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Research Article

Positive Solutions of Singular Initial-Boundary Value Problems to Second-Order Functional Differential Equations

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Positive solutions to the singular initial-boundary value problems $x'' = -f(t, x_t)$, $0 < t < 1$, $x_0 = 0$, $x(1) = 0$, are obtained by applying the Schauder fixed-point theorem, where $x_t(u) = x(t+u)$ $(0 \leq t \leq 1)$ on $[-r, 0]$ and $f(\cdot,\cdot) : (0,1) \times (C^+\setminus\{0\}) \to R^+ (C^+ = \{x \in C([-r,0], R),\ x(t) \geq 0,\ \forall t \in [-r,0]\})$ may be singular at $\varphi(u) = 0\,(-r \leq u \leq 0)$ and $t = 0$. As an application, an example is given to demonstrate our result.

1. Introduction

Recently, in [1–4], Erbe, Kong, Jiang, Wang, and Weng considered the following singular functional differential equations:

$$\begin{aligned} x'' &= -f(t, x(\tau(t))), \quad 0 < t < 1, \\ \alpha x(t) - \beta x'(t) &= \mu(t), \quad a \leq t \leq 0, \\ \gamma x(t) + \delta x'(t) &= \nu(t), \quad 1 \leq t \leq b, \end{aligned} \tag{1.1}$$

where $a = \min\{0, \inf\{\tau(t) : 0 \leq t \leq 1\}\}$, $b = \max\{1, \sup\{\tau(t) : 0 \leq t \leq 1\}\}$, and the existence of positive solutions to (1.1) is obtained. When $\tau(t) = t - r$ in (1.1), Agarwal and O'Regan in [5], Lin and Xu in [6] discussed the existence of positive solutions to (1.1) also. We notice that the nonlinearities $f(t, u)$ in all the above-mentioned references depend on $(t, u) \in (0,1) \times R$.

The more difficult case is that the term $f(t, \varphi)$ depends on $(t, \varphi) \in (0,1) \times C([0,1], R)$ for second-order functional differential equations with delay. When $f(t, \varphi)$ has no singularity at $t = 0$ and $\varphi = \theta$, there are many results on the following (1.2) (see [7–9] and references therein). Up to now, to our knowledge, there are fewer results on (1.2) when the term $f(t,\varphi)$ is allowed to possess singularity for the term $f(t,\varphi)$ at $t = 0$ and $\varphi = 0$, which is of more actual significance.

In this paper, motivated by above results, we consider the second-order initial-boundary value problems:

$$\begin{gathered} x'' = -f(t, x_t), \quad 0 < t < 1, \\ x_0 = 0, \\ x(1) = 0, \end{gathered} \tag{1.2}$$

where $f : (0,1) \times (C^+ \setminus 0) \to (0,\infty)$ $(C^+ = \{x \in C([-r,0],R),\ x(t) \geq 0,\ \forall t \in [-r,0]\})$, $x_t = x(t+u)$ $(-r \leq u \leq 0)$. By Leray-Schauder fixed-point theorem, the existence of positive solutions to (1.2) is obtained when $f(t,\varphi)$ is singular at $t = 0$ and $\varphi = 0$.

For $\varphi \in C([-r,0],R)$ and $x \in C([-r,1],R)$, let $\|\varphi\| = \max_{t\in[-r,0]}|\varphi(t)|$ and $\|x\| = \max_{t\in[-r,1]}|x(t)|$. Then, $C([-r,0],R)$ and $C([-r,1],R)$ are Banach spaces. Let $C^+ = \{x \in C([-r,0],R),\ x(t) \geq 0,\ \forall t \in [-r,0]\}$ and $P = \{x \in C([-r,1],R),\ x(t) \geq 0,\ \forall t \in [-r,1]\}$. Obviously, C^+ and P are cones in $C([-r,0],R)$ and $C([-r,1],R)$, respectively. Now, we give a new definition.

