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Research Article

Generalizations of the Lax-Milgram Theorem

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We prove a linear and a nonlinear generalization of the Lax-Milgram theorem. In particular, we give sufficient conditions for a real-valued function defined on the product of a reflexive Banach space and a normed space to represent all bounded linear functionals of the latter. We also give two applications to singular differential equations.

1. Introduction

The following generalization of the Lax-Milgram theorem was proved recently by An et al. in [1].

Theorem 1.1. *Let X be a reflexive Banach space over $\mathbb{R}$, let $\{X_n\}_{n\in\mathbb{N}}$ be an increasing sequence of closed subspaces of X and $V=\bigcup_{n\in\mathbb{N}}X_n$. Suppose that*

$$A: X\times V \longrightarrow \mathbb{R} \tag{1.1}$$

is a real-valued function on $X\times V$ for which the following hold:

(a) *$A_n = A|_{X_n\times X_n}$ is a bounded bilinear form, for all $n\in\mathbb{N}$;*

(b) *$A(\cdot, v)$ is a bounded linear functional on X, for all $v\in V$;*

(c) *A is coercive on V, that is, there exists $c>0$ such that*

$$A(v,v)\geq c\|v\|^2, \tag{1.2}$$

for all $v\in V$.

Then, for each bounded linear functional v^* on V, there exists $x \in X$ such that

$$A(x,v) = \langle v^*, v\rangle, \tag{1.3}$$

for all $v \in V$.

In this paper our aim is to prove a linear extension and a nonlinear extension of Theorem 1.1. In the linear case, we use a variant of a theorem due to Hayden [2, 3], and thus manage to substitute the coercivity condition in (c) of the previous theorem with a more general inf-sup condition. In the nonlinear case, we appropriately modify the notion of type M operator and use a surjectivity result for monotone, hemicontinuous, coercive operators. We also present two examples to illustrate the applicability of our results.

All Banach spaces considered are over $\mathbb{R}$. Given a Banach space X, X^* will denote its dual and $\langle \cdot, \cdot \rangle$ will denote their duality product. Moreover, if M is a subset of X, then $M^\perp$ will denote its annihilator in X^* and if N is a subset of X^*, then ${}^\perp N$ will denote its preannihilator in X.

2. The linear case

To prove our main result for the linear case, we need the following lemma which is a variant of [2, Theorem 12] and [3, Theorem 1].

Lemma 2.1. *Let X be a reflexive Banach space, let Y be a Banach space and let*

$$A : X \times Y \longrightarrow \mathbb{R} \tag{2.1}$$

be a bounded, bilinear form satisfying the following two conditions:

(a) *A is nondegenerate with respect to the second variable, that is, for each $y \in Y \setminus \{0\}$, there exists $x \in X$ with $A(x,y) \neq 0$;*

(b) *there exists $c > 0$ such that*

$$\sup_{\|y\|=1} |A(x,y)| \geq c\|x\|, \tag{2.2}$$

for all $x \in X$.

Then, for every $y^* \in Y^*$, there exists a unique $x \in X$ with

$$A(x,y) = \langle y^*, y\rangle, \tag{2.3}$$

for all $y \in Y$.

Proof. Let $T : X \to Y^*$ with $\langle Tx, y\rangle = A(x,y)$, for all $x \in X$ and all $y \in Y$. Obviously ,T is a bounded linear map. Since, by (b), $\|Tx\| \geq c\|x\|$, for all $x \in X$, T is one to one. To complete the proof, we need to show that T is onto.

Since A is nondegenerate with respect to the second variable, we have that

$${}^\perp T(X) = \{y \in Y \mid A(x,y) = 0,\ \forall x \in X\} = \{0\}. \tag{2.4}$$

Hence

$$({}^{\perp}T(X))^{\perp} = Y^*, \tag{2.5}$$

and so by [4, Proposition 2.6.6],

$$\overline{T(X)}^{w^*} = Y^*. \tag{2.6}$$

Thus to show that T maps X onto Y^*, we need to prove that $T(X)$ is w^*-closed in Y^*. To see that, let $\{Tx_\lambda\}_{\lambda\in\Lambda}$ be a net in $T(X)$ and let y^* be an element of Y^* such that

