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*Research Article*

# A Note on the Relaxation-Time Limit of the Isothermal Euler Equations

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This work is concerned with the relaxation-time limit of the multidimensional isothermal Euler equations with relaxation. We show that Coulombel-Goudon's results (2007) can hold in the *weaker* and *more general* Sobolev space of fractional order. The method of proof used is the Littlewood-Paley decomposition.



## 1. Introduction

The multidimensional isothermal Euler equation with relaxation describing the perfect gas flow is given by

$$\begin{gathered} n_t + \nabla \cdot (n\mathbf{u}) = 0, \\ (n\mathbf{u})_t + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla p(n) = -\frac{1}{\tau} n\mathbf{u} \end{gathered} \tag{1.1}$$

for $(t,x) \in [0,+\infty) \times \mathbb{R}^d$, $d \geq 3$, where $n$, $\mathbf{u} = (u^1, u^2, \dots, u^d)^\top$ ($\top$ represents transpose) denote the density and velocity of the flow, respectively, and the constant $\tau$ is the momentum relaxation time for some physical flow. Here, we assume that $0 < \tau \leq 1$. The pressure $p(n)$ satisfies $p(n) = An$, and $A > 0$ is a physical constant. The symbols $\nabla$, $\otimes$ are the gradient operator and the symbol for the tensor products of two vectors, respectively. The system is supplemented with the initial data

$$(n, \mathbf{u})(x, 0) = (n_0, \mathbf{u}_0)(x), \quad x \in \mathbb{R}^d. \tag{1.2}$$

To be concerned with the small relaxation-time analysis, we define the scaled variables

$$(n^\tau,\mathbf{u}^\tau)(x,s)=(n,\mathbf{u})\left(x,\frac{s}{\tau}\right). \tag{1.3}$$

Then the new variables satisfy the following equations:

$$\begin{gathered} n_s^\tau+\nabla\cdot\left(\frac{n^\tau\mathbf{u}^\tau}{\tau}\right)=0,\\ \tau^2\left(\frac{n^\tau\mathbf{u}^\tau}{\tau}\right)_s+\tau^2\left(\frac{n^\tau\mathbf{u}^\tau\otimes\mathbf{u}^\tau}{\tau^2}\right)+\frac{n^\tau\mathbf{u}^\tau}{\tau}=-A\nabla n^\tau \end{gathered} \tag{1.4}$$

with initial data

$$(n^\tau,\mathbf{u}^\tau)(x,0)=(n_0,\mathbf{u}_0). \tag{1.5}$$

Let $\tau\to 0$, formally, we obtain the heat equation

$$\begin{gathered} \mathcal{N}_s-A\Delta\mathcal{N}=0,\\ \mathcal{N}(x,0)=n_0. \end{gathered} \tag{1.6}$$

The above formal derivation of heat equation has been justified by many authors, see [1–3] and the references therein. In [2], Junca and Rascle studied the convergence of the solutions to (1.1) towards those of (1.6) for arbitrary large initial data in $BV(\mathbb{R})$ space. Marcati and Milani [3] showed the derivation of the porous media equation as the limit of the isentropic Euler equations in one space dimension. Recently, Coulombel and Goudon [1] constructed the uniform smooth solutions to (1.1) in the multidimensional case and proved this relaxation-time limit in some Sobolev space $H^k(\mathbb{R}^d)$ ($k>1+d/2$, $k\in\mathbb{N}$). In this paper, we *weaken* the regularity assumptions on the initial data and establish a similar relaxation result in the more general Sobolev space of fractional order ($H^{\sigma+\varepsilon}(\mathbb{R}^d)$, $\sigma=1+d/2$, $\varepsilon>0$) with the aid of Littlewood-Paley decomposition theory.

If fixed $\tau>0$, there are some efforts on the global existence of smooth solutions to the system (1.1)-(1.2) for the isentropic gas or the general hyperbolic system, the interested readers can refer to [4–7]. Now, we state main results as follows.

