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Research Article

Existence of Positive Solutions for Boundary Value Problems of Nonlinear Functional Difference Equation with p-Laplacian Operator

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The existence of positive solutions for boundary value problems of nonlinear functional difference equations with p-Laplacian operator is investigated. Sufficient conditions are obtained for the existence of at least one positive solution and two positive solutions.

1. Introduction

In recent years, boundary value problems of differential and difference equations have been studied widely and there are many excellent results (see Erbe and Wang [1], Grimm and Schmitt [2], Gustafson and Schmitt [3], Weng and Jiang [4], Weng and Tian [5], Wong [6], and Yang et al. [7]). Weng and Guo [8] considered two-point boundary value problem of a nonlinear functional difference equation with p-Laplacian operator

$$\begin{gathered}\Delta\Phi_p(\Delta x(t)) + r(t)f(x_t) = 0, \quad t \in [0,T], \\ x_0 = \varphi \in C^+, \qquad \Delta x(T+1) = 0,\end{gathered} \tag{1.1}$$

where $\Phi_p(u) = |u|^{p-2}u$, $p > 1$, $\phi(0) = 0$, $C^+ = \{\varphi \mid \varphi \in C,\ \varphi(k) \geq 0,\ k \in [-\tau, 0]\}$.

Ntouyas et al. [9] investigated the existence of solutions of a boundary value problem for functional differential equations

$$\begin{gathered}x''(t) = f(t, x_t, x'(t)), \quad t \in [0,T], \\ \alpha_0 x_0 - \alpha_1 x'(0) = \phi, \\ \beta_0 x(T) + \beta_1 x'(T) = A,\end{gathered} \tag{1.2}$$

where $f : [0,T] \times C_r \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\varphi \in C_r$, $A \in \mathbb{R}^n$, $C_r = C([-r,0],\mathbb{R}^n)$.

Let

$$\mathbb{R}^+ = \{x \mid x \in \mathbb{R},\ x \geq 0\},$$
$$[a,b] = \{a,\dots,b\}, \quad [a,b) = \{a,\dots,b-1\}, \quad [a,\infty) = \{a,a+1,\dots\} \tag{1.3}$$

for $a,b \in \mathbb{N}$ and $a < b$. For $\tau, T \in \mathbb{N}$ and $0 \leqslant \tau < T$, we define

$$\mathbb{C}_\tau = \{\varphi \mid \varphi : [-\tau,0] \longrightarrow \mathbb{R}\}, \qquad \mathbb{C}_\tau^+ = \{\varphi \in \mathbb{C}_\tau \mid \varphi(\vartheta) \geq 0,\ \vartheta \in [-\tau,0]\}. \tag{1.4}$$

Then $\mathbb{C}_\tau$ and $\mathbb{C}_\tau^+$ are both Banach spaces endowed with the max-norm

$$\|\varphi\|_\tau = \max_{k \in [-\tau,0]} |\varphi(k)|. \tag{1.5}$$

For any real function x defined on the interval $[-\tau,T]$ and any $t \in [0,T]$, we denote by x_t an element of $\mathbb{C}_\tau$ defined by $x_t(k) = x(t+k)$, $k \in [-\tau,0]$.

In this paper, we consider the following nonlinear difference boundary value problems:

$$\Delta\Phi_p(\Delta x(t)) + r(t) f(x(t),x_t) = 0, \quad t \in [1,T],$$
$$\alpha_0 x_0 - \alpha_1 \Delta x(0) = h, \quad t \in [-\tau,0], \tag{1.6}$$
$$\beta_0 x(T+1) + \beta_1 \Delta x(T+1) = A,$$

