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Research Article

A Note on Generalized Fractional Integral Operators on Generalized Morrey Spaces

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We show some inequalities for generalized fractional integral operators on generalized Morrey spaces. We also show the boundedness property of the generalized fractional integral operators on the predual of the generalized Morrey spaces.

1. Introduction

The present paper is an offspring of [1]. We obtain some inequalities for generalized fractional integral operators on generalized Morrey spaces. We also show the boundedness property of the generalized fractional integral operators on the predual of the generalized Morrey spaces. They generalize what was shown in [1]. We will go through the same argument as [1].

For $0 < \alpha < 1$ the classical fractional integral operator I_α and the classical fractional maximal operator M_α are given by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n(1-\alpha)}} dy,$$
$$M_\alpha f(x) := \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|^{1-\alpha}} \int_Q |f(y)| dy. \tag{1.1}$$

In the present paper, we generalize the parameter α. Let $\rho : [0,\infty) \to [0,\infty)$ be a suitable function. We define the generalized fractional integral operator T_ρ and the generalized fractional maximal operator M_ρ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy,$$
$$M_\rho f(x) := \sup_{x \in Q \in \mathcal{Q}} \frac{\rho(\ell(Q))}{|Q|} \int_Q |f(y)| dy. \tag{1.2}$$

Here, we use the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $\ell(Q)$, to denote the sidelength of Q and $|Q|$ to denote the volume of Q. If $\rho(t) \equiv t^{n\alpha}$, $0 < \alpha < 1$, then we have $T_\rho = I_\alpha$ and $M_\rho = M_\alpha$.

A well-known fact in partial differential equations is that I_α is an inverse of $(-\Delta)^{n\alpha/2}$. The operator $(1-\Delta)^{-1}$ admits an expression of the form T_ρ for some ρ. For more details of this operator we refer to [2]. As we will see, these operators will fall under the scope of our main results.

Among other function spaces, it seems that the Morrey spaces reflect the boundedness properties of the fractional integral operators. To describe the Morrey spaces we recall some definitions and notation. All cubes are assumed to have their sides parallel to the coordinate axes. For $Q \in \mathcal{Q}$ we use cQ to denote the cube with the same center as Q, but with sidelength of $c\ell(Q)$. $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$.

Let $0 < p < \infty$ and $\phi : [0,\infty) \to [0,\infty)$ be a suitable function. For a function f locally in $L^p(\mathbb{R}^n)$ we set

$$\|f\|_{p,\phi} := \sup_{Q \in \mathcal{Q}} \phi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(x)|^p dx \right)^{1/p}. \tag{1.3}$$

We will call the Morrey space $\mathcal{M}^{p,\phi}(\mathbb{R}^n) = \mathcal{M}^{p,\phi}$ the subset of all functions f locally in $L^p(\mathbb{R}^n)$ for which $\|f\|_{\mathcal{M}^{p,\phi}} = \|f\|_{p,\phi}$ is finite. Applying Hölder's inequality to (1.3), we see that $\|f\|_{p_1,\phi} \geq \|f\|_{p_2,\phi}$ provided that $p_1 \geq p_2 > 0$. This tells us that $\mathcal{M}^{p_1,\phi} \subset \mathcal{M}^{p_2,\phi}$ when $p_1 \geq p_2 > 0$. We remark that without the loss of generality we may assume

$$\phi(t) \text{ is nondecreasing but } \phi(t)^p t^{-n} \text{ is nonincreasing.} \tag{1.4}$$

(See [1].) Hereafter, we always postulate (1.4) on ϕ.

If $\phi(t) \equiv t^{n/p_0}$, $p_0 \geq p$, $\mathcal{M}^{p,\phi}$ coincides with the usual Morrey space and we write this for $\mathcal{M}^{p,p_0}$ and the norm for $\|\cdot\|_{\mathcal{M}^{p,p_0}}$. Then we have the inclusion

$$L^{p_0} = \mathcal{M}^{p_0,p_0} \subset \mathcal{M}^{p_1,p_0} \subset \mathcal{M}^{p_2,p_0} \tag{1.5}$$

when $p_0 \geq p_1 \geq p_2 > 0$.

In the present paper, we take up some relations between the generalized fractional integral operator T_ρ and the generalized fractional maximal operator M_ρ in the framework of the Morrey spaces $\mathcal{M}^{p,\phi}$ (Theorem 1.2). In the last section, we prove a dual version of Olsen's inequality on predual of Morrey spaces (Theorem 3.1). As a corollary (Corollary 3.2), we have the boundedness properties of the operator T_ρ on predual of Morrey spaces.

Let $\theta : [0,\infty) \to [0,\infty)$ be a function. By the Dini condition we mean that θ fulfills

$$\int_0^1 \frac{\theta(s)}{s} ds < \infty, \tag{1.6}$$

while the doubling condition on θ (with a doubling constant $C_1 > 0$) is that θ satisfies

$$\frac{1}{C_1} \leq \frac{\theta(s)}{\theta(t)} \leq C_1, \quad \text{if } \frac{1}{2} \leq \frac{s}{t} \leq 2. \tag{1.7}$$

We notice that (1.4) is stronger than the doubling condition. More quantitatively, if we assume (1.4), then ϕ satisfies the doubling condition with the doubling constant $2^{n/p}$. A simple consequence that can be deduced from the doubling condition of θ is that

$$\frac{\log 2}{C_1} \theta(t) \leq \int_{t/2}^{t} \frac{\theta(s)}{s} ds \leq \log 2 \cdot C_1 \theta(t) \quad \forall t > 0. \tag{1.8}$$

The key observation made in [1] is that it is frequently convenient to replace θ satisfying (1.6) and (1.7) by $\widetilde{\theta}$:

$$\widetilde{\theta}(t) = \int_0^t \frac{\theta(s)}{s} ds. \tag{1.9}$$

Before we formulate our main results, we recall a typical result obtained in [1].

