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Research Article

Unbounded Supersolutions of Nonlinear Equations with Nonstandard Growth

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We show that every weak supersolution of a variable exponent p-Laplace equation is lower semicontinuous and that the singular set of such a function is of zero capacity if the exponent is logarithmically Hölder continuous. As a technical tool we derive Harnack-type estimates for possibly unbounded supersolutions.

1. Introduction

The purpose of this work is to study regularity theory related to partial differential equations with nonstandard growth conditions. The principal prototype that we have in mind is the equation

$$\operatorname{div}\Big(p(x)\,|\,\nabla u(x)\,|^{\,p(x)-2}\nabla u(x)\Big) = 0, \tag{1.1}$$

which is the Euler-Lagrange equation of the variational integral

$$\int |\,\nabla u(x)\,|^{\,p(x)}\mathrm{d}x. \tag{1.2}$$

Here $p(\cdot)$ is a measurable function satisfying

$$1 < \inf_{x\in\mathbb{R}^n} p(x) \le p(x) \le \sup_{x\in\mathbb{R}^n} p(x) < \infty. \tag{1.3}$$

If $p(\cdot)$ is a constant function, then we have the standard p-Laplace equation and p-Dirichlet integral. This kind of variable exponent p-Laplace equation has first been considered by Zhikov [1] in connection with the Lavrentiev phenomenon for a Thermistor problem. By now there is an extensive literature on partial differential equations with nonstandard growth conditions; for example, see [2–6].

It has turned out that regularity results for weak solutions of (1.1) do not hold without additional assumptions on the variable exponent. In [1] Zhikov introduced a logarithmic condition on modulus of continuity. Variants of this condition have been expedient tools in the study of maximal functions, singular integral operators, and partial differential equations with nonstandard growth conditions on variable exponent spaces. Under this assumption Harnack's inequality and local Hölder continuity follow from Moser or DeGiorgi-type procedure; see [7, 8]. See also [9]. An interesting feature of this theory is that estimates are intrinsic in the sense that they depend on the solution itself. For example, supersolutions are assumed to be locally bounded and Harnack-type estimates in [7] depend on this bound.

In this work we are interested in possibly unbounded supersolutions of (1.1) and hence the previously obtained estimates are not immediately available for us. The main novelty of our approach is that instead of the boundedness we apply summability estimates for supersolutions. Roughly speaking we are able to replace L^∞-estimates with certain L^p-estimates for small values of p. The argument is a modification of Moser's iteration scheme presented in [7]. However, the modification is not completely straightforward and we have chosen to present all details here. As a by-product, we obtain refinements of results in [7, 9].

After these technical adjustments we are ready for our main results. Solutions are known to be continuous and hence it is natural to ask whether supersolutions are semicontinuous. Indeed, using Harnack-type estimates we show that every supersolution has a lower semicontinuous representative. Thus it is possible to study pointwise behavior of supersolutions. Our main result states that the singular set of a supersolution is of zero capacity. For the capacity theory in variable exponent spaces we refer to [10]. In fact we study a slightly more general class of functions than supersolutions which corresponds to the class of superharmonic functions in the case when $p(\cdot)$ is constant; see [11, 12].

2. Preliminaries

A measurable function $p\colon \mathbb{R}^n \to (1,\infty)$ is called a variable exponent. We denote

$$p_A^+ = \sup_{x\in A} p(x), \qquad p_A^- = \inf_{x\in A} p(x), \qquad p^+ = \sup_{x\in\mathbb{R}^n} p(x), \qquad p^- = \inf_{x\in\mathbb{R}^n} p(x) \tag{2.1}$$

and assume that $1 < p^- \le p^+ < \infty$.

Let Ω be an open subset of $\mathbb{R}^n$ with $n \ge 2$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty. \tag{2.2}$$

The Luxemburg norm on this space is defined as

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}. \tag{2.3}$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of a more general Orlicz-Musielak space studied in [13]. For a constant function $p(\cdot)$ the variable exponent Lebesgue space coincides with the standard Lebesgue space.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}. \tag{2.4}$$

For basic results on variable exponent spaces we refer to [14]. See also [15].

A somewhat unexpected feature of the variable exponent Sobolev spaces is that smooth functions need not be dense without additional assumptions on the variable exponent. This was observed by Zhikov in connection with the so-called Lavrentiev phenomenon. In [1] he introduced a logarithmic condition on modulus of continuity of the variable exponent. Next we briefly recall a version of this condition. The variable exponent p is said to satisfy a logarithmic Hölder continuity property, or briefly log-Hölder, if there is a constant $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)} \tag{2.5}$$

for all $x, y \in \Omega$ such that $|x-y| \le 1/2$. Under this condition smooth functions are dense in variable exponent Sobolev spaces and there is no confusion to define the Sobolev space with zero boundary values $W_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. We refer to [16, 17] for the details.

In this work we do not need any deep properties of variable exponent spaces. For our purposes, one of the most important facts about the variable exponent Lebesgue spaces is the following. If E is a measurable set with a finite measure, and p and q are variable exponents satisfying $q(x) \le p(x)$ for almost every $x \in E$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular this implies that every function $u \in W^{1,p(\cdot)}(\Omega)$ also belongs to $W_{\mathrm{loc}}^{1,p_\Omega^-}(\Omega)$ and to $W^{1,p_B^-}(B)$, where $B \subset \Omega$ is a ball. For all these facts we refer to [15, 14].

We say that a function $u \in W_{\mathrm{loc}}^{1,p(\cdot)}(\Omega)$ is a weak solution (supersolution) of (1.1), if

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = (\ge) 0 \tag{2.6}$$

for every test function $\varphi \in C_0^\infty(\Omega)$ $(\varphi \ge 0)$. When $1 < p^- \le p^+ < \infty$ the dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ obtained by conjugating the exponent pointwise, see [14]. This together with our definition $W_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ implies that we can also test with functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Our notation is rather standard. Various constants are denoted by C and the value of the constant may differ even on the same line. The quantities on which the constants depend are given in the statements of the theorems and lemmas. A dependence on p

includes dependence on the log-Hölder-constant of p. Note also that due to the local nature of the estimates, the constants depend only on the values of p in some ball.

3. Harnack estimates

In this section we prove a weak Harnack inequality for supersolutions. Throughout this section we write

$$v_\alpha = u + R^\alpha, \tag{3.1}$$

where u is a nonnegative supersolution.

We derive a suitable Caccioppoli-type estimate with variable exponents. Our aim is to combine this estimate with the standard Sobolev inequality. Thus we need a suitable passage between constant and variable exponents. This is accomplished in the following lemma.