Theorem 1.1. *Let $\bar{n}$ be a constant reference density. Suppose that $n_0-\bar{n}$ and $\mathbf{u}_0\in H^{\sigma+\varepsilon}(\mathbb{R}^d)$, there exist two positive constants $\delta_0$ and $C_0$ independent of $\tau$ such that if*

$$\|(n_0-\bar{n},\mathbf{u}_0)\|^2_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}\le\delta_0, \tag{1.7}$$

*then the system (1.1)-(1.2) admits a unique global solution $(n,\mathbf{u})$ satisfying*

$$(n-\bar{n},\mathbf{u})\in\mathcal{C}([0,\infty),H^{\sigma+\varepsilon}(\mathbb{R}^d)). \tag{1.8}$$

*Moreover, the uniform energy inequality holds:*

$$\begin{aligned}\|(n-\overline{n},\mathbf{u})(\cdot,t)\|^2_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}+\frac{1}{\tau}\int_0^t\|\mathbf{u}(\cdot,\varsigma)\|^2_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}d\varsigma+\tau\int_0^t\|(\nabla n,\nabla\mathbf{u})(\cdot,\varsigma)\|^2_{H^{\sigma-1+\varepsilon}(\mathbb{R}^d)}d\varsigma \\ \leq C_0\|(n_0-\overline{n},\mathbf{u}_0)\|^2_{H^{\sigma+\varepsilon}(\mathbb{R}^d)},\quad t\geq 0.\end{aligned}\tag{1.9}$$

Based on Theorem 1.1, using the standard weak convergence method and compactness theorem [8], we can obtain the following relaxation-time limit immediately.

Corollary 1.2. *Let $(n,\mathbf{u})$ be the global solution of Theorem 1.1, then*

$$\begin{aligned}&n^\tau-\overline{n}\text{ is uniformly bounded in }\mathscr{C}([0,\infty),H^{\sigma+\varepsilon}(\mathbb{R}^d)),\\ &\frac{n^\tau\mathbf{u}^\tau}{\tau}\text{ is uniformly bounded in }L^2([0,\infty),H^{\sigma+\varepsilon}(\mathbb{R}^d)).\end{aligned}\tag{1.10}$$

*Furthermore, there exists some function $\mathcal{N}\in\mathscr{C}([0,\infty),\overline{n}+H^{\sigma+\varepsilon}(\mathbb{R}^d))$ which is a global weak solution of (1.6). For any time $T>0$, we have $n^\tau(x,s)$ strongly converges to $\mathcal{N}(x,s)$ in $\mathscr{C}([0,T],(H^{\sigma'+\varepsilon}(\mathbb{R}^d))_{\mathrm{loc}})$ $(\sigma'<\sigma)$ as $\tau\to 0$.*

## 2. Preliminary lemmas

On the Littlewood-Paley decomposition and the definitions of Besov space, for brevity, we omit the details, see [9] or [7]. Here, we only present some useful lemmas.

Lemma 2.1 ([9, 7]). *Let $s>0$ and $1\leq p,r\leq\infty$. Then $B^s_{p,r}\cap L^\infty$ is an algebra and one has*

$$\|fg\|_{B^s_{p,r}}\lesssim\|f\|_{L^\infty}\|g\|_{B^s_{p,r}}+\|g\|_{L^\infty}\|f\|_{B^s_{p,r}}\quad\text{if } f,g\in B^s_{p,r}\cap L^\infty.\tag{2.1}$$

Lemma 2.2 [9, 7]. *Let $1\leq p,r\leq\infty$, and $I$ be open interval of $\mathbb{R}$. Let $s>0$ and $\ell$ be the smallest integer such that $\ell\geq s$. Let $F:I\to\mathbb{R}$ satisfy $F(0)=0$ and $F'\in W^{\ell,\infty}(I;\mathbb{R})$. Assume that $v\in B^s_{p,r}$ takes values in $J\subset\subset I$. Then $F(v)\in B^s_{p,r}$ and there exists a constant $C$ depending only on $s$, $I$, $J$, and $d$ such that*

$$\|F(v)\|_{B^s_{p,r}}\leq C(1+\|v\|_{L^\infty})^\ell\|F'\|_{W^{\ell,\infty}(I)}\|v\|_{B^s_{p,r}}.\tag{2.2}$$