Proposition 1.1 (see [1, Theorem 1.3]). *Let*

$$1 \leq p < \infty, \quad \begin{cases} p \leq q & \text{if } p = 1, \\ p < q & \text{if } p > 1, \end{cases} \tag{1.10}$$

$0 \leq b \leq 1$ *and* $b < a$. *Suppose that* $\widetilde{\rho}(t)^{\max(ap,bq)} t^{-n}$ *is nonincreasing. Then*

$$\left\| g \cdot T_\rho f \right\|_{p,\widetilde{\rho}^a} \leq C \left\| g \right\|_{q,\widetilde{\rho}^b} \left\| M_{\widetilde{\rho}^{1-b}} f \right\|_{p,\widetilde{\rho}^a}, \tag{1.11}$$

where the constant C *is independent of* f *and* g.

The aim of the present paper is to generalize the function spaces to which f and g belong. With theorem 1.2, which we will present just below, we can replace $\widetilde{\rho}^a$ with ϕ and $\widetilde{\rho}^b$ with η. We now formulate our main theorems. In the sequel we always assume that ρ satisfies (1.6) and (1.7), and C is used to denote various positive constants.

Theorem 1.2. *Let*

$$1 \le p < \infty, \quad \begin{cases} p \le q & \text{if } p = 1, \\ p < q & \text{if } p > 1. \end{cases} \tag{1.12}$$

Suppose that $\phi(t)$ *and* $\eta(t)$ *are nondecreasing but that* $\phi(t)^p t^{-n}$ *and* $\eta(t)^q t^{-n}$ *are nonincreasing. Assume also that*

$$\int_t^\infty \frac{\rho(s)\eta(s)}{s\tilde{\rho}(s)\phi(s)} ds \le C\frac{\eta(t)}{\phi(t)} \quad \forall t > 0, \tag{1.13}$$

then

$$\|g \cdot T_\rho f\|_{p,\phi} \le C\|g\|_{q,\eta}\|M_{\tilde{\rho}/\eta} f\|_{p,\phi}, \tag{1.14}$$

where the constant C *is independent of* f *and* g.

Remark 1.3. Let $0 \le b \le 1$ and $b < a$. Then $\phi = \tilde{\rho}^a$ and $\eta = \tilde{\rho}^b$ satisfy the assumption (1.13). Indeed,

$$\begin{aligned} \int_t^\infty \frac{\rho(s)\tilde{\rho}(s)^b}{s\tilde{\rho}(s)\tilde{\rho}(s)^a} ds &= \int_t^\infty \tilde{\rho}(s)^{b-a-1}\frac{\rho(s)}{s} ds \\ &= \int_t^\infty \frac{d}{ds}\left(\frac{1}{b-a}\tilde{\rho}(s)^{b-a}\right) ds \le \frac{1}{a-b}\tilde{\rho}(t)^{b-a}. \end{aligned} \tag{1.15}$$

Hence, Theorem 1.2 generalizes Proposition 1.1.

Letting $\eta(t) \equiv 1$ and $g(x) \equiv 1$ in Theorem 1.2, we obtain the result of how $M_{\tilde{\rho}}$ controls T_ρ.

Corollary 1.4. *Let* $1 \le p < \infty$. *Suppose that*

$$\int_t^\infty \frac{\rho(s)}{s\tilde{\rho}(s)\phi(s)} ds \le \frac{C}{\phi(t)} \quad \forall t > 0, \tag{1.16}$$

then

$$\|T_\rho f\|_{p,\phi} \le C\|M_{\tilde{\rho}} f\|_{p,\phi}. \tag{1.17}$$

Corollary 1.4 generalizes [3, Theorem 4.2]. Letting $\eta = \tilde{\rho}$ in Theorem 1.2, we also obtain the condition on g and ρ under which the mapping

$$f \in \mathcal{M}_{p,\phi} \longmapsto g \cdot T_\rho f \in \mathcal{M}_{p,\phi} \tag{1.18}$$

is bounded.

Corollary 1.5. *Let*

$$1 \le p < \infty, \qquad \begin{cases} p \le q & \text{if } p = 1, \\ p < q & \text{if } p > 1. \end{cases} \tag{1.19}$$

Suppose that

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)}ds \le C\frac{\tilde{\rho}(t)}{\phi(t)} \quad \forall t > 0, \tag{1.20}$$

then

$$\|g \cdot T_\rho f\|_{p,\phi} \le C\|g\|_{q,\tilde{\rho}}\|Mf\|_{p,\phi}. \tag{1.21}$$

In particular, if $1 < p < q < \infty$, *then*

$$\|g \cdot T_\rho f\|_{p,\phi} \le C\|g\|_{q,\tilde{\rho}}\|f\|_{p,\phi}. \tag{1.22}$$

Here, M denotes the Hardy-Littlewood maximal operator defined by

$$Mf(x) := \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|}\int_Q |f(y)|dy. \tag{1.23}$$

We will establish that M is bounded on $\mathcal{M}_{p,\phi}$ when $p > 1$ (Lemma 2.2). Therefore, the second assertion is immediate from the first one.

Theorem 1.6. *Let* $1 < p \le r < q < \infty$. *Suppose that* $\phi(t)$ *and* $\eta(t)$ *are nondecreasing but that* $\phi(t)^p t^{-n}$ *and* $\eta(t)^q t^{-n}$ *are nonincreasing. Suppose also that*

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}ds \le C\frac{\eta(t)}{\phi(t)^{p/r}} \quad \forall t > 0, \tag{1.24}$$

then

$$\|g \cdot T_\rho f\|_{r,\phi^{p/r}} \le C\|g\|_{q,\eta}\|f\|_{p,\phi}, \tag{1.25}$$

where the constant C *is independent of* f *and* g.

