\overline{y})(t)\right|&\leq\int_0^1|H(t,s)|\left|f(s,y(s))-f(s,\overline{y}(s))\right|ds\\&\leq\int_0^1|G(t,s)|l(s)\left|y(s)-\overline{y}(s)\right|ds\\&\quad+\frac{1}{g^*}\int_0^1 l(s)\left|y(s)-\overline{y}(s)\right|\left|g(r)\right|\int_0^1|G(r,s)|ds\,dr.\end{aligned} \tag{3.16}$$

Therefore

$$\|N(y) - N(\overline{y})\|_\infty \leq \frac{1}{4}\left(\|l\|_{L^1} + \frac{\|g\|_{L^1}\|l\|_{L^1}}{g^*}\right)\|y - \overline{y}\|_\infty, \tag{3.17}$$

showing that, $N$ is a contraction and hence it has a unique fixed point which is a solution to (1.1). The proof is completed. □

We now present an existence result for problem (1.1).

**Theorem 3.4.** *Suppose that hypotheses*

(H1) *The function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is an* $L^1$*-Carathéodory,*

(H2) *There exist functions* $p, \overline{q} \in L^1([0,1], \mathbb{R}_+)$ *and* $\alpha \in [0,1)$ *such that*

$$|f(t,u)| \leq p(t)|u|^\alpha + \overline{q}(t) \quad \text{for each } (t,u) \in [0,1] \times \mathbb{R}, \tag{3.18}$$

*are satisfied. Then the BVP* (1.1) *has at least one solution. Moreover the solution set*

$$S = \{y \in C([0,1], \mathbb{R}) : y \text{ solution of the problem } (1.1)\} \tag{3.19}$$

*is compact.*

*Proof.* Transform the BVP (1.1) into a fixed-point problem. Consider the operator $N$ as defined in Theorem 3.3. We will show that $N$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

*Step 1* (N is continuous). Let $\{y_m\}$ be a sequence such that $y_m \to y$ in $C([0,1], \mathbb{R})$. Then

$$|N(y_n)(t) - N(y)(t)| \leq \int_0^1 |H(t,s)||f(s, y_m(s)) - f(s, y(s))|ds. \tag{3.20}$$

Since $f$ is $L^1$-Carathéodory and $g \in L^1([0,1], \mathbb{R})$, then

$$\begin{aligned}\|N(y_m) - N(y)\|_\infty &\leq \frac{1}{4}\|f(\cdot, y_m(\cdot)) - f(\cdot, y(\cdot))\|_{L^1} \\ &\quad + \frac{\|g\|_{L^1}}{4g^*}\|f(\cdot, y_m(\cdot)) - f(\cdot, y(\cdot))\|_{L^1}.\end{aligned} \tag{3.21}$$

Hence

$$\|N(y_m) - N(y)\|_\infty \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{3.22}$$

*Step 2* ($N$ maps bounded sets into bounded sets in $C([0,1], \mathbb{R})$). Indeed, it is enough to show that there exists a positive constant $\ell$ such that for each $y \in B_q = \{y \in C([0,1], \mathbb{R}) : \|y\|_\infty \leq q\}$ one has $\|N(y)\|_\infty \leq \ell$.

Let $y \in B_q$. Then for each $t \in [0,1]$, we have

$$N(y)(t) = \int_0^1 H(t,s) f(s, y(s)) ds. \tag{3.23}$$

By (H2) we have for each $t \in [0,1]$

$$\begin{aligned} |N(y)(t)| &\le \int_0^1 |H(t,s)| |f(s,y(s))| ds \\ &\le \frac{1}{4}\left[\|\overline{q}\|_{L^1} + q^\alpha \|p\|_{L^1}\right] + \frac{\|g\|_{L^1}}{4g^*}\left[\|\overline{q}\|_{L^1} + q^\alpha \|p\|_{L^1}\right]. \end{aligned} \tag{3.24}$$

Then for each $y \in B_q$ we have

$$\|Ny\|_\infty \le \frac{1}{4}\left[\|\overline{q}\|_{L^1} + q^\alpha \|p\|_{L^1}\right] + \frac{\|g\|_{L^1}}{4g^*}\left[\|\overline{q}\|_{L^1} + q^\alpha \|p\|_{L^1}\right] := \ell. \tag{3.25}$$

*Step 3* ($N$ maps bounded set into equicontinuous sets of $C([0,1],\mathbb{R})$). Let $\tau_1, \tau_2 \in [0,1]$, $\tau_1 < \tau_2$ and $B_q$ be a bounded set of $C([0,1],\mathbb{R})$ as in Step 2. Let $y \in B_q$ and $t \in [0,1]$ we have

$$|N(y)(\tau_2) - N(y)(\tau_1)| \le \int_0^1 |H(\tau_2,s) - H(\tau_1,s)| \overline{q}(s) ds + q^\alpha \int_0^1 |H(\tau_2,s) - H(\tau_1,s)| p(s) ds. \tag{3.26}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero. Then $N(B_q)$ is equicontinuous. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem we can conclude that $N : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