Lemma 2.3 [7]. *Let $s>0$, $1<p<\infty$, the following inequalities hold.*

(I) $q\geq -1$:

$$2^{qs}\|[f,\Delta_q]\mathcal{A}g\|_{L^p}\leq\begin{cases}Cc_q\|f\|_{B^s_{p,2}}\|g\|_{B^s_{p,2}}, & f,g\in B^s_{p,2},\ s=1+\dfrac{d}{p}+\varepsilon\ (\varepsilon>0),\\ Cc_q\|f\|_{B^s_{p,2}}\|g\|_{B^{s+1}_{p,2}}, & f\in B^s_{p,2},\ g\in B^{s+1}_{p,2},\ s=\dfrac{d}{p}+\varepsilon\ (\varepsilon>0),\\ Cc_q\|f\|_{B^{s+1}_{p,2}}\|g\|_{B^s_{p,2}}, & f\in B^{s+1}_{p,2},\ g\in B^s_{p,2},\ s=\dfrac{d}{p}+\varepsilon\ (\varepsilon>0).\end{cases}\tag{2.3}$$

*If $f = g$, then*

$$2^{qs}\|[f,\Delta_q]\mathcal{A}g\|_{L^p} \leq Cc_q\|\nabla f\|_{L^\infty}\|g\|_{B^s_{p,2}}, \quad s > 0. \tag{2.4}$$

(II) $q = -1$*:*

$$2^{-s}\|[f,\Delta_q]\mathcal{A}g\|_{L^{2d/(d+2)}} \leq Cc_{-1}\|f\|_{B^s_{2,2}}\|g\|_{B^s_{2,2}}, \quad f,g \in B^s_{2,2},\ s = 1+\frac{d}{2}+\varepsilon\ (\varepsilon > 0), \tag{2.5}$$

*where the operator $\mathcal{A} = \mathrm{div}$ or $\nabla$, the commutator $[f,h] = fh - hf$, $C$ is a harmless constant, and $c_q$ denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$. (In particular, Besov space $B^s_{2,2} \equiv H^s$.)*

## 3. Reformulation and local existence

Let us introduce the enthalpy $\mathcal{H}(\varrho) = A\ln\varrho$ $(\varrho > 0)$, and set

$$m(t,x) = A^{-1/2}\big(\mathcal{H}(n(t,x)) - \mathcal{H}(\bar{n})\big). \tag{3.1}$$

Then (1.1) can be transformed into the symmetric hyperbolic form

$$\partial_t U + \sum_{j=1}^{d} A_j(\mathbf{u})\partial_{x_j} U = -\frac{1}{\tau}\begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix}, \tag{3.2}$$

where

$$U = \begin{pmatrix} m \\ \mathbf{u} \end{pmatrix}, \qquad A_j(\mathbf{u}) = \begin{pmatrix} u^j & \sqrt{A}e_j^\top \\ \sqrt{A}e_j & u^j \end{pmatrix}. \tag{3.3}$$

The initial data (1.2) become into

$$U_0 = \big(\sqrt{A}(\ln n_0 - \ln\bar{n}), \mathbf{u}_0\big)^\top. \tag{3.4}$$

*Remark 1.* The variable change is from the open set $\{(n,\mathbf{u}) \in (0,+\infty)\times\mathbb{R}^d\}$ to the whole space $\{(m,\mathbf{u}) \in \mathbb{R}^d\times\mathbb{R}^d\}$. It is easy to show that the system (1.1)-(1.2) is equivalent to (3.2)–(3.4) for classical solutions $(n,\mathbf{u})$ away from vacuum.

First, we recall a local existence and uniqueness result of classical solutions to (3.2)–(3.4) which has been obtained in [7].

Proposition 3.1. *For any fixed relaxation time $\tau > 0$, assume that $U_0 \in B^\sigma_{2,1}$, then there exist a time $T_0 > 0$ (only depending on the initial data $U_0$) and a unique solution $U(t,x)$ to (3.2)–(3.4) such that $U \in \mathcal{C}^1([0,T_0]\times\mathbb{R}^d)$ and $U \in \mathcal{C}([0,T_0],B^\sigma_{2,1}) \cap \mathcal{C}^1([0,T_0],B^{\sigma-1}_{2,1})$.*

## 4. A priori estimate and global existence

In this section, we will establish a uniform a priori estimate, which is used to derive the global existence of classical solutions to (3.2)–(3.4). Defining the energy function

$$E_\tau(T)^2 := \sup_{0\le t\le T} ||U(t)||^2_{H^{\sigma+\varepsilon}} + \frac{1}{\tau}\int_0^T ||\mathbf{u}(t)||^2_{H^{\sigma+\varepsilon}}dt + \tau\int_0^T ||\nabla_x U(t)||^2_{H^{\sigma-1+\varepsilon}}dt, \tag{4.1}$$

then we have the following a priori estimate.