Lemma 3.1. *Let E be a measurable subset of $\mathbb{R}^n$. For all nonnegative measurable functions f and g defined on E,*

$$\int_E f g^{p_E^-}\,\mathrm{d}x \le \int_E f\,\mathrm{d}x + \int_E f g^{p(x)}\,\mathrm{d}x. \tag{3.2}$$

Proof. The claim follows from an integration of the pointwise inequality

$$f(x)g(x)^{p_E^-} \le f(x) + f(x)g(x)^{p(x)}. \tag{3.3}$$

If $p(x) = p_E^-$ this is immediate. Otherwise we apply Young's inequality with the exponent $p(x)/p_E^- > 1$. □

Lemma 3.2 (Caccioppoli estimate). *Suppose that u is a nonnegative supersolution in B_{4R}. Let E be a measurable subset of B_{4R} and $\eta \in C_0^\infty(B_{4R})$ such that $0 \le \eta \le 1$. Then for every $\gamma_0 < 0$ there is a constant C depending on p and γ_0 such that the inequality*

$$\int_E v_\alpha^{\gamma-1} |\nabla u|^{p_E^-} \eta^{p_{B_{4R}}^+}\,\mathrm{d}x \le C \int_{B_{4R}} \left(\eta^{p_{B_{4R}}^+} v_\alpha^{\gamma-1} + v_\alpha^{\gamma+p(x)-1} |\nabla \eta|^{p(x)}\right)\mathrm{d}x \tag{3.4}$$

holds for every $\gamma < \gamma_0 < 0$ and $\alpha \in \mathbb{R}$.

Proof. Let $s = p_{B_{4R}}^+$. We want to test with the function $\psi = v_\alpha^\gamma \eta^s$. To this end we show that $\psi \in W_0^{1,p(\cdot)}(B_{4R})$. Since η has a compact support in B_{4R}, it is enough to show that $\psi \in W^{1,p(\cdot)}(\Omega)$. We observe that $\psi \in L^{p(\cdot)}(\Omega)$ since $|v_\alpha^\gamma|\eta^s \le R^{\alpha\gamma}$. Furthermore, we have

$$|\nabla \psi| \le \left| \gamma v_\alpha^{\gamma-1} \eta^s \nabla u + v_\alpha^\gamma s \eta^{s-1} \nabla \eta \right| \le |\gamma| R^{\alpha(\gamma-1)} |\nabla u| + s R^{\alpha\gamma} |\nabla \eta|, \tag{3.5}$$

from which we conclude that $|\nabla \psi| \in L^{p(\cdot)}(\Omega)$.

Using the facts that u is a supersolution and ψ is a nonnegative test function we find that

$$\begin{aligned} 0 &\le \int_{B_{4R}} p(x)\,|\,\nabla u(x)\,|^{\,p(x)-2}\nabla u(x)\cdot\nabla\psi(x)\,\mathrm{d}x \\ &= \int_{B_{4R}} p(x)\gamma|\nabla u|^{p(x)}\eta^{s}v_{\alpha}^{\gamma-1}\,\mathrm{d}x + \int_{B_{4R}} p(x)s|\nabla u|^{p(x)-2}v_{\alpha}^{\gamma}\eta^{s-1}\nabla u\cdot\nabla\eta\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Since γ is a negative number, this implies

$$|\,\gamma_0\,|\,p_{B_{4R}}^{-}\int_{B_{4R}}|\nabla u|^{p(x)}\eta^{s}v_{\alpha}^{\gamma-1}\,\mathrm{d}x \le s\int_{B_{4R}} p(x)|\nabla u|^{p(x)-2}v_{\alpha}^{\gamma}\eta^{s-1}\nabla u\cdot\nabla\eta\,\mathrm{d}x. \tag{3.7}$$

We denote the right-hand side of (3.7) by I. Since the left-hand side of (3.7) is nonnegative, so is I. Using the ε-version of Young's inequality we obtain

$$\begin{aligned} I &\le s\int_{B_{4R}} p(x)|\nabla u|^{p(x)-1}v_{\alpha}^{\gamma}\eta^{s-1}|\nabla\eta|\,\mathrm{d}x \\ &\le s\int_{B_{4R}}\left(\frac{1}{\varepsilon}\right)^{p(x)-1} p(x)\frac{\left(v_{\alpha}^{(\gamma+p(x)-1)/p(x)}|\nabla\eta|\eta^{s-s/p'(x)-1}\right)^{p(x)}}{p(x)} \\ &\qquad + \varepsilon p(x)\frac{\left(|\nabla u|^{p(x)-1}\eta^{s/p'(x)}v_{\alpha}^{\gamma-(\gamma+p(x)-1)/p(x)}\right)^{p'(x)}}{p'(x)}\,\mathrm{d}x \\ &\le s\left(\frac{1}{\varepsilon}\right)^{s-1}\int_{B_{4R}} v_{\alpha}^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\eta^{s-p(x)}\,\mathrm{d}x \\ &\qquad + s(s-1)\varepsilon\int_{B_{4R}}|\nabla u|^{p(x)}\eta^{s}v_{\alpha}^{\gamma-1}\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

By combining this with (3.7) we arrive at

$$\begin{aligned} &|\,\gamma_0\,|\,p_{B_{4R}}^{-}\int_{B_{4R}}|\nabla u|^{p(x)}\eta^{s}v_{\alpha}^{\gamma-1}\,\mathrm{d}x \\ &\le s\left(\frac{1}{\varepsilon}\right)^{s-1}\int_{B_{4R}} v_{\alpha}^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\eta^{s-p(x)}\,\mathrm{d}x + s(s-1)\varepsilon\int_{B_{4R}}|\nabla u|^{p(x)}\eta^{s}v_{\alpha}^{\gamma-1}\,\mathrm{d}x. \end{aligned} \tag{3.9}$$

By choosing

$$\varepsilon = \min\left\{1, \frac{|\gamma_0|\, p_{B_{4R}}^-}{2s(s-1)}\right\} \tag{3.10}$$

we can absorb the last term in (3.9) to the left-hand side and obtain

$$\int_{B_{4R}} |\nabla u|^{p(x)} \eta^s v_\alpha^{\gamma-1}\, \mathrm{d}x \le s\left(\frac{2s(s-1)}{|\gamma_0|\, p_{B_{4R}}^-} + 1\right)^{s-1} \frac{2}{|\gamma_0|\, p_{B_{4R}}^-} \int_{B_{4R}} v_\alpha^{\gamma+p(x)-1} |\nabla \eta|^{p(x)}\, \mathrm{d}x. \tag{3.11}$$

Taking $f = v_\alpha^{\gamma-1}\eta^s$ and $g = |\nabla u|$ in Lemma 3.1 and using inequality (3.11) we have the desired estimate. □

In the proof of the Caccioppoli estimate we did not use any other assumption on the variable exponent p except that $1 < p^- \le p^+ < \infty$. From now on we assume the logarithmic Hölder continuity. This is equivalent to the following estimate:

$$|B|^{-(p_B^+ - p_B^-)} \le C, \tag{3.12}$$

where $B \Subset \Omega$ is any ball; see for example [18].

The next two lemmas will be used to handle the right-hand side of the Caccioppoli estimate.

Lemma 3.3. *If the exponent $p(\cdot)$ is* log-*Hölder continuous,*

$$r^{-p(x)} \le C r^{-p_E^-} \tag{3.13}$$

provided $x \in E \subset B_r$.