Definition 1.1. $f(t,\varphi)$ is said to be singular at $t = 0$ for $\varphi \in (C^+ - \{0\})$, when $f(t,\varphi)$ satisfies $\lim_{t\to 0} f(t,\varphi) = +\infty$ for $\varphi \in (C^+ - \{0\})$ and $f(t,\varphi)$ is said to be singular at $\varphi = 0$ for $t \in (0,1)$ when $f(t,\varphi)$ satisfies $\lim_{\|\varphi\|\to 0} f(t,\varphi) = +\infty$ for $t \in (0,1)$.

And one defines some functions which one has to use in this paper.

Let

$$\begin{gathered} h(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ t(1-t), & 0 \leq t \leq 1, \end{cases} \\ G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \end{gathered} \tag{1.3}$$

where $G(t,s)$ is a Green's function. It is clear that $G(t,s) > 0$ for $(t,s) \in (0,1) \times (0,1)$ and $h(t)\,h(s) \leq G(t,s) \leq h(s)$ on $[0,1] \times [0,1]$.

We now introduce the definition of a solution to IBVP(1.2).

Definition 1.2. A function x is said to be a solution to IBVP(1.2) if it satisfies the following conditions:

(1) $x(t)$ is continuous and nonnegative on $[-r,1]$;

(2) $x_0 = 0$, $x(1) = 0$;

(3) $x'(t)$ and $x''(t)$ exist on $(0,1)$;

(4) $h(t)|x''(t)|$ is Lebesgue integrable on $[0,1]$;

(5) $x''(t) = -f(t, x_t)$ for $t \in (0,1)$.

Furthermore, a solution x is said to be positive if $x(t) > 0$ on $(0,1)$.

Let x be a solution to IBVP(1.2). Then, it can be represented as

$$x(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int_0^1 G(t,s) f(s,x_s)\,\mathrm{d}s, & 0 \le t \le 1. \end{cases} \tag{1.4}$$

It is clear that

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) f(s,x_s)\,\mathrm{d}s \le \int_0^1 h(s) f(s,x_s)\,\mathrm{d}s \quad \text{for } t \in [0,1], \\ x(t) &\ge h(t) \int_0^1 h(s) f(s,x_s)\,\mathrm{d}s \ge \|x\| h(t) \quad \text{on } [0,1], \end{aligned} \tag{1.5}$$

for all solutions, x, to IBVP(1.2), where $\|x\| = \max_{0\le t\le 1} x(t)$. For $\xi \in R^+$, let $\widetilde{\xi}(u) \equiv \xi$ on $[-r,1]$ throughout this paper. Obviously, $\widetilde{\xi} \in C^+([-r,1],R)$ and $\widetilde{\xi}_0 = \widetilde{\xi}_t$ for all $t \in (0,1]$.

Throughout this paper, we assume the following hypotheses hold.

(H_1) $f(t,\varphi)$ is continuous on $(0,1) \times (C^+ \setminus \{0\})$.

(H_2) There exists $\varepsilon > 0$, such that

$$\begin{aligned} f(t,\varphi) &\ge f(t,\widetilde{\varepsilon}_0), \quad \text{for } \|\varphi\| \le \varepsilon, \\ 0 &< \int_0^1 h(s) f(s,\widetilde{\varepsilon}_0)\,\mathrm{d}s < \infty. \end{aligned} \tag{1.6}$$

Lemma 1.3. *Assume that (H_1)-(H_2) hold, then there exists a $\theta^* > 0$, such that*

$$x(t) \ge \theta^* h(t), \quad \text{on } [0,1], \tag{1.7}$$

for all solutions, x, to (1.2).