Theorem 1.6 extends [4, Theorem 2], [1, Theorem 1.1], and [5, Theorem 1]. As the special case $\eta(t) \equiv 1$ and $g(x) \equiv 1$ in Theorem 1.6 shows, this theorem covers [1, Remark 2.8].

Corollary 1.7 (see [1, Remark 2.8], see also [6–8]). *Let* $1 < p \le r < \infty$. *Suppose that*

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}ds \le \frac{C}{\phi(t)^{p/r}} \quad \forall t > 0, \tag{1.26}$$

then

$$\|T_\rho f\|_{r,\phi^{p/r}} \leq C\|f\|_{p,\phi}. \tag{1.27}$$

Nakai generalized Corollary 1.7 to the Orlicz-Morrey spaces ([9, Theorem 2.2] and [10, Theorem 7.1]).

We dare restate Theorem 1.6 in the special case when T_ρ is the fractional integral operator I_α. The result holds by letting $\rho(t) \equiv t^{n\alpha}$, $\phi(t) \equiv t^{n/p_0}$, and $\eta(t) \equiv t^{n/q_0}$.

Proposition 1.8 (see [1, Proposition 1.7]). *Let* $0 < \alpha < 1$, $1 < p \leq p_0 < \infty$, $1 < q \leq q_0 < \infty$, *and* $1 < r \leq r_0 < \infty$. *Suppose that* $q > r$, $1/p_0 > \alpha$, $1/q_0 \leq \alpha$, $1/r_0 = 1/q_0 + 1/p_0 - \alpha$, *and* $r/r_0 = p/p_0$ *then*

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r,r_0}} \leq C\|g\|_{\mathcal{M}^{q,q_0}}\|f\|_{\mathcal{M}^{p,p_0}}, \tag{1.28}$$

where the constant C is independent of f and g.

Proposition 1.8 extends [4, Theorem 2] (see [1, Remark 1.9]).

Remark 1.9. The special case $q_0 = \infty$ and $g(x) \equiv 1$ in Proposition 1.8 corresponds to the classical theorem due to Adams (see [11]).

The fractional integral operator I_α, $0 < \alpha < 1$, is bounded from $\mathcal{M}^{p,p_0}$ to $\mathcal{M}^{r,r_0}$ if and only if the parameters $1 < p \leq p_0 < \infty$ and $1 < r \leq r_0 < \infty$ satisfy $1/r_0 = 1/p_0 - \alpha$ and $r/r_0 = p/p_0$.

Using naively the Adams theorem and Hölder's inequality, one can prove a minor part of q in Proposition 1.8. That is, the proof of Proposition 1.8 is fundamental provided $(p/p_0)q_0 \leq q \leq q_0$. Indeed, by virtue of the Adams theorem we have, for any cube $Q \in \mathcal{Q}$,

$$|Q|^{1/s_0}\left(\frac{1}{|Q|}\int_Q |I_\alpha f(x)|^s dx\right)^{1/s} \leq C\|f\|_{\mathcal{M}^{p,p_0}}, \quad \frac{1}{s} = \frac{p_0}{p}\frac{1}{s_0}, \quad \frac{1}{s_0} = \frac{1}{p_0} - \alpha. \tag{1.29}$$

The condition $r/r_0 = p/p_0$, $1/r_0 = 1/q_0 + 1/p_0 - \alpha$ reads

$$\frac{1}{r} = \frac{p_0}{p}\left(\frac{1}{q_0} + \frac{1}{p_0} - \alpha\right) = \frac{p_0}{p}\frac{1}{q_0} + \frac{1}{s}. \tag{1.30}$$

These yield

$$|Q|^{1/q_0+1/s_0}\left(\frac{1}{|Q|}\int_Q |g(x)I_\alpha f(x)|^r dx\right)^{1/r} \leq C\|g\|_{\mathcal{M}^{q,q_0}}\|f\|_{\mathcal{M}^{p,p_0}} \tag{1.31}$$

if $r/r_0 = p/p_0 = q/q_0$. In view of inclusion (1.5), the same can be said when $(p/p_0)q_0 \leq q \leq q_0$. Also observe that $1/r_0 = 1/q_0 + 1/p_0 - \alpha > 1/q_0$. Hence we have $q_0 > r_0$. Thus, since the condition $q > r$, Proposition 1.8 is significant only when $(p/p_0)r_0 < q < (p/p_0)q_0$. The case

$p/p_0 = r/r_0 = 1$ (the case of the Lebesgue spaces) corresponds (so-called) to the Fefferan-Phong inequality (see [12]). An inequality of the form

$$\int_{\mathbb{R}^n} |u(x)|^2 v(x)dx \le C_v \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx, \quad 0 \le u \in C_0^\infty(\mathbb{R}^n), \ v \ge 0 \tag{1.32}$$

is called the trace inequality and is useful in the analysis of the Schrödinger operators. For example, Kerman and Sawyer utilized an inequality of type (1.32) to obtain an eigenvalue estimates of the operators (see [13]). By letting $\alpha = 1/n$, we obtain a sharp estimate on the constant C_v in (1.32).

In [14], we characterized the range of I_α, which motivates us to consider Proposition 1.8.