Proof. For $r \ge 1$ we have $r^{-p(x)} \le r^{-p_E^-}$. Suppose then that $0 < r < 1$. Since $E \subset B_r$ implies $\operatorname{osc}_E p \le \operatorname{osc}_{B_r} p$, we obtain

$$r^{-p(x)} \le r^{-p_E^+} \le r^{-(\operatorname{osc}_E p)} r^{-p_E^-} \le r^{-(\operatorname{osc}_{B_r} p)} r^{-p_E^-} \le C r^{-p_E^-}, \tag{3.14}$$

where we used logarithmic Hölder continuity in the last inequality. □

In the following lemma the barred integral sign denotes the integral average.

Lemma 3.4. *Let f be a positive measurable function and assume that the exponent $p(\cdot)$ is* log-*Hölder continuous. Then*

$$\fint_{B_r} f^{p_{B_r}^+ - p_{B_r}^-}\, \mathrm{d}x \le C \|f\|_{L^s(B_r)}^{p_{B_r}^+ - p_{B_r}^-} \tag{3.15}$$

for any $s > p_{B_r}^+ - p_{B_r}^-$.

Proof. Let $q = p^+_{B_r} - p^-_{B_r}$. Hölder's inequality implies

$$\begin{aligned} \fint_{B_r} f^{p^+_{B_r} - p^-_{B_r}}\,dx &\le \frac{C}{r^n}\left(\int_{B_r} 1\,dx\right)^{1-q/s}\left(\int_{B_r} f^s\,dx\right)^{q/s} \\ &\le \frac{C}{r^n} r^{n(1-q/s)}\|f\|^q_{L^s(B_r)} \le C\|f\|^q_{L^s(B_r)}. \end{aligned} \tag{3.16}$$

Again we used the logarithmic Hölder continuity in the last inequality. □

Later we apply Lemma 3.4 with $f = u^{q'}$. In this case the upper bound written in terms of u is

$$C\|u\|^{q'(p^+_{B_r} - p^-_{B_r})}_{L^{q's}(B_r)}. \tag{3.17}$$

Now we have everything ready for the iteration. We write

$$\Phi(f,q,B_r) = \left(\fint_{B_r} f^q\,dx\right)^{1/q} \tag{3.18}$$

for a nonnegative measurable function f.

Lemma 3.5. *Assume that u is a nonnegative supersolution in B_{4R} and let $R \le \rho < r \le 3R$. Then the inequality*

$$\Phi(v_1,q\beta,B_r) \le C^{1/|\beta|}(1+|\beta|)^{p^+_{B_{4R}}/|\beta|}\left(\frac{r}{r-\rho}\right)^{p^+_{B_{4R}}/|\beta|}\Phi\left(v_1,\frac{\beta n}{n-1},B_\rho\right) \tag{3.19}$$

holds for every $\beta < 0$ and $1 < q < n/(n-1)$. The constant C depends on n, p, and the $L^{q's}(B_{4R})$-norm of u with $s > p^+_{B_{4R}} - p^-_{B_{4R}}$.

Proof. In Lemma 3.2 we take $E = B_{4R}$ and $\gamma = \beta - p^-_{B_{4R}} + 1$. Then $\gamma < 1 - p^-_{B_{4R}}$ and thus

$$\int_{B_{4R}} v_1^{\beta - p^-_{B_{4R}}}|\nabla u|^{p^-_{B_{4R}}}\eta^{p^+_{B_{4R}}}\,dx \le C\int_{B_{4R}}\left(\eta^{p^+_{4R}}v_1^{\beta - p^-_{B_{4R}}} + v_1^{\beta - p^-_{B_{4R}} + p(x)}|\nabla\eta|^{p(x)}\right)dx. \tag{3.20}$$

Next we take a cutoff function $\eta \in C_0^\infty(B_r)$ with $0 \le \eta \le 1$, $\eta = 1$ in B_ρ, and

$$|\nabla\eta| \le \frac{Cr}{R(r-\rho)}. \tag{3.21}$$

By Lemma 3.3 we have

$$|\nabla\eta|^{-p(x)} \le CR^{-p(x)}\left(\frac{r}{r-\rho}\right)^{p^+_{B_{4R}}} \le CR^{-p^-_{B_{4R}}}\left(\frac{r}{r-\rho}\right)^{p^+_{B_{4R}}}. \tag{3.22}$$

Using inequality (3.20) with this choice of η we have

$$\begin{aligned}
&\fint_{B_r} \left|\nabla\left(v_1^{\beta/p_{B_{4R}}^-}\eta^{p_{B_{4R}}^+/p_{B_{4R}}^-}\right)\right|^{p_{B_{4R}}^-} \mathrm{d}x \\
&\quad\le C\fint_{B_r} |\beta|^{p_{B_{4R}}^-} v_1^{\beta-p_{B_{4R}}^-} |\nabla u|^{p_{B_{4R}}^-}\eta^{p_{B_{4R}}^+}\,\mathrm{d}x + C\fint_{B_r} v_1^{\beta}\eta^{p_{B_{4R}}^+-p_{B_{4R}}^-}|\nabla\eta|^{p_{B_{4R}}^-}\,\mathrm{d}x \\
&\quad\le C|\beta|^{p_{B_{4R}}^-}\fint_{B_r}\left(\eta^{p_{B_{4R}}^+} v_1^{\beta-p_{B_{4R}}^-} + v_1^{\beta-p_{B_{4R}}^-+p(x)}|\nabla\eta|^{p(x)}\right)\mathrm{d}x + C\fint_{B_r} v_1^{\beta}\eta^{p_{B_{4R}}^+-p_{B_{4R}}^-}|\nabla\eta|^{p_{B_{4R}}^-}\,\mathrm{d}x \\
&\quad\le C(1+|\beta|)^{p_{B_{4R}}^+}\left(\fint_{B_r}\eta^{p_{B_{4R}}^+} v_1^{\beta-p_{B_{4R}}^-}\,\mathrm{d}x + \fint_{B_r} v_1^{\beta-p_{B_{4R}}^-+p(x)}|\nabla\eta|^{p(x)}\,\mathrm{d}x + \fint_{B_r} v_1^{\beta}|\nabla\eta|^{p_{B_{4R}}^-}\,\mathrm{d}x\right).
\end{aligned} \tag{3.23}$$

Now the goal is to estimate each integral in the parentheses by

$$\left(\fint_{B_r} v_1^{q\beta}\,\mathrm{d}x\right)^{1/q}. \tag{3.24}$$

The first integral can be estimated with Hölder's inequality. Since $v_1^{-p_{B_{4R}}^-} \le R^{-p_{B_{4R}}^-}$, we have

$$\fint_{B_r}\eta^{p_{B_{4R}}^+} v_1^{\beta-p_{B_{4R}}^-}\,\mathrm{d}x \le \left(\fint_{B_r} v_1^{q(\beta-p_{B_{4R}}^-)}\,\mathrm{d}x\right)^{1/q} \le R^{-p_{B_{4R}}^-}\left(\fint_{B_r} v_1^{q\beta}\,\mathrm{d}x\right)^{1/q}. \tag{3.25}$$