Proof. Suppose that the claim is false. (1.5) guarantees that there exists a sequence $\{x_m(t)\}$ of solutions to IBVP(1.2) such that

$$\lim_{m\to\infty} \|x_m\| = 0. \tag{1.8}$$

Without loss of generality, we may assume that

$$\varepsilon \ge \|x_m\| \ge \|x_{m+1}\| \quad \forall m \ge 1. \tag{1.9}$$

From (H_1), (H_2), and (1.5), it follows that

$$\begin{aligned} x_m\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(s, x_{m_s})\,\mathrm{d}s \\ &\ge h\left(\frac{1}{2}\right) \int_0^1 h(s) f(s, x_{m_s})\,\mathrm{d}s \\ &\ge h\left(\frac{1}{2}\right) \int_0^1 h(s) f(s, \widetilde{\varepsilon}_s)\,\mathrm{d}s \\ &> 0, \end{aligned} \tag{1.10}$$

which contradicts the assumption that $\lim_{m\to\infty}\|x_m\| = 0$ and hence the claim is true provided θ^* is suitably small. □

Remark 1.4. The following inequality

$$\int_0^1 h(s)f(s,\tilde{\varepsilon}_0)\mathrm{d}s \geq \theta \tag{1.11}$$

holds provided that $\theta < \min\{\varepsilon, \theta^*\}$ is sufficiently small, where θ^* is in Lemma 1.3.

(H_3) There exist a nonnegative continuous function $k(\cdot)$ defined on (0,1) and two nonnegative continuous functions $F_1(\varphi)$, $F_2(\varphi)$ defined on, respectively, $C^+ \setminus \{0\}$, C^+, such that

$$f(t,\varphi) \leq k(t)\left[F_1(\varphi) + F_2(\varphi)\right] \quad \text{for } (t,\varphi) \in (0,1) \times \left(C^+ \setminus \{0\}\right), \tag{1.12}$$

where $k(t), F_1(\varphi)$, and $F_2(\varphi)$ satisfy

$$\int_0^1 h(s)k(s)\mathrm{d}s < \infty, \qquad \int_0^1 h(s)k(s)F_1(\theta h_s)\mathrm{d}s < \infty, \qquad \lim_{\|\varphi\|\to\infty} \frac{|F_2(\varphi)|}{\|\varphi\|} = 0. \tag{1.13}$$

Furthermore, $F_1(\varphi)$ is nonincreasing and $F_2(\varphi)$ is nondecreasing, that is,

$$\begin{aligned} F_1(\varphi) &\geq F_1(\varphi) \quad \text{for } \varphi(u) \leq \varphi(u) \text{ on } [-r,0], \\ F_2(\varphi) &\leq F_2(\varphi) \quad \text{for } \varphi(u) \leq \varphi(u) \text{ on } [-r,0]. \end{aligned} \tag{1.14}$$

Lemma 1.5 (see [7]). *Let E be the Banach space and let X be any nonempty, convex, closed, and bounded subset of E. If T is a continuous mapping of X into itself and TX is relatively compact, then the mapping T has at least one fixed point (i.e., there exists an $x \in X$ with $x = Tx$) .*

Using Lemma 1.5, we present the existence of at least one positive solution to (1.2) when $f(t,\varphi)$ is singular at $\varphi = 0$ and $t = 0$ (notice the new Definition 1.1). To some extent, our paper complements and generalizes these in [1–6, 8–10].

2. Main results

Theorem 2.1. *Assume that (H_1)–(H_3) hold. Then, the IBVP*(1.2) *has at least one positive solution.*

Proof. Since $\lim_{\|\varphi\|\to\infty}(|F_2(\varphi)|/\|\varphi\|) = 0$, we can choose an $N > \varepsilon$ such that

$$F_2(\varphi) \leq \mu\|\varphi\|, \quad \text{for } \|\varphi\| \geq N, \tag{2.1}$$

where the positive number μ satisfies

$$0 < \mu\int_0^1 h(s)k(s)\mathrm{d}s = \sigma < 1. \tag{2.2}$$

Let

$$\begin{aligned} R &= \int_0^1 h(s)k(s)F_1\left(\theta h_s\right)\mathrm{d}s, \\ T &= \int_0^1 h(s)k(s)F_2\left(\widetilde{N}_s\right)\mathrm{d}s, \\ M^* &= \frac{R+T+N}{1-\sigma}. \end{aligned} \tag{2.3}$$