Proposition 1.10 (see [14]). *Let $1 < p \le p_0 < \infty$, $1 < s \le s_0 < \infty$, and $0 < \alpha < 1$. Assume that*

$$\frac{p}{p_0} = \frac{s}{s_0}, \qquad \frac{1}{s_0} = \frac{1}{p_0} - \alpha. \tag{1.33}$$

(1) *$I_\alpha : \mathcal{M}^{p,p_0} \longrightarrow \mathcal{M}^{s,s_0}$ is continuous but not surjective.*

(2) *Let $\varphi \in \mathcal{S}$ be an auxiliary function chosen so that $\varphi(x) = 1$, $2 \le |x| \le 4$ and that $\varphi(x) = 0$, $|x| \le 1$, $|x| \ge 8$. Then the norm equivalence*

$$\|f\|_{\mathcal{M}^{p,p_0}} \simeq \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2j(n-\alpha)} \left| \mathcal{F}\varphi(2^j \cdot) * I_\alpha f \right|^2 \right)^{1/2} \right\|_{\mathcal{M}^{p,p_0}} \tag{1.34}$$

holds for $f \in \mathcal{M}^{p,p_0}$, where $\mathcal{F}$ denotes the Fourier transform.

In view of this proposition $\mathcal{M}^{s,s_0}$ is not a good space to describe the boundedness of I_α, although we have (1.29). As we have seen by using Hölder's inequality in Remark 1.9, if we use the space $\mathcal{M}^{s,s_0}$, then we will obtain a result weaker than Proposition 1.8.

Finally it would be interesting to compare Theorem 1.2 with the following Theorem 1.11.

Theorem 1.11. *Let $0 < p < \infty$. Suppose that ρ, η, and ϕ are nondecreasing and that $\eta(t)^p t^{-n}$ and $\phi(t)^p t^{-n}$ are nonincreasing. Then*

$$\|g \cdot M_\rho f\|_{p,\phi} \le C\|g\|_{p,\eta} \|M_{\rho/\eta} f\|_{p,\phi}, \tag{1.35}$$

where the constant C is independent of f and g.

Theorem 1.11 generalizes [1, Theorem 1.7] and the proof remains unchanged except some minor modifications caused by our generalization of the function spaces to which f and g belong. So, we omit the proof in the present paper.

2. Proof of Theorems

For any $1 < p < \infty$ we will write p' for the conjugate number defined by $1/p + 1/p' = 1$. Hereafter, for the sake of simplicity, for any $Q \in \mathcal{Q}$ and $0 < p < \infty$ we will write

$$m_Q(f) := \frac{1}{|Q|}\int_Q f(x)dx, \qquad m_Q^{(p)}(f) := m_Q\left(|f|^p\right)^{1/p}. \tag{2.1}$$

2.1. Proof of Theorem 1.2

First, we will prove Theorem 1.2. Except for some sufficient modifications, the proof of the theorem follows the argument in [15]. We denote by $\mathfrak{D}$ the family of all dyadic cubes in $\mathbb{R}^n$. We assume that f and g are nonnegative, which may be done without any loss of generality thanks to the positivity of the integral kernel. We will denote by $B(x,r)$ the ball centered at x and of radius r. We begin by discretizing the operator $T_\rho f$ following the idea of Pérez (see [16]):

$$\begin{aligned} T_\rho f(x) &= \sum_{\nu\in\mathbb{Z}}\int_{2^{\nu-1}<|x-y|\le 2^\nu} f(y)\frac{\rho(|x-y|)}{|x-y|^n}dy \\ &\le C\sum_{\nu\in\mathbb{Z}}\frac{\rho(2^\nu)}{2^{n\nu}}\int_{B(x,2^\nu)} f(y)dy \\ &\le C\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathfrak{D}:Q\ni x,\ell(Q)=2^\nu}\frac{\rho(\ell(Q))}{|Q|}\int_{3Q} f(y)dy \\ &= C\sum_{Q\in\mathfrak{D}}\frac{\rho(\ell(Q))}{|Q|}\int_{3Q} f(y)dy\cdot\chi_Q(x) \\ &= C\sum_{Q\in\mathfrak{D}}\rho(\ell(Q))m_{3Q}(f)\cdot\chi_Q(x), \end{aligned} \tag{2.2}$$

where we have used the doubling condition of ρ for the first inequality. To prove Theorem 1.2, thanks to the doubling condition of ϕ, which holds by use of the facts that $\phi(t)$ is nondecreasing and that $\phi(t)^p t^{-n}$ is nonincreasing, it suffices to show

$$\left(\int_{Q_0}\left(g(x)T_\rho f(x)\right)^p dx\right)^{1/p} \le C\|g\|_{q,\eta}\|M_{\tilde{\rho}/\eta}f\|_{p,\phi}|Q_0|^{1/p}\phi(\ell(Q_0))^{-1}, \tag{2.3}$$

for all dyadic cubes Q_0. Hereafter, we let

$$\begin{aligned} \mathfrak{D}_1(Q_0) &:= \{Q\in\mathfrak{D} : Q\subset Q_0\}, \\ \mathfrak{D}_2(Q_0) &:= \{Q\in\mathfrak{D} : Q\supsetneq Q_0\}. \end{aligned} \tag{2.4}$$

Let us define for $i = 1, 2$

$$F_i(x) := \sum_{Q \in \mathfrak{D}_i(Q_0)} \rho(\ell(Q)) m_{3Q}(f) \chi_Q(x) \tag{2.5}$$

and we will estimate

$$\left(\int_{Q_0} (g(x) F_i(x))^p dx \right)^{1/p}. \tag{2.6}$$

The case $i = 1$ and $p = 1$ We need the following crucial lemma, the proof of which is straightforward and is omitted (see [15, 16]).