Proposition 4.1. *For any given time $T>0$, if $U\in\mathscr{C}([0,T],H^{\sigma+\varepsilon})$ is a solution to the system (3.2)–(3.4), then the following inequality holds:*

$$E_\tau(T)^2 \le C(S(T))\big(E_\tau(0)^2 + E_\tau(T)^2 + E_\tau(T)^4\big), \tag{4.2}$$

*where $S(T)=\sup_{0\le t\le T}\|U(\cdot,t)\|_{H^{\sigma+\varepsilon}}$, $C(S(T))$ denotes an increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, which is independent of $\tau,T,U$.*

*Proof.* The proof of Proposition 4.1 is divided into two steps. First, we estimate the $L^\infty([0,T],H^{\sigma+\varepsilon})$ norm of $U$, and the $L^2([0,T],H^{\sigma+\varepsilon})$ one of $\mathbf{u}$. Then, we estimate the $L^2([0,T],H^{\sigma-1+\varepsilon})$ norm of $\nabla U$.

*Step 1.* Applying the operator $\Delta_q$ to (3.2), multiplying the resulting equations by $\Delta_q m$ and $\Delta_q\mathbf{u}$, respectively, and then integrating them over $\mathbb{R}^d$, we get

$$\begin{aligned}
&\frac{1}{2}\Big(||\Delta_q m||^2_{L^2} + ||\Delta_q\mathbf{u}||^2_{L^2}\Big)\Big|_0^t + \frac{1}{\tau}\int_0^t ||\Delta_q\mathbf{u}(\varsigma)||^2_{L^2}d\varsigma \\
&\quad= \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\operatorname{div}\mathbf{u}\Big(\,|\Delta_q m|^2 + |\Delta_q\mathbf{u}|^2\Big)dx\,d\varsigma \\
&\qquad+ \int_0^t\int_{\mathbb{R}^d}\{[\mathbf{u},\Delta_q]\cdot\nabla m\Delta_q m + [\mathbf{u},\Delta_q]\cdot\nabla\mathbf{u}\Delta_q\mathbf{u}\}dx\,d\varsigma.
\end{aligned} \tag{4.3}$$

In what follows, we first deal with the low-frequency case. By performing integration by parts, then using Hölder- and Gagliardo-Nirenberg-Sobolev inequality, we have ($d\ge 3$)

$$\begin{aligned}
&\Big(||\Delta_{-1}m||^2_{L^2} + ||\Delta_{-1}\mathbf{u}||^2_{L^2}\Big)\Big|_0^t + \frac{2}{\tau}\int_0^t ||\Delta_{-1}\mathbf{u}(\varsigma)||^2_{L^2}d\varsigma \\
&\quad\le \int_0^t \Big(2\|\mathbf{u}\|_{L^d}||\Delta_{-1}m||_{L^{2d/(d-2)}}||\Delta_{-1}\nabla m||_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty}||\Delta_{-1}\mathbf{u}||^2_{L^2}\Big)d\varsigma \\
&\qquad+ 2\int_0^t \Big(||[\mathbf{u},\Delta_{-1}]\cdot\nabla m||_{L^{2d/(d+2)}}||\Delta_{-1}m||_{L^{2d/(d-2)}} + ||[\mathbf{u},\Delta_{-1}]\cdot\nabla\mathbf{u}||_{L^2}||\Delta_{-1}\mathbf{u}||_{L^2}\Big)d\varsigma \\
&\quad\le \int_0^t \Big(2||\mathbf{u}||_{L^d}||\Delta_{-1}\nabla m||^2_{L^2} + ||\nabla\mathbf{u}||_{L^\infty}||\Delta_{-1}\mathbf{u}||^2_{L^2}\Big)d\varsigma \\
&\qquad+ 2\int_0^t \Big(||[\mathbf{u},\Delta_{-1}]\cdot\nabla m||_{L^{2d/(d+2)}}||\Delta_{-1}\nabla m||_{L^2} + ||[\mathbf{u},\Delta_{-1}]\cdot\nabla\mathbf{u}||_{L^2}||\Delta_{-1}\mathbf{u}||_{L^2}\Big)d\varsigma.
\end{aligned} \tag{4.4}$$