For each $x \in P \subseteq C([-r,1],R)$, we define $x^*(t)$ by

$$x^*(t) = \begin{cases} 0, & -r \le t \le 0, \\ \theta h(t), & \text{if } x(t) < \theta h(t) \text{ on } (0,1], \\ x(t), & \text{if } \theta h(t) \le x(t) \le M^* \text{ on } (0,1], \\ M^*, & \text{if } x(t) > M^* \text{ on } (0,1], \end{cases}$$
$$f^*(t,x_t) = f(t,x_t^*) \quad \text{for } t \in (0,1). \tag{2.4}$$

It is obvious that $f^*(t,x_t)$ satisfies the hypotheses (H_1)–(H_3) and $M^* > N$.
We now consider the modified initial-boundary value problem:

$$\begin{gathered} x'' = -f^*(t,x_t), \quad 0 < t < 1, \\ x_0 = 0, \\ x(1) = 0. \end{gathered} \tag{2.5}$$

We claim that for all solutions, x, to IBVP(2.5),

$$x(t) \ge \theta h(t), \quad \text{on } [-r,1]. \tag{2.6}$$

Suppose that the claim is false. Then there exists $t' \in (0,1)$ such that

$$x(t') < \theta h(t'). \tag{2.7}$$

Since $x(t) = h(t)$ on $[-r,0]$, there are the following three cases.

Case 1. $x(t) < \theta h(t)$ for all $t \in (0,1)$.

The solution of IBVP(2.5) can be represented as (notice $\theta < \min\{\varepsilon,\theta^*\}$ Remark 1.4)

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) f^*(s,x_s)\,ds \\ &= \int_0^1 G(t,s) f(s,x_s^*)\,ds \\ &\ge \int_0^1 G(t,s) f(s,\theta h_s)\,ds \\ &\ge h(t)\int_0^1 h(s) f(s,\tilde{\varepsilon}_s)\,ds \ (\text{notice } H_2) \\ &= \int_0^1 h(s) f(s,\tilde{\varepsilon}_0)\,ds \\ &> \theta h(t), \quad t \in (0,1], \end{aligned} \tag{2.8}$$

which contradicts (2.7).

Case 2. There exists a $t_0 \in (0,1)$ such that $x(t_0) > \theta h(t_0)$ and $\|x\| < \theta$.

In this case, we have

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) f^*(s,x_s)\mathrm{d}s \\ &\geq h(t)\int_0^1 h(s) f(s,\tilde{\varepsilon}_s)\mathrm{d}s \\ &= h(t)\int_0^1 h(s) f(s,\tilde{\varepsilon}_0)\mathrm{d}s \\ &\geq \theta h(t), \quad t \in (0,1], \end{aligned} \tag{2.9}$$

which contradicts (2.7).

Case 3. There exists a $t_0 \in (0,1)$ such that $x(t_0) > \theta h(t_0)$ and $\|x\| \geq \theta$.

From (1.5), we get

$$x(t) \geq \|x\| h(t) \geq \theta h(t), \quad t \in (0,1], \tag{2.10}$$

which contradicts (2.7).

So we have

$$x(t) \geq \theta h(t) \quad \text{on } [-r,1]. \tag{2.11}$$

To prove the existence of positive solutions to IBVP(2.5), we seek to transform (2.5) into an integral equation via the use of Green's function and then find a positive solution by using Lemma 1.5.

Define a nonempty convex and closed subset of $C([-r,1],R)$ by

$$D = \{x \in C([-r,1],R) : 0 \leq x(t) \leq M^*,\ t \in [0,1],\ x(t) = 0,\ t \in [-r,0]\}. \tag{2.12}$$

Then, we define an operator $T : D \to C([-r,1],R)$ by

$$(Tx)(t) = \begin{cases} 0, & \text{if } -r \leq t \leq 0, \\ \displaystyle\int_0^1 G(t,s) f^*(s,x_s)\mathrm{d}s, & \text{if } 0 \leq t \leq 1. \end{cases} \tag{2.13}$$