where $\Phi_p(u) = |u|^{p-2}u$, $p > 1$, $q > 1$ are positive constants satisfying $1/p + 1/q = 1$, $\Delta x(t) = x(t+1) - x(t)$, $f : \mathbb{R} \times \mathbb{C}_\tau \to \mathbb{R}$ is a continuous function, $h \in \mathbb{C}_\tau^+$ and $h(t) \geq h(0) \geq 0$, $t \in [-\tau,0]$, $A \in \mathbb{R}^+$, α_0, α_1, $\beta_0 < \beta_1$ are nonnegative real constants such that

$$\alpha_0\beta_0 T + \alpha_0\beta_1 + \alpha_1\beta_0 \neq 0. \tag{1.7}$$

At this point, it is necessary to make some remarks on the first boundary condition in (1.6). This condition is a generalization of the classical condition

$$\alpha_0 x(0) - \alpha_1 \Delta x(0) = c \tag{1.8}$$

from ordinary difference equations. Here this condition connects the history x_0 with the single value $\Delta x(0)$. This is suggested by the well posedness of the BVP (1.6), since the function f depends on the terms x_t and $x(t)$.

The case $\alpha_0 = 0$ must be treated separately, since in this case, the BVP (1.6) is not well posed. Indeed, if $\alpha_0 = 0$, the first boundary condition yields

$$-\alpha_1 \Delta x(0) = h, \tag{1.9}$$

where now h must be a constant in $\mathbb{R}$ and $\alpha_1 \neq 0$, because of (1.7). In this case, we consider the next boundary conditions instead of the two boundary conditions in (1.6):

$$\begin{gathered} x_0 = x(0), \\ -\alpha_1 \Delta x(0) = h, \\ \beta_0 x(T) + \beta_1 \Delta x(T+1) = A. \end{gathered} \tag{1.10}$$

As usual, a sequence $\{u(-\tau),\dots,u(T+2)\}$ is said to be a positive solution of BVP (1.6) if it satisfies (1.6) with $u(k) > 0$ for $k \in \{1,\dots,T+1\}$.

We will need the following well-known lemma (See Guo [10]).

Lemma 1.1. *Assume that* $\mathbb{X}$ *is a Banach space and* $K \subset \mathbb{X}$ *is a cone in* $\mathbb{X}$. Ω_1, Ω_2 *are two open sets in* $\mathbb{X}$ *with* $0 \in \overline{\Omega}_1 \subset \Omega_2$. *Furthermore, assume that* $\Psi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ *is a completely continuous operator and satisfies one of the following two conditions:*

(1) $\|\Psi x\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$, $\|\Psi x\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_2$;

(2) $\|\Psi x\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$, $\|\Psi x\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_1$.

Then Ψ *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main results

Suppose that $x(t)$ is a solution of BVP (1.6).

If $h(0) = 0$, then

(i) if $\alpha_0 \neq 0$, $\beta_1 \neq 0$,

$$x(t) = \begin{cases} \displaystyle\sum_{m=0}^{t-1} \Phi_q\left(\sum_{n=m}^{T} r(n) f(x(n), x_n)\right) & \text{if } t \in [1, T+1], \\ \dfrac{\alpha_1}{\alpha_0 + \alpha_1} x(1) & \text{if } t = 0, \\ \dfrac{\alpha_1 \Delta x(0) + h(t)}{\alpha_0} & \text{if } t \in [-\tau, 0), \\ \dfrac{1}{\beta_1} A + \dfrac{\beta_1 - \beta_0}{\beta_1} x(T+1) & \text{if } t = T+2; \end{cases} \tag{2.1}$$

(ii) if $\alpha_0 \neq 0$, $\beta_1 = 0$,

$$x(t) = \begin{cases} \displaystyle\sum_{m=0}^{t-1} \Phi_q\left(\sum_{n=m}^{T} r(n) f(x(n), x_n)\right) & \text{if } t \in [1, T], \\ \dfrac{\alpha_1}{\alpha_0 + \alpha_1} x(1) & \text{if } t = 0, \\ \dfrac{\alpha_1 \Delta x(0) + h(t)}{\alpha_0} & \text{if } t \in [-\tau, 0), \\ \dfrac{1}{\beta_0} A & \text{if } t = T+1; \end{cases} \tag{2.2}$$