$$Tx_\lambda \xrightarrow{w^*} y^*. \tag{2.7}$$

Without loss of generality, we may assume, using the special case of the Krein-Šmulian theorem on w^*-closed linear subspaces (see [4, Corollary 2.7.12]), the proof of which is originally due to Banach [5, Theorem 5, page 124] for the separable case and due to Dieudonné [6, Theorem 23] for the general case, that $\{Tx_\lambda\}_{\lambda\in\Lambda}$ is bounded. Thus, since $\|Tx\| \geq c\|x\|$ for all $x \in X$, the net $\{x_\lambda\}_{\lambda\in\Lambda}$ is also bounded. Hence, since X is reflexive, there exist a subnet $\{x_{\lambda_\mu}\}_{\mu\in M}$ and an element x of X such that $\{x_{\lambda_\mu}\}_{\mu\in M}$ converges weakly to x. Since T is $w - w^*$ continuous, $Tx_{\lambda_\mu} \xrightarrow{w^*} Tx$. Hence $Tx = y^*$, and so $T(X)$ is w^*-closed. □

Remark 2.2. An alternative proof of the previous lemma can be obtained using the closed range theorem.

We are now in a position to prove our main result for the linear case.

Theorem 2.3. *Let X be a reflexive Banach space, let Y be a Banach space, let Λ be a directed set, let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subspaces of X, let $\{Y_\lambda\}_{\lambda\in\Lambda}$ be an upwards directed family of closed subspaces of Y, and let $V = \bigcup_{\lambda\in\Lambda} Y_\lambda$. Suppose that*

$$A : X \times V \longrightarrow \mathbb{R} \tag{2.8}$$

is a function for which the following hold:

(a) *$A_\lambda = A|_{X_\lambda\times Y_\lambda}$ is a bounded bilinear form, for all $\lambda \in \Lambda$;*
(b) *$A(\cdot, v)$ is a bounded linear functional on X, for all $v \in V$;*
(c) *A_λ is nondegenerate with respect to the second variable, for all $\lambda \in \Lambda$;*
(d) *there exists $c > 0$ such that for all $\lambda \in \Lambda$,*

$$\sup_{y\in Y_\lambda,\,\|y\|=1} \left|A_\lambda(x,y)\right| \geq c\|x\|, \tag{2.9}$$

for all $x \in X_\lambda$.

Then, for each bounded linear functional v^* on V, there exists $x \in X$ such that

$$A(x,v) = \langle v^*, v\rangle, \tag{2.10}$$

for all $v \in V$.

Proof. Let $v^* \in V^*$, and for each $\lambda \in \Lambda$, let $v^*_\lambda = v^*|_{Y_\lambda}$. For all $\lambda \in \Lambda$, v^*_λ is a bounded linear functional on Y_λ. By hypothesis, for all $\lambda \in \Lambda$, A_λ is a bounded bilinear form on $X_\lambda \times Y_\lambda$ satisfying the two conditions of Lemma 2.1. Since for all $\lambda \in \Lambda$, X_λ is a reflexive Banach space, we get that for each $\lambda \in \Lambda$, there exists a unique x_λ such that $A_\lambda(x_\lambda, y) = \langle v^*_\lambda, y\rangle$, for all $y \in Y_\lambda$. Since A satisfies condition (d), we get that for all $\lambda \in \Lambda$,

$$c\|x_\lambda\| \leq \sup_{y\in Y_\lambda, \|y\|=1} |A_\lambda(x_\lambda, y)| = \sup_{y\in Y_\lambda, \|y\|=1} |\langle v^*_\lambda, y\rangle| \leq \|v^*\|. \tag{2.11}$$

So $\{x_\lambda\}_{\lambda\in\Lambda}$ is a bounded net in X. Since X is reflexive, there exist a subnet $\{x_{\lambda_\mu}\}_{\mu\in M}$ of $\{x_\lambda\}_{\lambda\in\Lambda}$ and x in X such that $\{x_{\lambda_\mu}\}_{\mu\in M}$ converges weakly to x.

We are going to prove that $A(x, v) = \langle v^*, v\rangle$, for all $v \in V$. Take $v \in V$. Then there exists some $\lambda_0 \in \Lambda$ with $v \in Y_{\lambda_0}$. Since $\{x_{\lambda_\mu}\}_{\mu\in M}$ is a subnet of $\{x_\lambda\}_{\lambda\in\Lambda}$, there exists some $\mu_0 \in M$ with $\lambda_{\mu_0} \geq \lambda_0$. Hence, since the family $\{Y_\lambda\}_{\lambda\in\Lambda}$ is upwards directed,

$$v \in Y_{\lambda_\mu}, \tag{2.12}$$

for all $\mu \geq \mu_0$. Thus, for all $\mu \geq \mu_0$,

$$A_{\lambda_\mu}(x_{\lambda_\mu}, v) = \langle v^*_{\lambda_\mu}, v\rangle. \tag{2.13}$$

Therefore

$$\lim_{\mu\in M} A(x_{\lambda_\mu}, v) = \langle v^*, v\rangle. \tag{2.14}$$

Since $A(\cdot, v)$ is a bounded linear functional on X,

$$\lim_{\mu\in M} A(x_{\lambda_\mu}, v) = A(x, v). \tag{2.15}$$

Hence $A(x, v) = \langle v^*, v\rangle$. □

The following example illustrates the possible applicability of Theorem 2.3.