Multiplying the factor $2^{-2(\sigma+\varepsilon)}$ on both sides of (4.4), from Lemma 2.3 and Young inequality, we obtain

$$\begin{aligned}
&2^{-2(\sigma+\varepsilon)}\left(\|\Delta_{-1}m\|_{L^2}^2+\|\Delta_{-1}\mathbf{u}\|_{L^2}^2\right)\Big|_0^t+\frac{2}{\tau}\int_0^t 2^{-2(\sigma+\varepsilon)}\|\Delta_{-1}\mathbf{u}(\varsigma)\|_{L^2}^2d\varsigma\\
&\quad\le\int_0^t\left(\frac{1}{2}\|\mathbf{u}\|_{L^d}2^{-2(\sigma-1+\varepsilon)}\|\Delta_{-1}\nabla m\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^\infty}2^{-2(\sigma+\varepsilon)}\|\Delta_{-1}\mathbf{u}\|_{L^2}^2\right)d\varsigma\\
&\qquad+C\int_0^t\left(c_{-1}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}\|m\|_{H^{\sigma+\varepsilon}}2^{-(\sigma-1+\varepsilon)}\|\Delta_{-1}\nabla m\|_{L^2}+c_{-1}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 2^{-(\sigma+\varepsilon)}\|\Delta_{-1}\mathbf{u}\|_{L^2}\right)d\varsigma\\
&\quad\le\int_0^t\left(\frac{1}{2}\|\mathbf{u}\|_{L^d}2^{-2(\sigma-1+\varepsilon)}\|\Delta_{-1}\nabla m\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^\infty}2^{-2(\sigma+\varepsilon)}\|\Delta_{-1}\mathbf{u}\|_{L^2}^2\right)d\varsigma\\
&\qquad+C\int_0^t\|m\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}c_{-1}^2\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\tau 2^{-2(\sigma-1+\varepsilon)}\|\Delta_{-1}\nabla m\|_{L^2}^2\right)d\varsigma\\
&\qquad+C\int_0^t\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}c_{-1}^2\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\frac{1}{\tau}2^{-2(\sigma+\varepsilon)}\|\Delta_{-1}\mathbf{u}\|_{L^2}^2\right)d\varsigma\quad\left(\tau\le\frac{1}{\tau}\right),
\end{aligned}\tag{4.5}$$

where $C$ is some positive constant independent of $\tau$. For the high-frequency case, we can also achieve the similar inequality:

$$\begin{aligned}
&2^{2q(\sigma+\varepsilon)}\left(\|\Delta_q m\|_{L^2}^2+\|\Delta_q\mathbf{u}\|_{L^2}^2\right)\Big|_0^t+\frac{2}{\tau}\int_0^t 2^{2q(\sigma+\varepsilon)}\|\Delta_q\mathbf{u}(\varsigma)\|_{L^2}^2d\varsigma\\
&\quad\le C\int_0^t\|\nabla\mathbf{u}\|_{L^\infty}\left(2^{2q(\sigma-1+\varepsilon)}\|\Delta_q\nabla m\|_{L^2}^2+2^{2q(\sigma+\varepsilon)}\|\Delta_q\mathbf{u}\|_{L^2}^2\right)d\varsigma\\
&\qquad+C\int_0^t\|m\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}c_q^2\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\tau 2^{2q(\sigma-1+\varepsilon)}\|\Delta_q\nabla m\|_{L^2}^2\right)d\varsigma\\
&\qquad+C\int_0^t\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}c_q^2\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\frac{1}{\tau}2^{2q(\sigma+\varepsilon)}\|\Delta_q\mathbf{u}\|_{L^2}^2\right)d\varsigma\quad\left(\tau\le\frac{1}{\tau}\right),
\end{aligned}\tag{4.6}$$

where we have taken the advantage of the fact $\|\Delta_q\nabla m\|_{L^2}\approx 2^q\|\Delta_q m\|_{L^2}$ $(q\ge 0)$.