By (3.22), Hölder's inequality, and Lemma 3.4 for the second integral we have

$$\begin{aligned}
&\fint_{B_r} v_1^{\beta-p_{B_{4R}}^-+p(x)}|\nabla\eta|^{p(x)}\,\mathrm{d}x \\
&\qquad\le CR^{-p_{B_{4R}}^-}\left(\frac{r}{r-\rho}\right)^{p_{B_{4R}}^+}\fint_{B_r} v_1^{\beta-p_{B_{4R}}^-+p(x)}\,\mathrm{d}x \\
&\qquad\le CR^{-p_{B_{4R}}^-}\left(\frac{r}{r-\rho}\right)^{p_{B_{4R}}^+}\left(\fint_{B_r} v_1^{q'(p(x)-p_{B_{4R}}^-)}\,\mathrm{d}x\right)^{1/q'}\left(\fint_{B_r} v_1^{q\beta}\,\mathrm{d}x\right)^{1/q} \\
&\qquad\le CR^{-p_{B_{4R}}^-}\left(\frac{r}{r-\rho}\right)^{p_{B_{4R}}^+}\left(1+\|v_1\|_{L^{q's}(B_{4R})}^{q'(p_{B_{4R}}^+-p_{B_{4R}}^-)}\right)^{1/q'}\left(\fint_{B_r} v_1^{q\beta}\,\mathrm{d}x\right)^{1/q}.
\end{aligned} \tag{3.26}$$

Finally, for the third integral we have by Hölder's inequality,

$$\fint_{B_r} v_1^{\beta}|\nabla\eta|^{p_{B_{4R}}^-}\,\mathrm{d}x \le CR^{-p_{B_{4R}}^-}\left(\frac{r}{r-\rho}\right)^{p_{B_{4R}}^-}\left(\fint_{B_r} v_1^{q\beta}\,\mathrm{d}x\right)^{1/q}. \tag{3.27}$$

Now we have arrived at the inequality

$$\begin{aligned}
&\fint_{B_r} \left| \nabla \left(v_1^{\beta/p^-_{B_{4R}}} \eta^{p^+_{B_{4R}}/p^-_{B_{4R}}} \right) \right|^{p^-_{B_{4R}}} \mathrm{d}x \\
&\quad \le C(1+|\beta|)^{p^+_{B_{4R}}} \left(1 + \|v_1\|^{q'(p^+_{B_{4R}}-p^-_{B_{4R}})}_{L^{q's}(B_{4R})} \right)^{1/q'} R^{-p^-_{B_{4R}}} \left(\frac{r}{r-\rho} \right)^{p^+_{B_{4R}}} \left(\fint_{B_r} v_1^{q\beta} \,\mathrm{d}x \right)^{1/q}.
\end{aligned} \tag{3.28}$$

By the Sobolev inequality

$$\left(\fint_{B_r} |u|^{na/(n-1)} \,\mathrm{d}x \right)^{(n-1)/na} \le CR \left(\fint_{B_r} |\nabla u|^a \,\mathrm{d}x \right)^{1/a}, \tag{3.29}$$

where $u \in W_0^{1,a}(B_r)$ and $a = p^-_{B_{4R}}$, and (3.28) we obtain

$$\begin{aligned}
\left(\fint_{B_\rho} v_1^{\beta n/(n-1)} \,\mathrm{d}x \right)^{(n-1)/n} &\le \left(C \fint_{B_r} \left(v_1^{\beta/p^-_{B_{4R}}} \eta^{p^+_{B_{4R}}/p^-_{B_{4R}}} \right)^{np^-_{B_{4R}}/(n-1)} \mathrm{d}x \right)^{(n-1)/n} \\
&\le CR^{p^-_{B_{4R}}} \fint_{B_r} \left| \nabla \left(v_1^{\beta/p^-_{B_{4R}}} \eta^{p^+_{B_{4R}}/p^-_{B_{4R}}} \right) \right|^{p^-_{B_{4R}}} \mathrm{d}x \\
&\le C(1+|\beta|)^{p^+_{B_{4R}}} \left(\frac{r}{r-\rho} \right)^{p^+_{B_{4R}}} \left(\fint_{B_r} v_1^{q\beta} \,\mathrm{d}x \right)^{1/q}.
\end{aligned} \tag{3.30}$$

The claim follows from this since β is a negative number. □

The next lemma is the crucial passage from positive exponents to negative exponents in the Moser iteration scheme.

Lemma 3.6. *Assume that u is a nonnegative supersolution in B_{4R} and $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. Then there exist constants $q_0 > 0$ and C depending on n, p, and $L^s(B_{4R})$-norm of u such that*

$$\Phi(v_1, q_0, B_{3R}) \le C\Phi(v_1, -q_0, B_{3R}). \tag{3.31}$$

Proof. Choose a ball $B_{2r} \subset B_{4R}$ and a cutoff function $\eta \in C_0^\infty(B_{2r})$ such that $\eta = 1$ in B_r and $|\nabla \eta| \le C/r$. Taking $E = B_r$ and $\gamma = 1 - p^-_{B_r}$ in Lemma 3.2 we have

$$\fint_{B_r} |\nabla \log v_1|^{p^-_{B_r}} \,\mathrm{d}x \le C \left(\fint_{B_{2r}} v_1^{-p^-_{B_r}} + \fint_{B_{2r}} v_1^{p(x)-p^-_{B_r}} r^{-p(x)} \,\mathrm{d}x \right). \tag{3.32}$$

Using Lemmas 3.3 and 3.4 and the estimate $v_1^{-p_{B_r}^-} \leq R^{-p_{B_r}^-} \leq r^{-p_{B_r}^-}$ we have

$$\begin{aligned} \fint_{B_r} |\nabla \log v_1|^{p_{B_r}^-}\,\mathrm{d}x &\leq C\Big(r^{-p_{B_r}^-} + r^{-p_{B_{2r}}^-}\fint_{B_{2r}} v_1^{p(x)-p_{B_r}^-}\,\mathrm{d}x\Big) \\ &\leq C\Big(r^{-p_{B_r}^-} + r^{-p_{B_{2r}}^-}\Big(1+\|v_1\|_{L^s(B_{4R})}^{p_{B_{4R}}^+-p_{B_{4R}}^-}\Big)\Big). \end{aligned} \tag{3.33}$$

Let $f = \log v_1$. By the Poincaré inequality and the above estimate we obtain

$$\begin{aligned} \fint_{B_r} |f - f_{B_r}|\,\mathrm{d}x &\leq \Big(r^{p_{B_r}^-}\fint_{B_r} |\nabla f|\,\mathrm{d}x\Big)^{1/p_{B_r}^-} \\ &\leq C\Big(1 + r^{p_{B_r}^- - p_{B_{2r}}^-}\Big(1+\|v_1\|_{L^s(B_{4R})}^{p_{B_{4R}}^+-p_{B_{4R}}^-}\Big)\Big)^{1/p_{B_r}^-}. \end{aligned} \tag{3.34}$$

Note that $p_{B_r}^- \geq p_{B_{2r}}^-$ since $B_r \subset B_{2r}$, so that the right-hand side of (3.34) is bounded.