Example 2.4. Let $a \in C^1(0,1)$ be a decreasing function with $\lim_{t\to 0} a(t) = \infty$ and $a(t) \geq 0$, for all $t \in (0,1)$. We will establish the existence of a solution for the following Cauchy problem:

$$\begin{gathered} u' + a(t)u = f \quad \text{a.e. on } (0,1), \\ u(0) = 0, \end{gathered} \tag{2.16}$$

where $f \in L^2(0,1)$.

Let $X = \{u \in H^1(0,1) \mid u(0) = 0\}$ be equipped with the norm $\|u\| = (\int_0^1 |u'|^2 dt)^{1/2}$, which is equivalent to the original Sobolev norm, and $Y = L^2(0,1)$. Note that X is a reflexive Banach space, being a closed subspace of $H^1(0,1)$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $(0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Define

$$X_n = \{u \in H^1(\alpha_n, 1) \mid u(\alpha_n) = 0\}, \qquad Y_n = L^2(\alpha_n, 1) \tag{2.17}$$

(we can consider X_n and Y_n as closed subspaces of X and Y, resp., by extending their elements by zero outside $(\alpha_n,1)$). Also let $V=\bigcup_{n=1}^{\infty} Y_n$.

Let $A: X\times V\to\mathbb{R}$ be the bilinear map defined by

$$A(u,v)=\int_0^1 u'v\,dt+\int_0^1 a(t)uv\,dt. \tag{2.18}$$

A is well defined and $A(\cdot,v)$ is a bounded linear functional on X for any $v\in V$.

Let $A_n=A|_{X_n\times Y_n}$. A_n be a bounded bilinear form since

$$|A_n(u,v)|\le(1+M_n)\|u\|_{X_n}\|v\|_{Y_n}, \tag{2.19}$$

where M_n is the bound of a on $[\alpha_n,1]$. It should be noted that A is not bounded on the whole of $X\times V$.

To show that A_n is nondegenerate, let $v\in Y_n$ and assume that $A_n(u,v)=0$ for all $u\in X_n$, that is,

$$\int_{\alpha_n}^1 (u'+a(t)u)v\,dt=0, \quad \forall u\in X_n. \tag{2.20}$$

It is easy to see that the above implies that

$$\int_{\alpha_n}^1 wv\,dt=0, \tag{2.21}$$

for any continuous function w, and therefore $v=0$.

We next show that

$$\sup_{\|v\|=1,\, v\in Y_n} |A_n(u,v)|\ge\|u\|_{X_n}. \tag{2.22}$$

Define $T_n: X_n\to Y_n^*$ by $\langle T_n u,v\rangle=A_n(u,v)$. T_n is a well-defined bounded linear operator and $T_n u=u'+a(t)u$. Hence

$$\begin{aligned}||T_n u||^2&=\int_{\alpha_n}^1 |u'+a(t)u|^2dt\\&=\int_{\alpha_n}^1 |u'|^2dt+\int_{\alpha_n}^1 a^2(t)|u|^2dt+\int_{\alpha_n}^1 a(t)(u^2)'dt\\&=\int_{\alpha_n}^1 |u'|^2dt+\int_{\alpha_n}^1 (a^2(t)-a'(t))|u|^2dt+a(1)u^2(1)\ge\|u\|_{X_n}^2,\end{aligned} \tag{2.23}$$

since $u(\alpha_n)=0$, a is decreasing and $a(t)\ge 0$ for all $t\in(0,1)$.

All the hypotheses of Theorem 2.3 are hence satisfied and so if $F\in V^*$ is defined by $F(v)=\int_0^1 fv\,dt$, then there exists $u\in X$ such that

$$A(u,v)=F(v), \quad \forall v\in V. \tag{2.24}$$

Thus u satisfies (2.16).

3. The nonlinear case

We start by recalling some well-known definitions.