From (H_1)–(H_3) and the definition of T, we have, for every $x \in D$,

$$(Tx)(t) \in C[-r,1], \quad (Tx)(t) \geq 0 \quad \text{on } [0,1], \tag{2.14}$$

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s) f^*(s,x_s)\mathrm{d}s \\
&\leq \int_0^1 h(s) f^*(s,x_s)\mathrm{d}s \\
&\leq \int_0^1 h(s) f(s,x_s^*)\mathrm{d}s \\
&\leq \int_0^1 h(s)k(s)\left[F_1(x_s^*) + F_2(x_s^*)\right]\mathrm{d}s \\
&\leq \int_0^1 h(s)k(s)\left[F_1(\theta h_s) + F_2(x_s^*)\right]\mathrm{d}s \\
&\leq \int_0^1 h(s)k(s)F_1(\theta h_s)\mathrm{d}s + \int_0^1 h(s)k(s)F_2(x_s^*)\mathrm{d}s \\
&\leq R + \int_0^1 h(s)k(s)F_2(x_s^*)\mathrm{d}s \\
&\leq R + \int_0^1 h(s)k(s)F_2(\widetilde{M^*}_s)\mathrm{d}s \\
&\leq R + \int_0^1 h(s)k(s)\mu M^*\mathrm{d}s \\
&\leq R + \sigma M^* \\
&\leq M^*, \quad t \in (0,1].
\end{aligned} \tag{2.15}$$

Together with the definition of D, we get $T(D) \subset D$.
Also,

$$(Tx)'(t) = -\int_0^t s f^*(s,x_s)\mathrm{d}s + \int_t^1 (1-s) f^*(s,x_s)\mathrm{d}s \tag{2.16}$$

is continuous in (0,1), and

$$(Tx)''(t) = -f^*(t,x_t) \leq 0 \quad \text{in } (0,1). \tag{2.17}$$

From H_3 and (2.15), we can get

$$\begin{aligned}
\int_0^1 h(t)\left|(Tx)''(t)\right|\mathrm{d}t &= \int_0^1 h(t) f^*(t,x_t)\mathrm{d}t \\
&\leq M^* < +\infty,
\end{aligned} \tag{2.18}$$

which implies that $h(t)|(Tx)''(t)|$ is integrable on $[0,1]$.

Now, we claim that $T(D)$ is equicontinuous on $[-r,1]$. We will prove the claim. For any $x \in D$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s) f^*(s,x_s)\,\mathrm{d}s \\ &\le \int_0^1 G(t,s)k(s)\left[F_1(\theta h_s) + F_2(\widetilde{M}_s^*)\right]\mathrm{d}s \\ &= U(t), \quad 0 \le t \le 1.\end{aligned} \tag{2.19}$$

Since $U(t)$ is continuous on $[0,1]$ and $U(0) = U(1) = 0$, then for any $\varepsilon_0 > 0$, there is a $\delta \in (0,1/4)$ such that

$$0 \le (Tx)(t) \le U(t) < \frac{\varepsilon_0}{2}, \quad t \in [0,2\delta] \cup [1-2\delta,1]. \tag{2.20}$$

By (2.6), we have, for $t \in [\delta, 1-\delta]$,

$$\begin{aligned}\left|(Tx)'(t)\right| &\le \left| -\int_0^t s f^*(s,x_s)\,\mathrm{d}s + \int_t^1 (1-s) f^*(s,x_s)\,\mathrm{d}s \right| \\ &\le \int_0^{1-\delta} s f^*(s,x_s)\,\mathrm{d}s + \int_\delta^1 (1-s) f^*(s,x_s)\,\mathrm{d}s \\ &\le \int_0^{1-\delta} s k(s)\left[F_1(\theta h_s) + F_2(\widetilde{M^*}_s)\right]\mathrm{d}s + \int_\delta^1 (1-s)k(s)\left[F_1(\theta h_s) + F_2(\widetilde{M^*}_s)\right]\mathrm{d}s \\ &\le \frac{1}{\delta}\int_0^{1-\delta} (1-s)sk(s)\left[F_1(\theta h_s)+F_2(\widetilde{M^*}_s)\right]\mathrm{d}s + \frac{1}{\delta}\int_\delta s(1-s)k(s)\left[F_1(\theta h_s)+F_2(\widetilde{M^*}_s)\right]\mathrm{d}s \\ &\le \frac{2}{\delta}\int_0^1 h(s)k(s)\left[F_1(\theta h_s) + F_2(\widetilde{M^*}_s)\right]\mathrm{d}s \\ &= \frac{2}{\delta}K \\ &= L,\end{aligned} \tag{2.21}$$