The rest of the proof is standard. Since (3.34) holds for all balls $B_{2r} \subset B_{4R}$, by the John-Nirenberg lemma there exist positive constants C_1 and C_2 depending on the right-hand side of (3.34) such that

$$\fint_{B_{3R}} e^{C_1|f-f_{B_{3R}}|}\,\mathrm{d}x \leq C_2. \tag{3.35}$$

Using (3.35) we can conclude that

$$\begin{aligned} \Big(\fint_{B_{3R}} e^{C_1 f}\,\mathrm{d}x\Big)\Big(\fint_{B_{3R}} e^{-C_1 f}\,\mathrm{d}x\Big) &= \Big(\fint_{B_{3R}} e^{C_1(f-f_{B_{3R}})}\,\mathrm{d}x\Big)\Big(\fint_{B_{3R}} e^{-C_1(f-f_{B_{3R}})}\,\mathrm{d}x\Big) \\ &\leq \Big(\fint_{B_{3R}} e^{C_1|f-f_{B_{3R}}|}\,\mathrm{d}x\Big)^2 \leq C_2^2, \end{aligned} \tag{3.36}$$

which implies that

$$\begin{aligned} \Big(\fint_{B_{3R}} v_1^{C_1}\,\mathrm{d}x\Big)^{1/C_1} &= \Big(\fint_{B_{3R}} e^{C_1 f}\,\mathrm{d}x\Big)^{1/C_1} \\ &\leq C_2^{2/C_1}\Big(\fint_{B_{3R}} e^{-C_1 f}\,\mathrm{d}x\Big)^{-1/C_1} \\ &= C_2^{2/C_1}\Big(\fint_{B_{3R}} v_1^{-C_1}\,\mathrm{d}x\Big)^{-1/C_1}, \end{aligned} \tag{3.37}$$

so that we can take $q_0 = C_1$. □

Note that the exponent q_0 in Lemma 3.6 also depends on the $L^s(B_{4R})$-norm of u. More precisely, the constant C_1 obtained from the John-Nirenberg lemma is a universal constant divided by the right-hand side of (3.34). Thus we have

$$q_0 = \frac{C}{C' + \|u\|_{L^s(B_{4R})}^{p^+_{B_{4R}} - p^-_{B_{4R}}}}. \tag{3.38}$$

The following weak Harnack inequality is the main result of this section. It applies also for unbounded supersolutions.

Theorem 3.7 (weak Harnack inequality). *Assume that u is a nonnegative supersolution in B_{4R}, $1 < q < n/(n-1)$ and $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. Then*

$$\left(\fint_{B_{2R}} u^{q_0}\,\mathrm{d}x\right)^{1/q_0} \le C\left(\operatorname*{ess\,inf}_{B_R} u(x) + R\right), \tag{3.39}$$

where q_0 is the exponent from Lemma 3.6 and C depends on n, p, q, and $L^{q's}(B_{4R})$-norm of u.

Remark 3.8. (1) The main difference compared to Alkhutov's result in [7, 9] is that the constant and the exponent depend on the $L^{q's}(B_{4R})$-norm of u instead of the essential supremum of u in B_{4R}. This is a crucial advantage for us since we are interested in supersolutions which may be unbounded.

(2) Since the exponent $p(\cdot)$ is uniformly continuous, we can take for example $q's = p^-_\Omega$ by choosing R small enough. Thus the constants in the estimates are finite for all supersolutions u in a scale that depends only on $p(\cdot)$.

Proof. Let $R \le \rho < r \le 3R$, $r_j = \rho + 2^{-j}(r-\rho)$, and

$$\xi_j = -\left(\frac{n}{(n-1)q}\right)^j q_0 \tag{3.40}$$

for $j = 0,1,2,\ldots$. By Lemma 3.5 we have

$$\Phi(v_1,\xi_j,B_{r_j}) \le C^{1/|\xi_j|}\left(1+|\xi_j|\right)^{p^+_{B_{4R}}/|\xi_j|}\left(\frac{r_j}{r_j - r_{j+1}}\right)^{p^+_{B_{4R}}/|\xi_j|}\Phi(v_1,\xi_{j+1},B_{r_{j+1}}). \tag{3.41}$$

An iteration of this inequality yields

$$\begin{aligned}
\Phi(v_1,-q_0,B_r) &\le \prod_{j=0}^{\infty} C^{1/|\xi_j|}\left(1+|\xi_j|\right)^{p^+_{B_{4R}}/|\xi_j|}\left(\frac{r_j}{r_j-r_{j+1}}\right)^{p^+_{B_{4R}}/|\xi_j|}\operatorname*{ess\,inf}_{x\in B_\rho} v_1(x)\\
&\le C^{\sum_{j=0}^\infty 1/|\xi_j|}2^{\sum_{j=0}^\infty j p^+_{B_{4R}}/|\xi_j|}\left(\frac{r}{r-\rho}\right)^{\sum_{j=0}^\infty p^+_{B_{4R}}/|\xi_j|}\\
&\quad\times\prod_{j=0}^{\infty}\left(1+|\xi_j|\right)^{p^+_{B_{4R}}/|\xi_j|}\operatorname*{ess\,inf}_{x\in B_\rho} v_1(x).
\end{aligned} \tag{3.42}$$

We estimate the remaining product by using the fact that $|\xi_j| > 1$ when $j > j_0$ and $|\xi_j| \le 1$ when $j \le j_0$ for some j_0. This implies that

$$\begin{aligned}\prod_{j=0}^{\infty}\left(1+|\xi_j|\right)^{p^+_{B_{4R}}/|\xi_j|} &\le 2^{\sum_{j=0}^{j_0} p^+_{B_{4R}}/|\xi_j|}\, 2^{\sum_{j=j_0+1}^{\infty} p^+_{B_{4R}}/|\xi_j|}\left(\frac{n}{(n-1)q}\right)^{p^+_{B_{4R}} q_0 \sum_{j=j_0+1}^{\infty} j((n-1)q/n)^j} \\ &\le 2^{\sum_{j=0}^{\infty} p^+_{B_{4R}}/|\xi_j|}\left(\frac{n}{(n-1)q}\right)^{p^+_{B_{4R}} q_0 \sum_{j=0}^{\infty} j((n-1)q/n)^j}.\end{aligned} \tag{3.43}$$

All the series in the above estimates are convergent by the root test, so we obtain

$$\Phi(v_1, -q_0, B_r) \le C \operatorname*{ess\,inf}_{x\in B_\rho} v_1(x). \tag{3.44}$$

Next we choose $\rho = R$ and $r = 3R$ and use Lemma 3.6 to get

$$\Phi(v_1, q_0, B_{3R}) \le C \operatorname*{ess\,inf}_{x\in B_R} v_1(x). \tag{3.45}$$

Finally we observe that

$$\Phi(v_1, q_0, B_{2R}) \le C\Phi(v_1, q_0, B_{3R}). \tag{3.46}$$

This completes the proof. □

Lemma 3.4 can be used in the proof of the supremum estimate in [7] in the same way as in the proof of Lemma 3.5. Combining this with the weak Harnack inequality above one obtains the full Harnack inequality with the constant depending on the $L^{q's}(B_{4R})$-norm of the solution instead of the supremum. This implies the local Hölder continuity of solutions by the standard technique; see [19]. Summing up, we have the following theorem.