Lemma 2.1. *For a nonnegative function h in $L^\infty(Q_0)$ one lets $\gamma_0 := m_{Q_0}(h)$ and $c := 2^{n+1}$. For $k = 1, 2, \ldots$ let*

$$D_k := \bigcup_{Q \in \mathfrak{D}_1(Q_0) : m_Q(h) > \gamma_0 c^k} Q. \tag{2.7}$$

Considering the maximal cubes with respect to inclusion, one can write

$$D_k = \bigcup_j Q_{k,j}, \tag{2.8}$$

where the cubes $\{Q_{k,j}\} \subset \mathfrak{D}_1(Q_0)$ are nonoverlapping. By virtue of the maximality of $Q_{k,j}$ one has that

$$\gamma_0 c^k < m_{Q_{k,j}}(h) \leq 2^n \gamma_0 c^k. \tag{2.9}$$

Let

$$E_0 := Q_0 \setminus D_1, \qquad E_{k,j} := Q_{k,j} \setminus D_{k+1}. \tag{2.10}$$

Then $\{E_0\} \cup \{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0 and satisfies

$$|Q_0| \leq 2|E_0|, \qquad |Q_{k,j}| \leq 2|E_{k,j}|. \tag{2.11}$$

Also, one sets

$$\begin{aligned} \mathfrak{D}_0 &:= \{Q \in \mathfrak{D}_1(Q_0) : m_Q(h) \leq \gamma_0 c\}, \\ \mathfrak{D}_{k,j} &:= \left\{Q \in \mathfrak{D}_1(Q_0) : Q \subset Q_{k,j},\ \gamma_0 c^k < m_Q(h) \leq \gamma_0 c^{k+1}\right\}. \end{aligned} \tag{2.12}$$

Then

$$\mathfrak{D}_1(Q_0) := \mathfrak{D}_0 \cup \bigcup_{k,j} \mathfrak{D}_{k,j}. \tag{2.13}$$

With Lemma 2.1 in mind, let us return to the proof of Theorem 1.2. We need only to verify that

$$\int_{Q_0} g(x)F_1(x)dx \le C\|g\|_{q,\eta}\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)dx. \tag{2.14}$$

Inserting the definition of F_1, we have

$$\int_{Q_0} g(x)F_1(x)dx = \sum_{Q\in\mathfrak{D}_1(Q_0)} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)dx. \tag{2.15}$$

Letting $h = g$, we will apply Lemma 2.1 to estimate this quantity. Retaining the same notation as Lemma 2.1 and noticing (2.13), we have

$$\int_{Q_0} g(x)F_1(x)dx = \sum_{Q\in\mathfrak{D}_0} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)dx + \sum_{k,j}\sum_{Q\in\mathfrak{D}_{k,j}} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)dx. \tag{2.16}$$

We first evaluate

$$\sum_{Q\in\mathfrak{D}_{k,j}} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)dx. \tag{2.17}$$

It follows from the definition of $\mathfrak{D}_{k,j}$ that (2.17) is bounded by

$$C\gamma_0 c^{k+1}\sum_{Q\in\mathfrak{D}_{k,j}} \rho(\ell(Q))\int_{3Q} f(y)dy. \tag{2.18}$$

By virtue of the support condition and (1.8) we have

$$\begin{aligned}
\sum_{Q\in\mathfrak{D}_{k,j}} \rho(\ell(Q))\int_{3Q} f(y)dy &= \sum_{\nu=-\infty}^{\log_2\ell(Q_{k,j})} \rho(2^\nu)\left(\sum_{Q\in\mathfrak{D}_{k,j}:\ell(Q)=2^\nu}\int_{3Q} f(y)dy\right)\\
&\le C\int_{3Q_{k,j}} f(y)dy\left(\sum_{\nu=-\infty}^{\log_2\ell(Q_{k,j})}\rho(2^\nu)\right)\\
&\le C\int_{3Q_{k,j}} f(y)dy\left(\int_0^{\ell(Q_{k,j})}\frac{\rho(s)}{s}ds\right)\\
&= C\tilde{\rho}(\ell(Q_{k,j}))\int_{3Q_{k,j}} f(y)dy.
\end{aligned} \tag{2.19}$$

If we invoke relations $|Q_{k,j}| \le 2|E_{k,j}|$ and $\gamma_0 c^k < m_{Q_{k,j}}(g)$, then (2.17) is bounded by

$$C\tilde{\rho}(\ell(Q_{k,j}))m_{3Q_{k,j}}(f)m_{Q_{k,j}}(g)|E_{k,j}|. \tag{2.20}$$

Now that we have from the definition of the Morrey norm

$$m_{Q_{k,j}}(g) \le m^{(q)}_{Q_{k,j}}(g) \le \|g\|_{q,\eta}\eta(\ell(Q_{k,j}))^{-1}, \tag{2.21}$$

we conclude that

$$(2.17) \le C\|g\|_{q,\eta}\frac{\tilde{\rho}(\ell(Q_{k,j}))}{\eta(\ell(Q_{k,j}))}m_{3Q_{k,j}}(f)|E_{k,j}| \le C\|g\|_{q,\eta}\int_{E_{k,j}} M_{\tilde{\rho}/\eta}f(x)dx. \tag{2.22}$$

Here, we have used the fact that $\tilde{\rho}$ is nondecreasing, that η satisfies the doubling condition and that

$$\frac{\tilde{\rho}(\ell(3Q_{k,j}))}{\eta(\ell(3Q_{k,j}))}m_{3Q_{k,j}}(f) \le \inf_{y\in Q_{k,j}} M_{\tilde{\rho}/\eta}f(y). \tag{2.23}$$

Similarly, we have

$$\sum_{Q\in\mathfrak{D}_0}\rho(\ell(Q))m_{3Q}(f)\int_Q g(x)dx \le C\|g\|_{q,\eta}\int_{E_0} M_{\tilde{\rho}/\eta}f(x)dx. \tag{2.24}$$

Summing up all factors, we obtain (2.14), by noticing that $\{E_0\}\cup\{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0.