where $K = \int_0^1 h(s)k(s)[F_1(\theta h_s) + F_2(\widetilde{M^*}_s)]\mathrm{d}s < \infty$ is a constant number.

Put $\delta_1 = \varepsilon_0/L$, then for $t_1, t_2 \in [\delta, 1-\delta]$, $|t_1 - t_2| < \delta_1$,

$$\left|(Tx)(t_1) - (Tx)(t_2)\right| \le L\left|t_1 - t_2\right| < \varepsilon_0. \tag{2.22}$$

Set $\delta_0 = \min\{\delta, \delta_1\}$. Then for $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta_0$, and

$$\left|(Tx)(t_1) - (Tx)(t_2)\right| < \varepsilon_0. \tag{2.23}$$

Since $(Tx)(t) = 0$ on $t \in [-r,0]$, the above inequality holds for $t \in [-r,1]$.

Thus, $T(D)$ is a relative compact subset of D. That is, $T : D \to D$ is a compact operator.

We are now going to prove that the mapping T is continuous on D.

Let $\{x_n(t)\}_{n=0}^{\infty} \subset D$ be arbitrarily chosen and let $x_n(t)$ converge to $x_0(t)$ uniformly on $[-r, 1]$ as $n\to\infty$. Now, we claim that $x_n^*(t)$ converge to $x_0^*(t)$ uniformly as $n\to\infty$. From the definition of $x^*(t)$, we get

$$
\begin{aligned}
x_n^*(t) &= \frac{x_n(t)+\theta h(t)}{2} + \frac{|x_n(t)-\theta h(t)|}{2}, \quad t \in [-r,1], \\
x_0^*(t) &= \frac{x_0(t)+\theta h(t)}{2} + \frac{|x_0(t)-\theta h(t)|}{2}, \quad t \in [-r,1].
\end{aligned}
\tag{2.24}
$$

Thus,

$$
\begin{aligned}
\left|x_n^*(t) - x_0^*(t)\right| &= \left|\frac{x_n(t)+\theta h(t)}{2} + \frac{|x_n(t)-\theta h(t)|}{2} - \frac{x_0(t)+\theta h(t)}{2} - \frac{|x_0(t)-\theta h(t)|}{2}\right| \\
&\le \left|\frac{x_n(t)-x_0(t)}{2} + \frac{|x_n(t)+\theta h(t)| - |x_0(t)+\theta h(t)|}{2}\right| \\
&\le \left|\frac{x_n(t)-x_0(t)}{2}\right| + \left|\frac{|x_n(t)+\theta h(t)| - |x_0(t)+\theta h(t)|}{2}\right| \\
&\le \left|\frac{x_n(t)-x_0(t)}{2}\right| + \left|\frac{x_n(t)-x_0(t)}{2}\right| \\
&= \left|x_n(t)-x_0(t)\right|, \quad t \in [-r,1],
\end{aligned}
\tag{2.25}
$$

that is, the claim is true.