Theorem 3.9 (the Harnack inequality). *Let u be a nonnegative solution in B_{4R}, $1 < q < n/(n-1)$, and $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. Then*

$$\operatorname*{ess\,sup}_{x\in B_R} u(x) \le C\Big(\operatorname*{ess\,inf}_{x\in B_R} u(x) + R\Big), \tag{3.47}$$

where the constant C depends on n, p, and the $L^{q's}(B_{4R})$-norm of u.

The main difference compared to earlier results is that the constant depends on the $L^{q's}$-norm instead of the essential supremum. The following example shows that the constant in the Harnack inequality cannot be independent of u even if the exponent is Lipschitz continuous.

Example 3.10. Let $p\colon (0,1) \to (1,\infty)$ be defined by

$$p(x) = \begin{cases} 3 & \text{for } 0 < x \le \dfrac{1}{2}, \\ 3 - 2\left(x - \dfrac{1}{2}\right) & \text{for } \dfrac{1}{2} < x < 1. \end{cases} \tag{3.48}$$

Suppose that $u_a \in W^{1,p(\cdot)}(0,1)$ is the minimizer of the Dirichlet energy integral with the boundary values 0 and $a > 0$. Then u_a is a solution with the same boundary values by [20, Theorem 5.7].

Theorem 3.2 of [21] gives

$$u_a(x) = \int_0^x \left(\frac{C_a}{p(y)}\right)^{1/(p(y)-1)} \mathrm{d}y, \tag{3.49}$$

where C_a is a constant obtained from the equation

$$\int_0^1 \left(\frac{C_a}{p(y)}\right)^{1/(p(y)-1)} \mathrm{d}y = a. \tag{3.50}$$

Note that if $a \to \infty$, then $C_a \to \infty$. In $(0,1/2)$ the minimizer is linear, $u_a(x) = \sqrt{(C_a/3)}x$. In $(1/2,3/5)$ the gradient of u_a increases from $\sqrt{C_a/3}$ to $(5C_a/14)^{5/9}$. In $11/20$, the midpoint of $(1/2,3/5)$, the gradient of u_a is $(10C_a/29)^{10/19}$. Hence we find that

$$u_a\left(\frac{3}{5}\right) \ge \sqrt{\frac{C_a}{3}}\frac{1}{2} + \frac{1}{20}\left(\frac{10C_a}{29}\right)^{10/19}. \tag{3.51}$$

Let $B = B(1/2,1/10) = (2/5,3/5)$. Then we obtain

$$\begin{aligned} \frac{\operatorname{ess\,sup}_{x\in B} |u_a(x)|}{\operatorname{ess\,inf}_{x\in B} |u_a(x)|} &\ge \frac{\sqrt{(C_a/3)}(1/2) + (1/20)(10C_a/29)^{10/19}}{\sqrt{(C_a/3)}(2/5)} \\ &= \frac{5}{4} + \frac{1}{8}\frac{1}{\sqrt{3}}\left(\frac{10}{29}\right)^{10/19} C_a^{1/38} \longrightarrow \infty \end{aligned} \tag{3.52}$$

as $a \to \infty$.

This example can be extended to the planar case by studying functions $f_a(x,y) = u_a(x)$ in $\{(x,y) : 0 < x < 1, 0 < y < 1\}$ with the exponent $q(x,y) = p(x)$.

4. The singular set of a supersolution

First we prove that every supersolution has a lower semicontinuous representative if the exponent $p(\cdot)$ is log-Hölder. For this purpose, we need the fact that supersolutions are locally bounded from below. This is true because subsolutions are locally bounded above, which can be seen from the proof of Theorem 1 in [7].

We set

$$u^*(x) = \operatorname{ess\,liminf}_{y\to x} u(y) = \lim_{r\to 0} \operatorname*{ess\,inf}_{y\in B(x,r)} u(y). \tag{4.1}$$

Theorem 4.1. *Let u be a function defined on Ω such that*

(1) *u is finite almost everywhere, and*

(2) $\min\{u,\lambda\}$ *is a supersolution for every $\lambda > 0$.*

Then u^* is lower semicontinuous and

$$u^*(x) = u(x) \quad \text{for almost every } x \in \Omega. \tag{4.2}$$

Remark 4.2. Observe that all supersolutions satisfy the assumptions of the previous theorem. We present the result in a slightly more general case, since we would like to include functions which are increasing limits of supersolutions. For bounded supersolutions the theorem has been studied in [9].

Proof. Let $\Omega' \Subset \Omega$ and first assume that u is bounded above. Pick a point $x \in \Omega'$, choose R such that $B(x,2R) \subset \Omega'$ and let

$$M = \operatorname*{ess\,sup}_{\Omega'} u + 1. \tag{4.3}$$

For $0 < r \le 2R$ denote $m(r) = \operatorname{ess\,inf}_{y \in B(x,r)} u(y)$. Since supersolutions are locally bounded below, we have $m(r) > -\infty$ for $0 < r \le 2R$.

The function u^* is lower semicontinuous since $u_r^*(x) = \operatorname{ess\,inf}_{y \in B(x,r)} u(y)$ is lower semicontinuous and u^* is an increasing limit of the functions u_r^*.

We will complete the proof for bounded functions u by showing that

$$u^*(x) = \lim_{r \to 0} \fint_{B(x,r)} u(y)\,\mathrm{d}y. \tag{4.4}$$

For every $0 < 5r \le R$ the function $u - m(5r)$ is a nonnegative supersolution in $B(x,4r)$. Thus the weak Harnack inequality implies that

$$\begin{aligned} m(r) - m(5r) &\ge C\left(\left(\fint_{B_{2r}} (u - m(5r))^{q_0}\,\mathrm{d}x\right)^{1/q_0} - r\right) \\ &\ge C\left((M - m(5r))^{(q_0-1)/q_0}\left(\fint_{B_{2r}} (u - m(5r))\,\mathrm{d}x\right)^{1/q_0} - r\right), \end{aligned} \tag{4.5}$$

where we assumed that $q_0 < 1$. This implies that

$$\begin{aligned} 0 &\le \fint_{B(x,2r)} u\,\mathrm{d}y - m(5r) \\ &\le C(M - m(5r))^{1-q_0}(m(r) - m(5r) + Cr)^{q_0}. \end{aligned} \tag{4.6}$$

Since $m(r) - m(5r) + Cr$ tends to zero as $r \to 0$, the above estimate implies (4.4).