The case $i = 1$ and $p > 1$ In this case we establish

$$\left(\int_{Q_0}(g(x)F_1(x))^p dx\right)^{1/p} \le C\|g\|_{q,\eta}\left(\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)^p dx\right)^{1/p}, \tag{2.25}$$

by the duality argument. Take a nonnegative function $w \in L^{p'}(Q_0)$, $1/p + 1/p' = 1$, satisfying that $\|w\|_{L^{p'}(Q_0)} = 1$ and that

$$\left(\int_{Q_0}(g(x)F_1(x))^p dx\right)^{1/p} = \int_{Q_0} g(x)F_1(x)w(x)dx. \tag{2.26}$$

Letting $h = gw$, we will apply Lemma 2.1 to estimation of this quantity. First, we will insert the definition of F_1,

$$\begin{aligned}\int_{Q_0} g(x)F_1(x)w(x)dx &= \sum_{Q\in\mathfrak{D}_1(Q_0)} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)w(x)dx \\ &= \sum_{Q\in\mathfrak{D}_0} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)w(x)dx \\ &\quad+ \sum_{Q\in\mathfrak{D}_{j,k}} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)w(x)dx.\end{aligned} \tag{2.27}$$

First, we evaluate

$$\sum_{Q\in\mathfrak{D}_{k,j}} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)w(x)dx. \tag{2.28}$$

Going through the same argument as the above, we see that (2.28) is bounded by

$$C\tilde{\rho}(\ell(Q_{k,j}))m_{3Q_{k,j}}(f)m_{Q_{k,j}}(gw)|E_{k,j}|. \tag{2.29}$$

Using Hölder's inequality, we have

$$m_{Q_{k,j}}(gw) \le m^{(q)}_{Q_{k,j}}(g)m^{(q')}_{Q_{k,j}}(w) \le \|g\|_{q,\eta}\eta(\ell(Q_{k,j}))^{-1}m^{(q')}_{Q_{k,j}}(w). \tag{2.30}$$

These yield

$$\begin{aligned}(2.29) &\le C\|g\|_{q,\eta}\frac{\tilde{\rho}(\ell(Q_{k,j}))}{\eta(\ell(Q_{k,j}))}m_{3Q_{k,j}}(f)m^{(q')}_{Q_{k,j}}(w)|E_{k,j}| \\ &\le C\|g\|_{q,\eta}\int_{E_{k,j}} M_{\tilde{\rho}/\eta}f(x)Mw^{q'}(x)^{1/q'}dx.\end{aligned} \tag{2.31}$$

Similarly, we have

$$\sum_{Q\in\mathfrak{D}_0} \rho(\ell(Q))m_{3Q}(f)\int_Q g(x)w(x)dx \le C\|g\|_{q,\eta}\int_{E_0} M_{\tilde{\rho}/\eta}f(x)Mw^{q'}(x)^{1/q'}dx. \tag{2.32}$$

Summing up all factors we obtain

$$(2.28) \le C\|g\|_{q,\eta}\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)Mw^{q'}(x)^{1/q'}dx. \tag{2.33}$$

Another application of Hölder's inequality gives us that

$$(2.28) \le C\|g\|_{q,\eta}\left(\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)^p dx\right)^{1/p}\left(\int_{Q_0} Mw^{q'}(x)^{p'/q'} dx\right)^{1/p'}. \tag{2.34}$$

Now that $p' > q'$, the maximal operator M is $L^{p'/q'}$-bounded. As a result we have

$$\begin{aligned}(2.28) &\le C\|g\|_{q,\eta}\left(\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)^p dx\right)^{1/p}\left(\int_{Q_0} w(x)^{p'} dx\right)^{1/p'} \\ &= C\|g\|_{q,\eta}\left(\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)^p dx\right)^{1/p}.\end{aligned} \tag{2.35}$$

This is our desired inequality.

The case $i = 2$ and $p \ge 1$ By a property of the dyadic cubes, for all $x \in Q_0$ we have

$$\begin{aligned}F_2(x) &= \sum_{Q\in\mathfrak{D}_2(Q_0)} \rho(\ell(Q))m_{3Q}(f) \\ &= \sum_{Q\in\mathfrak{D}_2(Q_0)} \rho(\ell(Q))\frac{\eta(\ell(Q))}{\tilde{\rho}(\ell(Q))}\cdot\frac{\tilde{\rho}(\ell(Q))}{\eta(\ell(Q))}m_{3Q}(f) \\ &\le C\sum_{Q\in\mathfrak{D}_2(Q_0)} \rho(\ell(Q))\frac{\eta(\ell(Q))}{\tilde{\rho}(\ell(Q))}m_Q(M_{\tilde{\rho}/\eta}f).\end{aligned} \tag{2.36}$$

As a consequence we obtain

$$m_Q(M_{\tilde{\rho}/\eta}f) \le m_Q^{(p)}(M_{\tilde{\rho}/\eta}f) \le \|M_{\tilde{\rho}/\eta}f\|_{p,\phi}\phi(\ell(Q))^{-1}. \tag{2.37}$$

In view of the definition of $\mathfrak{D}_2$, for each $\nu \in \mathbb{Z}$ with $\nu \ge 1 + \log_2\ell(Q_0)$ there exists a unique cube in $\mathfrak{D}_2$ whose length is 2^ν. Hence, inserting these estimates, we obtain

$$\begin{aligned}F_2(x) &\le C\|M_{\tilde{\rho}/\eta}f\|_{p,\phi}\sum_{Q\in\mathfrak{D}_2(Q_0)}\frac{\rho(\ell(Q))\eta(\ell(Q))}{\tilde{\rho}(\ell(Q))\phi(\ell(Q))} \\ &= C\|M_{\tilde{\rho}/\eta}f\|_{p,\phi}\sum_{\nu=1+\log_2\ell(Q_0)}^{\infty}\frac{\rho(2^\nu)\eta(2^\nu)}{\tilde{\rho}(2^\nu)\phi(2^\nu)} \\ &\le C\|M_{\tilde{\rho}/\eta}f\|_{p,\phi}\int_{\ell(Q_0)}^{\infty}\frac{\rho(s)\eta(s)}{s\tilde{\rho}(s)\phi(s)}ds.\end{aligned} \tag{2.38}$$