Since $f(t, \varphi)$ is continuous with respect to φ for $t \in (0,1)$, we have

$$
\lim_{n\to\infty} G(t,s) f^*(s, x_{ns}) = G(t,s) f^*(s, x_{0s}) \quad \text{on } [0,1],
\tag{2.26}
$$

for each fixed $t \in [0,1]$. From the definition of f^* and (H_3), we know that

$$
0 \le f^*(t, x_{nt}) \le k(t)\left[F_1(\theta h_t) + F_2(\widetilde{M^*}_t)\right],
\tag{2.27}
$$

and hence

$$
0 \le G(t,s) f^*(s, x_{ns}) \le h(s)k(s)\left[F_1(\theta h_s) + F_2(\widetilde{M^*}_s)\right], \quad \text{for } (t,s) \in (0,1)\times(0,1),
\tag{2.28}
$$

where $h(s)k(s)[F_1(\theta h_s)+F_2(\widetilde{M^*}_s)]$ is a Lebesgue integrable function defined on $[0,1]$ because of (H_3). Consequently, we apply the dominated convergence theorem to get

$$
\begin{aligned}
\lim_{n\to\infty}\left|(Tx_n)(t) - (Tx_0)(t)\right| &= \lim_{n\to\infty}\max_{t\in[0,1]}\left|\int_0^1 G(t,s)\left[f^*(s,x_{ns}) - f^*(s,x_{0s})\right]\mathrm{d}s\right| \\
&\le \int_0^1 \max_{t\in[0,1]} G(t,s) \lim_{n\to\infty}\left|\left[f^*(s,x_{ns}) - f^*(s,x_{0s})\right]\right|\mathrm{d}s \\
&= 0,
\end{aligned}
\tag{2.29}
$$

which shows that the mapping T is continuous on D.

Then from Lemma 1.5, we get that there exists at least one positive solution, x, to IBVP(2.5) in D. The solution can be represented by (1.4), where f is replaced with f^*. So, (2.6) holds. Furthermore, from the definition of D, we can get

$$
x(t) \le M^*.
\tag{2.30}
$$

Thus, the solution of IBVP(2.5) is also the one of (1.2). The proof is complete. □

3. Application

Example 3.1. Consider the singular IBVP(3.1):

$$x'' + \frac{1}{t^{\alpha}\left(\int_{-r}^{0} x(t+u)\mathrm{d}u\right)^{\beta}} + \sin(\pi t) + \left[\max\left\{x(t+u) : -r \le u \le 0\right\}\right]^{\gamma} = 0, \quad 0 < t < 1,$$
$$x_0 = 0,$$
$$x(1) = 0, \tag{3.1}$$

where $\alpha > 0$, $\beta > 0$, $0 < \gamma < 1$, $\alpha + \beta < 1$.

4. Conclusion

Equation (3.1) has at least one positive solution.

Now, we will check that (H_1)–(H_3) hold in (3.1).

In IBVP(3.1), $f(t,\varphi) = (1/t^{\alpha}[\int_{-r}^{0}\varphi(u)du]^{\beta}) + \sin(\pi t) + [\max\{\varphi(u) : -r \le u \le 0\}]^{\gamma}$. It is clear that $f : (0,1] \times C^{+} \to (0,\infty)$ is continuous and singular at $t = 0$ and $\varphi = 0$. For (H_3), we choose

$$k(t) = \frac{1}{t^{\alpha}}, \qquad F_1(\varphi) = \frac{1}{\left[\int_{-r}^{0}\varphi(u)\mathrm{d}u\right]^{\beta}}, \qquad F_2(\varphi) = \left[\max\left\{\varphi(u) : -r \le u \le 0\right\}\right]^{\gamma} + 1, \tag{4.1}$$

when $\alpha > 0$, $\beta > 0$, $0 < \gamma < 1$, $\alpha + \beta < 1$; by simple computation, we can get

$$\int_0^1 h(s)k(s)\mathrm{d}s < \infty, \quad \int_0^1 h(s)k(s)F_1(s,\theta h_s)\mathrm{d}s < \infty \quad \text{for } 0 < \theta < +\infty, \quad \lim_{\|\varphi\|\to\infty}\frac{|F_2(\varphi)|}{\|\varphi\|} = 0. \tag{4.2}$$

It is obvious that $F_1(\varphi)$ is nonincreasing and $F_2(\varphi)$ is nondecreasing.