For the general case, denote $u_i = \min\{u,i\}$ for $i = 1,2,\dots$ and observe that

$$u^*(x) = \lim_{i \to \infty} u_i^*(x). \tag{4.7}$$

To see that $u = u^*$ almost everywhere, consider the sets

$$
\begin{aligned}
E &= \{x \in \Omega : u(x) < \infty, u^*(x) \neq u(x)\}, \\
F &= \{x \in \Omega : u(x) = \infty, u^*(x) \neq u(x)\}.
\end{aligned}
\tag{4.8}
$$

Then $|F| = 0$ since u is assumed to be finite almost everywhere. For the set E we have $E \subset \cup_i E_i$, where

$$
E_i = \{x \in \Omega : u_i^*(x) \neq u_i(x)\}, \tag{4.9}
$$

$|E_i| = 0$ by the first part of the theorem, and the claim follows. □

Our next goal is to obtain estimates for the singular set of a supersolution. To this end, we derive two Caccioppoli-type estimates for a supersolution.

Lemma 4.3. *Let u be a nonnegative supersolution in B_{4R}, $\eta \in C_0^\infty(B_{4R})$ such that $0 \leq \eta \leq 1$ and $\gamma < \gamma_0 < 0$. Then there is a constant C depending on p and γ_0 such that*

$$
\int_{B_{4R}} |\nabla u|^{p(x)} \eta^{p^+_{B_{4R}}} u^{\gamma-1}\, dx \leq C \int_{B_{4R}} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)}\, dx. \tag{4.10}
$$

Proof. Denote $u_k = u + 1/k$. Testing with $\eta^{p^+_{B_{4R}}} u_k^\gamma$ gives

$$
\int_{B_{4R}} |\nabla u|^{p(x)} \eta^{p^+_{B_{4R}}} u_k^{\gamma-1}\, dx \leq C \int_{B_{4R}} u_k^{\gamma+p(x)-1} |\nabla \eta|^{p(x)}\, dx \tag{4.11}
$$

as in the proof of inequality (3.11) in Lemma 3.2. $u_k^{\gamma-1} \to u^{\gamma-1}$ monotonically as $k \to \infty$, and similarly $u_k^{\gamma-1+p(x)} \to u^{\gamma-1+p(x)}$ monotonically when $\gamma - 1 + p(x) < 0$. If $\gamma - 1 + p(x) \geq 0$, we have

$$
\begin{aligned}
u_k^{\gamma-1+p(x)} |\nabla \eta|^{p(x)} &\leq C(1 + u^{\gamma-1+p(x)}) |\nabla \eta|^{p(x)} \\
&\leq C(2 + u^{p(x)}) |\nabla \eta|^{p(x)}.
\end{aligned}
\tag{4.12}
$$

since γ is negative. Now we can let $k \to \infty$ in the above inequality, obtaining the claim by the monotone converge theorem and the dominated converge theorem. □

In the following two theorems, q is an exponent such that $1 < q < n/(n-1)$, $s > p^+_{B_{4R}} - p^-_{B_{4R}}$, and $q_0 > 0$ is an exponent for which the weak Harnack inequality holds for the function under consideration.

Theorem 4.4. *Let u be a nonnegative supersolution in B_{4R}, $B_{2r} \subset B_{4R}$, $\gamma < \gamma_0 < 0$, $\eta \in C_0^\infty(B_{4R})$ such that $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq C/r$. Then*

$$
\int_{B_r \cap \{u \leq \lambda\}} |\nabla u|^{p(x)} \eta^{p^+_{B_{4R}}}\, dx \leq C \lambda^{1-\gamma} r^{n-p^-_{B_{2r}}} \left(\operatorname*{ess\,inf}_{B_r} u + r \right)^{(\gamma-1+p^-_{B_{2r}})}, \tag{4.13}
$$

where γ is chosen so that $q_0 \geq q(\gamma - 1 + p^-_{B_{2r}}) > 0$ and the constant C depends on n, p, γ_0, and the $L^{q's}(B_{4R})$-norm of u.

Proof. We have $u/\lambda \leq 1$ whenever $u \leq \lambda$. Using this fact, Lemmas 3.3, 3.4, and 4.3, the Hölder inequality, and the weak Harnack inequality we obtain

$$\begin{aligned}
&\int_{B_r \cap \{u \leq \lambda\}} |\nabla u|^{p(x)} \eta^{p^+_{B_{4R}}} \,\mathrm{d}x \\
&\quad \leq \int_{B_r \cap \{u \leq \lambda\}} \left(\frac{u}{\lambda}\right)^{\gamma - 1} |\nabla u|^{p(x)} \eta^{p^+_{B_{4R}}} \,\mathrm{d}x \\
&\quad \leq C\lambda^{1-\gamma} \int_{B_{2r}} u^{\gamma + p(x) - 1} |\nabla \eta|^{p(x)} \,\mathrm{d}x \\
&\quad \leq C\lambda^{1-\gamma} r^{-p^-_{B_{2r}}} \int_{B_{2r}} u^{\gamma - 1 + p(x)} \,\mathrm{d}x \\
&\quad \leq C\lambda^{1-\gamma} r^{n - p^-_{B_{2r}}} \left(\fint_{B_{2r}} u^{q'(p(x) - p^-_{B_{2r}})} \,\mathrm{d}x \right)^{1/q'} \left(\fint_{B_{2r}} u^{q(\gamma - 1 + p^-_{B_{2r}})} \,\mathrm{d}x \right)^{1/q} \\
&\quad \leq C\lambda^{1-\gamma} r^{n - p^-_{B_{2r}}} \left(1 + \|u\|^{q'(p^+_{B_{4R}} - p^+_{B_{4R}})}_{L^{q's}(B_{4R})} \right)^{1/q'} \left(\fint_{B_{2r}} u^{q(\gamma - 1 + p^-_{B_{2r}})} \,\mathrm{d}x \right)^{1/q} \\
&\quad \leq C\lambda^{1-\gamma} r^{n - p^-_{B_{2r}}} \left(\operatorname*{ess\,inf}_{B_r} u + r \right)^{(\gamma - 1 + p^-_{B_{2r}})}.
\end{aligned} \tag{4.14}$$

□

The Sobolev $p(\cdot)$-capacity of a set $E \subset \mathbb{R}^n$ is defined as

$$C_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) \mathrm{d}x, \tag{4.15}$$

where the infimum is taken over the set of admissible functions

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geq 1 \text{ in an open set containing } E\}. \tag{4.16}$$

This definition gives a Choquet capacity; for this and other properties of $C_{p(\cdot)}$, see [10].

The following theorem is our main result.

Theorem 4.5. *Let u be a nonnegative function such that*

(1) *u is lower semicontinuous,*

(2) *$\min\{u, \lambda\}$ is a supersolution for each $\lambda > 0$, and*

(3) *$u \in L^t_{\mathrm{loc}}(\Omega)$ for some $t > 0$.*

Denote

$$E_\lambda = \{x \in B(x_0, r) : u(x) > \lambda\}, \tag{4.17}$$

where $B(x_0,r)$ is a ball with $B_{4r} = B(x_0,4r) \Subset \Omega$. Then

$$C_{p(\cdot)}(E_\lambda) \le Cr^{n-p^-_{B_{2r}}}\lambda^{-q_0/q}\Big(\inf_{B_r} u + r\Big)^{q_0/q}, \tag{4.18}$$

where the constant C depends on p, n, and the $L^t(B_{4r})$-norm of u.