Definition 3.1. Let $T : X \to X^*$ be an operator. Then T is said to be

(i) monotone if $\langle Tx - Ty, x - y\rangle \geq 0$, for all $x, y \in X$;

(ii) hemicontinuous if for all $x, y \in X$, $T(x + ty) \xrightarrow{w} Tx$ as $t \to 0^+$;

(iii) coercive if

$$\lim_{\|x\| \to \infty} \frac{\langle Tx, x\rangle}{\|x\|} = \infty. \tag{3.1}$$

We also need the following generalization of the notion of type M operator (for the classical definition, see [7] or [8]).

Definition 3.2. Let X be a Banach space, let V be a linear subspace of X, and let

$$A : X \times V \longrightarrow \mathbb{R} \tag{3.2}$$

be a function. Then A is said to be of type M with respect to V if for any net $\{v_\lambda\}_{\lambda \in \Lambda}$ in $V, x \in X$ and $v^* \in V^*$;

(a) $v_\lambda \xrightarrow{w} x$;

(b) $A(v_\lambda, v) \to \langle v^*, v\rangle$, for all $v \in V$;

(c) $A(v_\lambda, v_\lambda) \to \langle \hat{v}^*, x\rangle$, where $\hat{v}^*$ is the extension of v^* on the closure of V,

imply that $A(x, v) = \langle v^*, v\rangle$, for all $v \in V$.

Our result is the following.

Theorem 3.3. *Let X be a reflexive Banach space, let Λ be a directed set, let $\{X_\lambda\}_{\lambda \in \Lambda}$ be an upwards directed family of closed subspaces of X, and let $V = \bigcup_{\lambda \in \Lambda} X_\lambda$. Suppose that*

$$A : X \times V \longrightarrow \mathbb{R} \tag{3.3}$$

is a function for which the following hold:

(a) *A is of type M with respect to V;*

(b) $\lim_{\|x\| \to \infty} A(x, x)/\|x\| = \infty$*;*

(c) *$A_\lambda(x, \cdot) \in X_\lambda^*$, for all $\lambda \in \Lambda$ and all $x \in X_\lambda$, where A_λ is the restriction of A on $X_\lambda \times X_\lambda$;*

(d) *the operator $T_\lambda : X_\lambda \to X_\lambda^*$, defined by $\langle T_\lambda x, y\rangle = A_\lambda(x, y)$ for all $x, y \in X_\lambda$, is monotone and hemicontinuous for all $\lambda \in \Lambda$.*

Then for each $v^* \in V^*$, there exists $x \in X$ such that

$$A(x, v) = \langle v^*, v\rangle, \tag{3.4}$$

for all $v \in V$.

Proof. As in the proof of Theorem 2.3, for each $\lambda \in \Lambda$, let $v_\lambda^* = v^*|_{X_\lambda}$. By the Browder-Minty theorem (see [8, Theorem 26.A]), a monotone, coercive, and hemicontinuous operator, from a real reflexive Banach space into its dual, is onto. Thus, by (b) and (d), for each $\lambda \in \Lambda$, the operator T_λ is onto and so there exists $x_\lambda \in X_\lambda$ such that

$$A_\lambda(x_\lambda, y) = \langle v_\lambda^*, y\rangle, \tag{3.5}$$

for all $y \in X_\lambda$. In particular $A_\lambda(x_\lambda,x_\lambda) = \langle v_\lambda^*,x_\lambda\rangle$, and hence by (b), we get that the net $\{x_\lambda\}_{\lambda\in\Lambda}$ is bounded. Continuing as in the proof of Theorem 2.3 and applying the fact that A is of type M with respect to V, we get the required result. □

Remark 3.4. It should be noted that since a crucial point in the above proof is the existence and boundedness of the net $\{x_\lambda\}_{\lambda\in\Lambda}$, variants of the previous theorem could be obtained using in (b) and (d) alternative conditions corresponding to other surjectivity results.

We now apply Theorem 3.3 to a singular Dirichlet problem.