By summing (4.6) on $q\in\mathbb{N}\cup\{0\}$ and adding (4.5) together, then according to the imbedding property in Sobolev space, we have

$$\begin{aligned}
&\left(\|m\|_{H^{\sigma+\varepsilon}}^2+\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2\right)\Big|_0^t+\frac{2}{\tau}\int_0^t\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2d\varsigma\\
&\quad\le C\int_0^t\|m\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\tau\|\nabla m\|_{H^{\sigma-1+\varepsilon}}^2\right)d\varsigma+C\int_0^t\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}\frac{1}{\tau}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2d\varsigma\\
&\qquad+C\int_0^t\|m\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\tau\|\nabla m\|_{H^{\sigma-1+\varepsilon}}^2\right)d\varsigma\\
&\qquad+C\int_0^t\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}\left(\frac{1}{\tau}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2+\frac{1}{\tau}\|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2\right)d\varsigma.
\end{aligned}\tag{4.7}$$

Therefore, for any $t \in [0,T]$, the following inequality holds:

$$||U(t)||^2_{H^{\sigma+\varepsilon}} + \frac{2}{\tau}\int_0^t \|\mathbf{u}\|^2_{H^{\sigma+\varepsilon}} d\varsigma \le C(S(t))\left(E_\tau(0)^2 + E_\tau(t)^2\right). \tag{4.8}$$

*Step 2.* Thanks to the important skew-symmetric lemma developed in [1, 6, 10], we are going to estimate the $L^2([0,T],H^{\sigma-1+\varepsilon})$ norm of $\nabla U$.

LEMMA 4.2 (Shizuta-Kawashima). *For all $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the system (3.2) admits a real skew-symmetric smooth matrix $K(\xi)$ which is defined in the unit sphere $\mathbf{S}^{d-1}$:*

$$K(\xi) = \begin{pmatrix} 0 & \dfrac{\xi^\top}{|\xi|} \\ -\dfrac{\xi}{|\xi|} & 0 \end{pmatrix}, \tag{4.9}$$

*then*

$$K(\xi)\sum_{j=1}^d \xi_j A_j(0) = \begin{pmatrix} \sqrt{A}|\xi| & 0 \\ 0 & -\sqrt{A}\dfrac{\xi\otimes\xi}{|\xi|} \end{pmatrix}. \tag{4.10}$$

The system (3.2) can be written as the linearized form

$$\partial_t U + \sum_{j=1}^d A_j(0)\partial_{x_j} U = \sum_{j=1}^d \{A_j(0) - A_j(\mathbf{u})\}\partial_{x_j} U - \frac{1}{\tau}\begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix}. \tag{4.11}$$

Let

$$\mathcal{G} = \sum_{j=1}^d \{A_j(0) - A_j(\mathbf{u})\}\partial_{x_j} U. \tag{4.12}$$

From Lemma 2.1, we have

$$\|\mathcal{G}\|_{H^{\sigma-1+\varepsilon}} \le C\|\mathbf{u}\|_{H^{\sigma-1+\varepsilon}}\|\nabla U\|_{H^{\sigma-1+\varepsilon}}. \tag{4.13}$$

Apply the operator $\Delta_q$ to the system (4.11) to get

$$\partial_t \Delta_q U + \sum_{j=1}^d A_j(0)\partial_{x_j}\Delta_q U = \Delta_q\mathcal{G} - \frac{1}{\tau}\begin{pmatrix} 0 \\ \Delta_q\mathbf{u} \end{pmatrix}. \tag{4.14}$$

By performing the Fourier transform with respect to the space variable $x$ for (4.14) and multiplying the resulting equation by $-i\tau(\widehat{\Delta_q U})^* K(\xi)$, "$*$" represents transpose and conjugator, then taking the real part of each term in the equality, we can obtain

$$\begin{aligned} &\tau\,\mathrm{Im}\left((\widehat{\Delta_q U})^* K(\xi)\frac{d}{dt}\widehat{\Delta_q U}\right) + \tau(\widehat{\Delta_q U})^* K(\xi)\left(\sum_{j=1}^d \xi_j A_j(0)\right)\widehat{\Delta_q U} \\ &\quad = -\mathrm{Im}\left((\widehat{\Delta_q m})^*\frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{u}}\right) + \tau\,\mathrm{Im}\left((\widehat{\Delta_q U})^* K(\xi)(\widehat{\Delta_q \mathcal{G}})\right). \end{aligned} \tag{4.15}$$