Here, in the last inequality we have used the doubling condition (1.8) and the facts that $\widetilde{\rho}$, ϕ, and η are nondecreasing and that $\widetilde{\rho}$ and ϕ satisfy the doubling condition. Thus, we obtain

$$F_2(x) \le C\left\| M_{\widetilde{\rho}/\eta} f \right\|_{p,\phi} \frac{\eta(\ell(Q_0))}{\phi(\ell(Q_0))} \tag{2.39}$$

for all $x \in Q_0$. Inserting this pointwise estimate, we obtain

$$\begin{aligned}\left(\int_{Q_0} (g(x)F_2(x))^p dx\right)^{1/p} &\le C m_{Q_0}^{(p)}(g) \left\| M_{\widetilde{\rho}/\eta} f \right\|_{p,\phi} \eta(\ell(Q_0)) \phi(\ell(Q_0))^{-1} |Q_0|^{1/p} \\ &\le C\|g\|_{q,\eta} \left\| M_{\widetilde{\rho}/\eta} f \right\|_{p,\phi} \phi(\ell(Q_0))^{-1} |Q_0|^{1/p}.\end{aligned} \tag{2.40}$$

This is our desired inequality.

2.2. *Proof of Theorem 1.6*

We need some lemmas.

Lemma 2.2 (see [1, Lemma 2.2]). *Let $p > 1$. Suppose that ϕ satisfies* (1.4), *then*

$$\|Mf\|_{p,\phi} \le C\|f\|_{p,\phi}. \tag{2.41}$$

Lemma 2.3. *Let $1 < p \le r < \infty$. Suppose that ϕ satisfies* (1.4), *then*

$$\left\| M_{\phi^{1-p/r}} f \right\|_{r,\phi^{p/r}} \le C\|f\|_{p,\phi}. \tag{2.42}$$

Proof. Let $x \in \mathbb{R}^n$ be a fixed point. For every cube $Q \ni x$ we see that

$$\begin{aligned}\phi(\ell(Q))^{1-p/r} m_Q(|f|) &\le \min\left(\phi(\ell(Q))^{1-p/r} Mf(x), \phi(\ell(Q))^{-p/r} \|f\|_{p,\phi}\right) \\ &\le \sup_{t \ge 0} \min\left(t^{1-p/r} Mf(x), t^{-p/r} \|f\|_{p,\phi}\right) \\ &= \|f\|_{p,\phi}^{1-p/r} Mf(x)^{p/r}.\end{aligned} \tag{2.43}$$

This implies

$$M_{\phi^{1-p/r}} f(x)^r \le \|f\|_{p,\phi}^{r-p} Mf(x)^p. \tag{2.44}$$

It follows from Lemma 2.2 that for every cube Q_0

$$m_{Q_0}^{(r)}\left(M_{\phi^{1-p/r}}f\right) \le \|f\|_{p,\phi}^{1-p/r} m_{Q_0}^{(p)}(Mf)^{p/r} \le C\|f\|_{p,\phi}\phi(\ell(Q_0))^{-p/r}. \tag{2.45}$$

The desired inequality then follows. □

Proof of Theorem 1.6. We use definition (2.5) again and will estimate

$$\left(\int_{Q_0} (g(x)F_i(x))^r dx\right)^{1/r} \tag{2.46}$$

for $i = 1, 2$.

The case $i = 1$ In the course of the proof of Theorem 1.2, we have established (2.25)

$$\left(\int_{Q_0} (g(x)F_1(x))^p dx\right)^{1/p} \le C\|g\|_{q,\eta}\left(\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)^p dx\right)^{1/p}. \tag{2.47}$$

We will use it with $p = r$

$$\left(\int_{Q_0} (g(x)F_1(x))^r dx\right)^{1/r} \le C\|g\|_{q,\eta}\left(\int_{Q_0} M_{\tilde{\rho}/\eta}f(x)^r dx\right)^{1/r}. \tag{2.48}$$

The case $i = 2$ It follows that

$$\rho(\ell(Q))m_{3Q}(f) \le C\|f\|_{p,\phi}\frac{\rho(\ell(Q))}{\phi(\ell(Q))} \tag{2.49}$$

from the Hölder inequality and the definition of the norm $\|f\|_{p,\phi}$. As a consequence we have

$$\begin{aligned} F_2(x) &\le C\|f\|_{p,\phi}\sum_{Q\in\mathfrak{D}_2(Q_0)}\frac{\rho(\ell(Q))}{\phi(\ell(Q))} \le C\|f\|_{p,\phi}\sum_{\nu=1+\log_2\ell(Q_0)}^{\infty}\frac{\rho(2^\nu)}{\phi(2^\nu)} \\ &\le C\|f\|_{p,\phi}\int_{\ell(Q_0)}^{\infty}\frac{\rho(s)}{s\phi(s)}ds \le C\|f\|_{p,\phi}\frac{\eta(t)}{\phi(t)^{p/r}}. \end{aligned} \tag{2.50}$$

Here, we have used the doubling condition (1.8) and the fact that ϕ is nondecreasing in the third inequality. Hence it follows that

$$\begin{aligned} \left(\int_{Q_0} (g(x)F_2(x))^r dx\right)^{1/r} &\le Cm_{Q_0}^{(r)}(g)\|f\|_{p,\phi}\frac{\eta(\ell(Q_0))}{\phi(\ell(Q_0))^{p/r}}|Q_0|^{1/r} \\ &\le C\|g\|_{q,\eta}\|f\|_{p,\phi}\phi(\ell(Q_0))^{-p/r}|Q_0|^{1/r}. \end{aligned} \tag{2.51}$$