(iii) if $\alpha_0 = 0, \beta_1 \neq 0$,

$$x(t) = \begin{cases} \sum_{m=0}^{t-1} \Phi_q\left(\sum_{n=m}^{T} r(n) f(x(n), x_n)\right) & \text{if } t \in [1, T+1], \\ x(1) + \frac{1}{\alpha_1} h & \text{if } t \in [-\tau, 0], \\ \frac{1}{\beta_1} A + \frac{\beta_1 - \beta_0}{\beta_1} x(T+1) & \text{if } t = T+2; \end{cases} \tag{2.3}$$

(iv) if $\alpha_0 = 0, \beta_1 = 0$,

$$x(t) = \begin{cases} \sum_{m=0}^{t-1} \Phi_q\left(\sum_{n=m}^{T} r(n) f(x(n), x_n)\right) & \text{if } t \in [1, T], \\ x(1) + \frac{1}{\alpha_1} h & \text{if } t \in [-\tau, 0], \\ \frac{1}{\beta_0} A & \text{if } t = T+2. \end{cases} \tag{2.4}$$

We only prove (i), the proofs of (ii)–(iv) are similar and we will omit them.

Assume that $f \equiv 0$, then BVP (1.6) may be rewritten as

$$\begin{gathered} \Delta\Phi_p(\Delta x(t)) = 0, \quad t \in [1, T], \\ \alpha_0 x_0 - \alpha_1 \Delta x(0) = h, \quad t \in [-\tau, 0], \\ \beta_0 x(T+1) + \beta_1 \Delta x(T+1) = A. \end{gathered} \tag{2.5}$$

Assume that $\overline{x}(t)$ is a solution of system (2.5), then

$$\overline{x}(t) = \begin{cases} 0 & \text{if } t \in [0, T+1], \\ \frac{1}{\alpha_0} h(t) & \text{if } t \in [-\tau, 0), \\ \frac{1}{\beta_1} A & \text{if } t = T+2. \end{cases} \tag{2.6}$$

Assume that $x(t)$ is a solution of BVP (1.6). Let $u(t) = x(t) - \overline{x}(t)$. Then for $t \in [1, T+1]$, we have $u(t) \equiv x(t)$, and

$$u(t) = \begin{cases} \sum_{m=0}^{t-1} \Phi_q\left(\sum_{n=m}^{T} r(n) f(u(n) + \overline{x}(n), u_n + \overline{x}_n)\right) & \text{if } t \in [1, T+1], \\ \frac{\alpha_1}{\alpha_0 + \alpha_1} u(1) & \text{if } t \in [-\tau, 0], \\ \frac{\beta_1 - \beta_0}{\beta_1} u(T+1) & \text{if } t = T+2. \end{cases} \tag{2.7}$$

Let

$$\begin{aligned}\|u\| &= \max_{t\in[-\tau,T+2]} |u(t)|,\\ E &= \{y \mid y:[-\tau,T+2] \longrightarrow \mathbb{R}\},\\ K &= \Big\{y \mid y\in E: y(t)=\frac{\alpha_1}{\alpha_0+\alpha_1}y(1) \text{ for } t\in[-\tau,0],\\ &\qquad y(t) \geq \frac{\beta_1-\beta_0}{\beta_1(T+1)}\|y\| \text{ for } t\in[1,T+2]\Big\}.\end{aligned} \tag{2.8}$$

Then E is a Banach space endowed with norm $\|\cdot\|$ and K is a cone in E.