*Step 4* (A priori bounds on solutions). Let $y = \gamma N(y)$ for some $0 < \gamma < 1$. This implies by (H2) that for each $t \in [0,1]$ we have

$$|y(t)| \le \frac{1}{4}\int_0^1 p(s)|y(s)|^\alpha ds + \frac{1}{4}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\int_0^1 p(s)|y(s)|^\alpha ds. \tag{3.27}$$

Then

$$\|y\|_\infty \le \frac{1}{4}\|p\|_{L^1}\|y\|_\infty^\alpha + \frac{1}{4}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|p\|_{L^1}\|y\|_\infty^\alpha. \tag{3.28}$$

If $\|y\|_\infty > 1$, we have

$$\|y\|_\infty^{1-\alpha} \leq \frac{1}{4}\|p\| + \frac{1}{4}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|p\|_{L^1}. \tag{3.29}$$

Thus

$$\|y\|_\infty \leq \left(\frac{1}{4}\|p\| + \frac{1}{4}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|\overline{q}\|_{L^1} + \frac{\|g\|_{L^1}}{4g^*}\|p\|_{L^1}\right)^{1/(1-\alpha)} := \varphi_*. \tag{3.30}$$

Hence

$$\|y\|_\infty \leq \max(1, \varphi_*) := M. \tag{3.31}$$

Set

$$U := \{y \in C([0,1], \mathbb{R}) : \|y\|_\infty < M + 1\}, \tag{3.32}$$

and consider the operator $N : \overline{U} \to C([0,1], \mathbb{R})$. From the choice of $U$, there is no $y \in \partial U$ such that $y = \gamma N(y)$ for some $\gamma \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [15], we deduce that $N$ has a fixed point $y$ in $U$ which is a solution of the problem (1.1).

□

Now, prove that $S$ is compact. Let $\{y_m\}_{m\geq 1}$ be a sequence in $S$, then

$$y_m(t) = \int_0^1 H(t,s) f(s, y_m(s))\, ds, \quad m \geq 1,\ t \in [0,1]. \tag{3.33}$$

As in Steps 3 and 4 we can easily prove that there exists $M > 0$ such that

$$\|y_m\|_\infty < M, \quad \forall m \geq 1, \tag{3.34}$$

and the set $\{y_m : m \geq 1\}$ is equicontinuous in $C([0,1], \mathbb{R})$, hence by Arzela-Ascoli theorem we can conclude that there exists a subsequence of $\{y_m : m \geq 1\}$ converging to $y$ in $C([0,1], \mathbb{R})$. Using that fast that $f$ is an $L^1$-Carathédory we can prove that

$$y(t) = \int_0^1 H(t,s) f(s, y(s))\, ds, \quad t \in [0,1]. \tag{3.35}$$

Thus $S$ is compact.

## 4. Examples

We present some examples to illustrate the applicability of our results.

*Example 4.1.* Consider the following BVP

$$-y''(t) = \frac{1}{5e^{t+1}} \frac{1}{1+|y(t)|}, \quad \text{a.e. } t \in (0,1),$$
$$y(0) = 0, \qquad y(1) = \int_0^1 \frac{s+1}{2} y(s)ds. \tag{4.1}$$

Set

$$f(t,y) = \frac{1}{5e^{t+1}} \frac{1}{1+|y|}, \quad (t,y) \in [0,1] \times \mathbb{R}. \tag{4.2}$$

We can easily show that conditions (A1), (3.14) are satisfied with

$$l(t) = \frac{1}{5e^{t+1}},$$
$$g(t) = \frac{s+1}{2}, \tag{4.3}$$
$$\|l\|_{L^1} = \frac{1-e^{-1}}{5e}, \qquad \|g\|_{L^1} = \frac{3}{4}, \qquad g^* = \frac{5}{12}.$$

Hence, by Theorem 3.3, the BVP (4.1) has a unique solution on $[0,1]$.

*Example 4.2.* Consider the following BVP

$$-y''(t) = 5e^t \frac{1+2|y(t)|^{1/3}}{1+|y(t)|}, \quad \text{a.e. } t \in (0,1),$$
$$y(0) = 0, \qquad y(1) = \int_0^1 s^2 y(s)ds. \tag{4.4}$$

Set

$$f(t,y) = 5e^t \frac{1+2|y|^{1/3}}{1+|y|}, \quad (t,y) \in [0,1] \times \mathbb{R}. \tag{4.5}$$

We can easily show that conditions (H1), (H2) are satisfied with

$$\alpha = \frac{1}{3}, \quad p(t) = 10e^t, \quad \overline{q}(t) = 5e^t, \quad t \in [0,1]. \tag{4.6}$$

Hence, by Theorem 3.4, the BVP (4.4) has at least one solution on $[0, 1]$. Moreover, its solutions set is compact.

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