Example 3.5. Let Ω be a bounded domain in $\mathbb{R}^N$. We consider the Dirichlet problem

$$\begin{gathered} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u}{\partial x_i}\right) + f(x,u) = 0 \quad \text{a.e. on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.6}$$

where $a \in L^\infty_{\text{loc}}(\Omega)$ and there exists $c_1 > 0$ such that $a(x) \geq c_1$ a.e. on Ω, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone increasing (with respect to its second variable for each fixed $x \in \Omega$) Carathéodory function, for which there exist $h \in L^2(\Omega)$ and $c_2 > 0$ such that

$$|f(x,u)| \leq h(x) + c_2|u|, \quad \forall x \in \Omega,\ u \in \mathbb{R}. \tag{3.7}$$

We will show that if the above hypotheses on a and f hold, then problem (3.6) has a weak solution, that is, that there exists a function $u \in H_0^1(\Omega)$ with

$$\int_\Omega a(x)\nabla u\nabla v\,dx + \int_\Omega f(x,u)v\,dx = 0, \quad \forall v \in C_0^\infty(\Omega). \tag{3.8}$$

To this end, let $X = H_0^1(\Omega)$, let $\{\Omega_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of Ω such that $\overline{\Omega_n} \subseteq \Omega_{n+1}$ and

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \tag{3.9}$$

and $X_n = H_0^1(\Omega_n)$, for each $n \in \mathbb{N}$. Observe that we can consider each X_n as a closed subspace of X by extending its elements by zero outside Ω_n and let

$$V = \bigcup_{n=1}^{\infty} X_n. \tag{3.10}$$

Finally, let

$$A : X \times V \longrightarrow \mathbb{R} \tag{3.11}$$

be the function defined by

$$A(u,v) = \int_\Omega a(x)\nabla u \nabla v\,dx + \int_\Omega f(x,u)v\,dx. \tag{3.12}$$

By $a(x) \geq c_1$ a.e. on Ω, the monotonicity of f, and the growth condition (3.7), we have

$$\begin{aligned} A(u,u) &= \int_\Omega a(x)|\nabla u|^2 dx + \int_\Omega f(x,u)u\,dx \\ &= \int_\Omega a(x)|\nabla u|^2 dx + \int_\Omega (f(x,u) - f(x,0))u\,dx + \int_\Omega f(x,0)u\,dx \\ &\geq c_1 \|\nabla u\|^2_{L^2(\Omega)} - \|h\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)}. \end{aligned} \tag{3.13}$$

Since by the Poincaré inequality $\|\nabla u\|_{L^2(\Omega)}$ is equivalent to the norm of X, it follows that A is coercive.

Let $A_n = A|_{X_n \times X_n}$. Then, since $a \in L^\infty_{\text{loc}}(\Omega)$, it follows that $a \in L^\infty(\Omega_n)$, for all $n \in \mathbb{N}$. Combining this with (3.7), we have that

$$|A_n(u,v)| \leq c(u,n)\|v\|_{X_n}, \tag{3.14}$$

where $c(u,n)$ is a positive constant depending on n and u. So the operator

$$T_n : X_n \longrightarrow X_n^*, \tag{3.15}$$

with $\langle T_n u, v\rangle_{X_n} = A_n(u,v)$, is well defined for all $n \in \mathbb{N}$. Let

$$T_{1,n}, T_{2,n} : X_n \longrightarrow X_n^* \tag{3.16}$$

be the operators defined by

$$\langle T_{1,n}u, v\rangle_{X_n} = \int_{\Omega_n} a(x)\nabla u \nabla v\,dx, \qquad \langle T_{2,n}u, v\rangle_{X_n} = \int_{\Omega_n} f(x,u)v\,dx. \tag{3.17}$$

Then $T_{1,n}$ is a monotone bounded linear operator. Using the monotonicity of f, it is easy to see that $T_{2,n}$ is monotone. Finally, recalling that the Nemytskii operator corresponding to f is continuous (see, e.g., [8, Proposition 26.7]) and that the embedding of X_n into $L^2(\Omega_n)$ is compact, we have that $T_{2,n}$ is hemicontinuous. Thus $T_n = T_{1,n} + T_{2,n}$ is monotone and hemicontinuous for all $n \in \mathbb{N}$.

To finish the proof, let $u_n \overset{w}{\to} u$ in X. Then since for all $v \in V$,

$$u \longmapsto \int_\Omega a(x)\nabla u \nabla v\,dx \tag{3.18}$$

is a bounded linear functional and, by the continuity of the Nemytskii operator and the compactness of the embedding of X into $L^2(\Omega)$,

$$\int_\Omega f(x,u_n)v\,dx \longrightarrow \int_\Omega f(x,u)v\,dx, \tag{3.19}$$

for all $v \in V$, we get that

$$A(u_n, v) \longrightarrow A(u, v), \quad \forall v \in V. \tag{3.20}$$

Thus A is of type M with respect to V. Applying now Theorem 3.3 we get that there exists $u \in X$ such that $A(u, v) = 0$ for all $v \in V$. Observing that $C_0^\infty(\Omega)$ is contained in V, we get that u is the required weak solution of (3.6).

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