Combining (2.48) and (2.51), we obtain

$$\|g \cdot T_\rho f\|_{r,\phi^{p/r}} \le C\|g\|_{q,\eta}\left(\|M_{\widetilde{\rho}/\eta} f\|_{r,\phi^{p/r}} + \|f\|_{p,\phi}\right). \tag{2.52}$$

We note that the assumption (1.24) implies $\widetilde{\rho}(t)/\eta(t) \le C\phi(t)^{1-p/r}$. Hence we arrive at the desired inequality by using Lemma 2.3. □

3. A Dual Version of Olsen's Inequality

In this section, as an application of Theorem 1.6, we consider a dual version of Olsen's inequality on predual of Morrey spaces (Theorem 3.1). As a corollary (Corollary 3.2), we have the boundedness properties of the operator T_ρ on predual of Morrey spaces. We will define the block spaces following [17].

Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Suppose that ϕ satisfies (1.4). We say that a function b on $\mathbb{R}^n$ is a (p',ϕ)-block provided that b is supported on a cube $Q \subset \mathbb{R}^n$ and satisfies

$$m_Q^{(p')}(b) \le \frac{\phi(\ell(Q))}{|Q|}. \tag{3.1}$$

The space $\mathcal{B}^{p',\phi}(\mathbb{R}^n) = \mathcal{B}^{p',\phi}$ is defined by the set of all functions f locally in $L^{p'}(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathcal{B}^{p',\phi}} := \inf\left\{ \|\{\lambda_k\}\|_{l^1} : f = \sum_k \lambda_k b_k \right\} < \infty, \tag{3.2}$$

where each b_k is a (p',ϕ)-block and $\|\{\lambda_k\}\|_{l^1} = \sum_k |\lambda_k| < \infty$, and the infimum is taken over all possible decompositions of f. If $\phi(t) \equiv t^{n/p_0}$, $p_0 \ge p$, $\mathcal{B}^{p',\phi}$ is the usual block spaces, which we write for $\mathcal{B}^{p',p_0'}$ and the norm for $\|\cdot\|_{\mathcal{B}^{p',p_0'}}$, because the right-hand side of (3.1) is equal to $|Q|^{1/p_0-1} = |Q|^{-1/p_0'}$. It is easy to prove

$$L^{p_0'} = \mathcal{B}^{p_0',p_0'} \supset \mathcal{B}^{p_1',p_0'} \supset \mathcal{B}^{p_2',p_0'} \tag{3.3}$$

when $1 < p_0' \le p_1' \le p_2' < \infty$. In [17, Theorem 1] and [18, Proposition 5], it was established that the predual space of $\mathcal{M}^{p,\phi}$ is $\mathcal{B}^{p',\phi}$. More precisely, if $g \in \mathcal{M}^{p,\phi}$, then $f \in \mathcal{B}^{p',\phi} \mapsto \int_{\mathbb{R}^n} f(x)g(x)dx$ is an element of $(\mathcal{B}^{p',\phi})^*$. Conversely, any continuous linear functional in $\mathcal{B}^{p',\phi}$ can be realized with some $g \in \mathcal{M}^{p,\phi}$.

Theorem 3.1. *Let $1 < p \le r < q < \infty$. Suppose that $\phi(t)$ and $\eta(t)$ are nondecreasing but that $\phi(t)^p t^{-n}$ and $\eta(t)^q t^{-n}$ are nonincreasing. Suppose also that*

$$\frac{\widetilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)} ds \le C\frac{\eta(t)}{\phi(t)^{p/r}} \quad \forall t > 0, \tag{3.4}$$

then

$$\|T_\rho(gf)\|_{\mathcal{B}^{p',\phi}} \le C\|g\|_{\mathcal{M}^{q,\eta}}\|f\|_{\mathcal{B}^{r',\phi^{p/r}}}, \tag{3.5}$$

if g is a continuous function.

Theorem 3.1 generalizes [1, Theorem 3.1], and its proof is similar to that theorem, hence omitted. As a special case when $g(x) \equiv 1$ and $\eta(t) \equiv 1$, we obtain the following.

Corollary 3.2. *Let $1 < p < \infty$. Suppose that ϕ is nondecreasing but that $\phi(t)^p t^{-n}$ is nonincreasing. Suppose also that*

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}ds \le \frac{C}{\phi(t)^{p/r}} \quad \forall t > 0, \tag{3.6}$$

then

$$\|T_\rho f\|_{\mathcal{B}^{p',\phi}} \le C\|f\|_{\mathcal{B}^{r',\phi^{p/r}}}. \tag{3.7}$$

We dare restate Corollary 3.2 in terms of the fractional integral operator I_α. The results hold by letting $\rho(t) \equiv t^{n\alpha}$, $\phi(t) \equiv t^{n/p_0}$, $\eta(t) \equiv 1$, and $g(x) \equiv 1$.

Proposition 3.3 (see [1, Proposition 3.8]). *Let $0 < \alpha < 1$, $1 < p \le p_0 < \infty$, and $1 < r \le r_0 < \infty$. Suppose that $1/p_0 > \alpha$, $1/r_0 = 1/p_0 - \alpha$, and $r/r_0 = p/p_0$, then*

$$\|I_\alpha f\|_{\mathcal{B}^{p',p'_0}} \le C\|f\|_{\mathcal{B}^{r',r'_0}}. \tag{3.8}$$

Remark 3.4 (see [1, Remark 3.9]). In Proposition 3.3, if $r/r_0 = p/p_0$ is replaced by $1/r = 1/p - \alpha$, then, using the Hardy-Littlewood-Sobolev inequality locally and taking care of the larger scales by the same manner as the proof of Theorem 3.1, one has a naive bound for I_α.

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