Using the skew-symmetry of $K(\xi)$, we have

$$\operatorname{Im}\left((\widehat{\Delta_q U})^* K(\xi)\frac{d}{dt}\widehat{\Delta_q U}\right) = \frac{1}{2}\frac{d}{dt}\operatorname{Im}\left((\widehat{\Delta_q U})^* K(\xi)\widehat{\Delta_q U}\right). \tag{4.16}$$

Substituting (4.10) into the second term on the left-hand side of (4.15), it is not difficult to get

$$\begin{aligned}\tau\operatorname{Im}\left((\widehat{\Delta_q U})^* K(\xi)\frac{d}{dt}\widehat{\Delta_q U}\right) &+ \tau(\widehat{\Delta_q U})^* K(\xi)\left(\sum_{j=1}^{d}\xi_j A_j(0)\right)\widehat{\Delta_q U} \\ &\geq \frac{\tau}{2}\frac{d}{dt}\operatorname{Im}\left((\widehat{\Delta_q U})^* K(\xi)\widehat{\Delta_q U}\right) + \tau\sqrt{A}|\xi|\,|\widehat{\Delta_q U}|^2 - 2\sqrt{A}|\xi|\,|\widehat{\Delta_q \mathbf{u}}|^2.\end{aligned} \tag{4.17}$$

With the help of Young inequality, the right-hand side of (4.15) can be estimated as

$$\begin{aligned}&-\operatorname{Im}\left((\widehat{\Delta_q m})^* \frac{\xi^\top}{|\xi|}\widehat{\Delta_q \mathbf{u}}\right) + \tau\operatorname{Im}\left((\widehat{\Delta_q U})^* K(\xi)(\widehat{\Delta_q \mathcal{G}})\right) \\ &\quad\leq \tau\frac{\sqrt{A}}{2}|\xi|\,|\widehat{\Delta_q U}|^2 + \frac{C}{\tau|\xi|}\,|\widehat{\Delta_q \mathbf{u}}|^2 + \frac{C\tau}{|\xi|}\,|(\widehat{\Delta_q \mathcal{G}})|^2,\end{aligned} \tag{4.18}$$

where the positive constant $C$ is independent of $\tau$. Combining with the equality (4.15) and the inequalities (4.17)-(4.18), we deduce

$$\tau\frac{\sqrt{A}}{2}|\xi|\,|\widehat{\Delta_q U}|^2 \leq \frac{C}{\tau}\left(|\xi| + \frac{1}{|\xi|}\right)|\widehat{\Delta_q \mathbf{u}}|^2 + \frac{C\tau}{|\xi|}\,|(\widehat{\Delta_q \mathcal{G}})|^2 - \frac{\tau}{2}\frac{d}{dt}\operatorname{Im}\left((\widehat{\Delta_q U})^* K(\xi)\widehat{\Delta_q U}\right). \tag{4.19}$$

Multiplying (4.19) by $|\xi|$ and integrating it over $[0,t]\times\mathbb{R}^d$, from Plancherel's theorem, we reach

$$\begin{aligned}\tau\int_0^t \|\Delta_q \nabla U\|_{L^2}^2 d\varsigma &\leq \frac{C}{\tau}\int_0^t\left(\|\Delta_q \mathbf{u}\|_{L^2}^2 + \|\Delta_q \nabla\mathbf{u}\|_{L^2}^2\right)d\varsigma + C\tau\int_0^t \|\Delta_q \mathcal{G}\|_{L^2}^2 d\varsigma \\ &\quad - \frac{\tau}{2}\operatorname{Im}\int_{\mathbb{R}^d}|\xi|\left((\widehat{\Delta_q U})^* K(\xi)\widehat{\Delta_q U}\right)d\xi\,\Big|_0^t \\ &\leq \frac{C}{\tau}\int_0^t 2^{2q}\|\Delta_q \mathbf{u}\|_{L^2}^2 d\varsigma + C\tau\int_0^t \|\Delta_q \mathcal{G}\|_{L^2}^2 d\varsigma \\ &\quad + C\tau 2^{2q}\left(\|\Delta_q U(t)\|_{L^2}^2 + \|\Delta_q U(0)\|_{L^2}^2\right),\end{aligned} \tag{4.20}$$

where we have used the uniform boundedness of the matrix $K(\xi)$ $(\xi \neq 0)$.