Remark 4.6. (1) Observe that all supersolutions satisfy the assumptions of the previous theorem.

(2) For constant $p(\cdot)$, the class of functions which satisfy (1)–(3) is called p-superharmonic functions. This is a strictly bigger class of functions than supersolutions. Indeed, the nonlinear counterpart of a fundamental solution is the prime example of such a function.

Proof. Denote $u_\lambda = \min\{u,\lambda\}$ and choose $\varphi \in C_0^\infty(B(x_0,2r))$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B(x_0,r)$, and $|\nabla\varphi| \le C/r$. For sufficiently small radii r, we can choose

$$q_0 = \frac{C}{C + \|u\|_{L^t(B_{4R})}^{p^+_{B_{4R}} - p^-_{B_{4R}}}} \tag{4.19}$$

since we can take $q's = t$ with a suitable choice of $s > p^+_{B_{4r}} - p^-_{B_{4r}}$. Then the weak Harnack inequality holds for u_λ with an exponent and a constant independent of λ. Further, we choose the parameter γ in Theorem 4.4 so that $q_0/q = \gamma - 1 + p^-_{B_{2r}}$. This is always possible, since we can take a smaller q_0 if necessary.

Since u is lower semicontinuous, the set E_λ is open. Further, $u_\lambda\varphi/\lambda = 1$ in E_λ, so we can test the capacity of E_λ with $u_\lambda\varphi/\lambda$. This gives

$$\begin{aligned} C_{p(\cdot)}(E_\lambda) &\le \int_{B(x_0,2r)} \left(\left|\frac{u_\lambda\varphi}{\lambda}\right|^{p(x)} + \left|\frac{\nabla(u_\lambda\varphi)}{\lambda}\right|^{p(x)}\right) \mathrm{d}x \\ &\le \lambda^{-p^-_{B_{2r}}} \int_{B(x_0,2r)} \left(|u_\lambda\varphi|^{p(x)} + |\nabla(u_\lambda\varphi)|^{p(x)}\right) \mathrm{d}x. \end{aligned} \tag{4.20}$$

For the first term in the integral above we have by the Hölder inequality, Lemma 3.4, and the weak Harnack inequality that

$$\begin{aligned} \int_{B(x_0,2r)} |u_\lambda|^{p(x)}\,\mathrm{d}x &= Cr^n \fint_{B(x_0,2r)} |u_\lambda|^{p(x)-p^-_{B_{2r}}} |u_\lambda|^{p^-_{B_{2r}}}\,\mathrm{d}x \\ &\le Cr^n \left(\fint_{B(x_0,2r)} |u_\lambda|^{q'(p(x)-p^-_{B_{2r}})}\right)^{1/q'} \left(\fint_{B(x_0,2r)} |u_\lambda|^{qp^-_{B_{2r}}}\,\mathrm{d}x\right)^{1/q} \\ &\le Cr^n \left(1 + \|u\|_{L^{q's}(B_{4r})}^{q'(p^+_{B_{4r}} - p^-_{B_{4r}})}\right) \left(\fint_{B(x_0,2r)} |u_\lambda|^{qp^-_{B_{2r}}}\,\mathrm{d}x\right)^{1/q} \\ &\le Cr^n \lambda^{p^-_{B_{2r}} - q_0/q} \left(\fint_{B(x_0,2r)} |u_\lambda|^{q_0}\,\mathrm{d}x\right)^{1/q} \\ &\le Cr^n \lambda^{p^-_{B_{2r}} - q_0/q} \left(\inf_{B(x_0,r)} u + r\right)^{q_0/q}. \end{aligned} \tag{4.21}$$

For the second term, we get by the product rule

$$|\nabla(u_\lambda\varphi)|^{p(x)} \leq C\Big(|\varphi\nabla u_\lambda|^{p(x)} + |u_\lambda\nabla\varphi|^{p(x)}\Big). \tag{4.22}$$

Using Lemma 3.3 and estimating the average of $|u_\lambda|^{p(x)}$ as above we have

$$\begin{aligned}\int_{B(x_0,2r)} |u_\lambda\nabla\varphi|^{p(x)}\,\mathrm{d}x &\leq Cr^{n-p^-_{B_{2r}}}\left(\fint_{B(x_0,2r)} |u_\lambda|^{p(x)}\,\mathrm{d}x\right)\\ &\leq Cr^{n-p^-_{B_{2r}}}\lambda^{p^-_{B_{2r}}-q_0/q}\left(\inf_{B(x_0,r)} u+r\right)^{q_0/q}.\end{aligned} \tag{4.23}$$

We estimate the remaining term by using Theorem 4.4. To this end, choose a function $\eta \in C_0^\infty(B_{3r})$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0,2r)$, and $|\nabla\eta| \leq C/r$. Then

$$\begin{aligned}\int_{B(x_0,2r)} |\varphi\nabla u_\lambda|^{p(x)}\,\mathrm{d}x &\leq \int_{B(x_0,2r)} |\nabla u_\lambda|^{p(x)}\,\mathrm{d}x\\ &= \int_{B(x_0,2r)} |\nabla u_\lambda|^{p(x)}\eta^{p^+_{B_{4r}}}\,\mathrm{d}x\\ &\leq C\lambda^{1-\gamma}r^{n-p^-_{B_{2r}}}\left(\inf_{B_r} u+r\right)^{q(\gamma-1+p^-_{B_{2r}})}.\end{aligned} \tag{4.24}$$

Combining the above estimates we have

$$C_{p(\cdot)}(E_\lambda) \leq Cr^{n-p^-_{B_{2r}}}\left(\lambda^{-q_0/q}+\lambda^{1-\gamma-p^-_{B_{2r}}}\right)\left(\inf_{B_r} u+r\right)^{q_0/q}. \tag{4.25}$$

Now our choice of γ gives the claim. □

The previous result implies that the singularity set of a supersolution is of zero capacity.

Corollary 4.7. *For functions u satisfying the assumptions of Theorem 4.5,*

$$C_{p(\cdot)}(\{x \in \Omega : u(x) = \infty\}) = 0. \tag{4.26}$$

Proof. Fix a ball $B(x_0,r)$ as in Theorem 4.5 and let

$$\begin{aligned}E_i &= \{x \in B(x_0,r) : u(x) > i\}, \quad i = 1,2,\dots,\\ E &= \{x \in B(x_0,r) : u(x) = \infty\}.\end{aligned} \tag{4.27}$$

Since $E = \cap_i E_i$ and $E_1 \supset E_2 \supset \cdots$, we get by the monotonicity of the capacity and Theorem 4.5 that

$$C_{p(\cdot)}(E) \leq \lim_{i \to \infty} C_{p(\cdot)}(E_i) = 0. \tag{4.28}$$

Since Ω can be covered by a countable number of balls for which (4.28) holds, the subadditivity of the capacity implies the claim. □

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