Now, we check (H_2). For any $\varepsilon > 0$, $\varphi \in C^{+}$, $\|\varphi\| \le \varepsilon$ (notice the definition of $\|\cdot\|$), we have

$$0 \le \left[\int_{-r}^{0}\varphi(u)\mathrm{d}u\right]^{\beta} \le \left[\int_{-r}^{0}\varepsilon\mathrm{d}u\right]^{\beta} = (r\varepsilon)^{\beta}, \tag{4.3}$$

$$\begin{aligned} f(t,\varphi) - f(t,\tilde{\varepsilon}_0) &= \frac{1}{t^{\alpha}}\left[\frac{1}{\left[\int_{-r}^{0}\varphi(u)\mathrm{d}u\right]^{\beta}} - \frac{1}{(r\varepsilon)^{\beta}}\right] + (\|\varphi\|)^{\gamma} - (\varepsilon)^{\gamma} \\ &\ge \frac{1}{\left[\int_{-r}^{0}\varphi(u)\mathrm{d}u\right]^{\beta}} - \frac{1}{(r\varepsilon)^{\beta}} + (\|\varphi\|)^{\gamma} - (\varepsilon)^{\gamma} \quad \text{(notice (3.4))} \\ &\ge \frac{1}{(\|\varphi\| r)^{\beta}} + (\|\varphi\|)^{\gamma} - \left[\frac{1}{(r\varepsilon)^{\beta}} + (\varepsilon)^{\gamma}\right]. \end{aligned} \tag{4.4}$$

We define

$$g(x) = \frac{1}{(rx)^{\beta}} + (x)^{\gamma}, \quad \text{for } x \in (0,+\infty). \tag{4.5}$$

Now, we will prove that there exists $\varepsilon > 0$ such that $g(\cdot)$ is decreasing on $(0, \varepsilon]$. Obviously,

$$g'(x) = \frac{\gamma r^{\beta} x^{1+\beta} - \beta x^{1-\gamma}}{r^{\beta} x^{1-\gamma} x^{1+\beta}}. \tag{4.6}$$

Put $g_1(x) = \gamma r^{\beta} x^{1+\beta} - \beta x^{1-\gamma}$, then

$$\begin{gathered} g_1(0) = 0, \\ {g_1}'(x) = \gamma(1+\beta)(rx)^{\beta} - (1-\gamma)\beta x^{-\gamma}, \\ \lim_{t \to 0^+} {g_1}'(x) = -\infty. \end{gathered} \tag{4.7}$$

From the continuity of ${g_1}'(x)$, we can find $\varepsilon > 0$ such that ${g_1}'(x) < 0$ on $(0, \varepsilon]$. Then, $g'(x) < 0$ on $(0, \varepsilon]$. That is, $g(x)$ is decreasing on $(0, \varepsilon]$.

Furthermore, we have

$$\begin{aligned} \int_0^1 h(s) f(s, \tilde{\varepsilon}_s) \mathrm{d}s &= \int_0^1 s(1-s) f(s, \tilde{\varepsilon}_s) \mathrm{d}s \\ &= \int_0^1 s(1-s) \left[\frac{1}{s^{\alpha}} \frac{1}{\left[\int_{-r}^{0} \varepsilon \mathrm{d}u\right]^{\beta}} + \varepsilon + \sin(\pi s) \right] \mathrm{d}s \\ &= \int_0^1 s^{1-\alpha}(1-s) \frac{1}{(r\varepsilon)^{\beta}} \mathrm{d}s + \int_0^1 s(1-s)\varepsilon \,\mathrm{d}s + \int_0^1 s(1-s) \sin(\pi s) \mathrm{d}s. \end{aligned} \tag{4.8}$$

Thus,

$$0 < \int_0^1 h(s) f(s, \tilde{\varepsilon}_s) \mathrm{d}s < \infty, \tag{4.9}$$

which implies that (H_2) holds.

So, from Theorem 2.1, IBVP(3.1) has at least one positive solution.

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