Multiplying the factor $2^{2q(\sigma-1+\varepsilon)}$ $(q \geq -1)$ on both sides of (4.20) and summing it on $q$, we have

$$\begin{aligned}\tau\int_0^t \|\nabla U\|_{H^{\sigma-1+\varepsilon}}^2 d\varsigma &\leq \frac{C}{\tau}\int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 d\varsigma + C\tau\int_0^t \|\mathcal{G}\|_{H^{\sigma-1+\varepsilon}}^2 d\varsigma + C\tau\left(\|U(t)\|_{H^{\sigma+\varepsilon}}^2 + \|U(0)\|_{H^{\sigma+\varepsilon}}^2\right) \\ &\leq C(S(t))\left(E_\tau(0)^2 + E_\tau(t)^2 + E_\tau(t)^4\right).\end{aligned} \tag{4.21}$$

Together with the inequalities (4.8) and (4.21), (4.2) follows immediately, which completes the proof of Proposition 4.1. □

*Proof of Theorem 1.1.* In fact, Proposition 3.1 also holds on the framework of the functional space $H^{\sigma+\varepsilon}(\equiv B^{\sigma+\varepsilon}_{2,2})$. There exists a sufficiently small number $\epsilon_0$ independent of $\tau$ such that $E_\tau(T) \leq \epsilon_0 \leq 1$ from (4.1), we have

$$E_\tau(T)^2 \leq \widetilde{C}(E_\tau(0)^2 + E_\tau(T)^3), \tag{4.22}$$

where the constant $\widetilde{C}$ is independent of $\tau$. Without loss of generality, we may assume $\widetilde{C} \geq 1$. Similar to that in [1], we achieve that

$$E_\tau(t) \leq \min\left\{\epsilon_0, \frac{1}{2\widetilde{C}}, \sqrt{2\widetilde{C}}E_\tau(0)\right\} \tag{4.23}$$

for any $t \geq 0$ if

$$||U_0||_{H^{\sigma+\varepsilon}} \leq \frac{1}{2(2\widetilde{C})^{3/2}}. \tag{4.24}$$

Note that the density

$$n - \overline{n} = \overline{n}\{\exp(A^{-1/2}m) - 1\}; \tag{4.25}$$

from Lemma 2.2, the definition of $E_\tau(t)$, and the standard continuity argument, we can obtain the following result: there exist two positive constants $\delta_0$, $C_0$ independent of $\tau$ if the initial data satisfy

$$||n_0 - \overline{n}||^2_{H^{\sigma+\varepsilon}} + ||\mathbf{u}_0||^2_{H^{\sigma+\varepsilon}} \leq \delta_0, \tag{4.26}$$

then the system (1.1)-(1.2) exists as a unique global solution $(n, \mathbf{u})$. Moreover, the uniform energy estimate holds:

$$\begin{aligned} &||(n-\overline{n},\mathbf{u})(\cdot,t)||^2_{H^{\sigma+\varepsilon}} + \frac{1}{\tau}\int_0^t ||\mathbf{u}(\cdot,\varsigma)||^2_{H^{\sigma+\varepsilon}} d\varsigma + \tau\int_0^t ||(\nabla n,\nabla\mathbf{u})(\cdot,\varsigma)||^2_{H^{\sigma-1+\varepsilon}} d\varsigma \\ &\quad \leq C_0||(n_0-\overline{n},\mathbf{u}_0)||^2_{H^{\sigma+\varepsilon}}, \quad t \geq 0, \end{aligned} \tag{4.27}$$

which completes the proof of Theorem 1.1. □

The proof of Corollary 1.2 is similar to that in [1]; here, we omit the details, the interested readers can refer to [1].

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