For $y\in K$, we have $y(t)=(\alpha_1/(\alpha_0+\alpha_1))y(1)$ for $t\in[-\tau,0]$. So,

$$\|y\| = \max_{t\in[-\tau,T+2]} |y(t)| = \max_{t\in[1,T+2]} |y(t)|. \tag{2.9}$$

Define an operator $\Psi: K\to E$,

$$\Psi y(t) = \begin{cases} \sum_{m=0}^{t-1}\Phi_q\Big(\sum_{n=m}^{T} r(n)f\big(y(n)+\overline{x}(n), y_n+\overline{x}_n\big)\Big) & \text{if } t\in[1,T+1],\\ \dfrac{\alpha_1}{\alpha_0+\alpha_1}\Psi y(1) & \text{if } t\in[-\tau,0],\\ \dfrac{\beta_1-\beta_0}{\beta_1}\Psi y(T+1) & \text{if } t=T+2.\end{cases} \tag{2.10}$$

Then we may transform our existence problem of BVP (1.6) into a fixed point problem of the operator (2.10).

By (2.10), we have

$$\begin{aligned}\|\Psi y\| = (\Psi y)(T+1) &= \sum_{m=0}^{T}\Phi_q\Bigg(\sum_{n=m}^{T} r(n)f\big(y(n)+\overline{x}(n), y_n+\overline{x}_n\big)\Bigg)\\ &\leqslant (T+1)\Phi_q\Bigg(\sum_{n=0}^{T} r(n)f\big(y(n)+\overline{x}(n), y_n+\overline{x}_n\big)\Bigg).\end{aligned} \tag{2.11}$$

Lemma 2.1. $\Psi(K)\subset K$.

Proof. If $t\in[-\tau,0]$, then $\Psi y(t)=(\alpha_1/(\alpha_0+\alpha_1))\Psi y(1)$.

If $t\in[1,T+1]$, then by (2.10) and (2.11), we have

$$\begin{aligned}\Psi y(t) &\geq \Phi_q\Bigg(\sum_{n=0}^{T} r(n)f\big(y(n)+\overline{x}(n), y_n+\overline{x}_n\big)\Bigg)\\ &\geq \frac{1}{T+1}\|\Psi y\| \geq \frac{\beta_1-\beta_0}{\beta_1(T+1)}\|\Psi y\|.\end{aligned} \tag{2.12}$$

If $t = T+2$, then

$$\Psi y(T+2) = \frac{\beta_1 - \beta_0}{\beta_1}\Psi y(T+1) \geq \frac{\beta_1 - \beta_0}{\beta_1(T+1)}\|\Psi y\|. \tag{2.13}$$

So, by the definition of K, we have $\Psi(K) \subset K$. □

LEMMA 2.2. *$\Psi : K \to K$ is completely continuous.*

Proof. Notice that $y_n + \overline{x}_n = (y(n-\tau)+\overline{x}(n-\tau),\dots,y(n)+\overline{x}(n))$. So $f : \mathbb{R}^{\tau+2} \to \mathbb{R}$. Then by [10, Theorem 2.6, page 33], f is completely continuous. Hence, Ψ is completely continuous. □

In this paper, we always assume that
(H$_1$) $\sum_{n=\tau+1}^{T} r(n) > 0$,
(H$_2$) $f : \mathbb{R}^+ \times \mathbb{C}_\tau^+ \to \mathbb{R}^+$
hold.

Then we have the following main results.

THEOREM 2.3. *Assume that (H$_1$), (H$_2$) hold. Then BVP (1.6) has at least one positive solution if the following conditions are satisfied:*
(H$_3$) there exist $\varrho_1 > 0$, such that if $\|\varphi\| \leqslant \varrho_1 + \varrho_0$, then

$$f(\varphi(n),\varphi_n) \leqslant (b\varrho_1)^{p-1}; \tag{2.14}$$

(H$_4$) there exists $\varrho_2 > \varrho_1 + 2$, such that if $\|\varphi\| \geq \varrho_2$, then

$$f(\varphi(n),\varphi_n) \geq (B\varrho_2)^{p-1} \tag{2.15}$$

or
(H$_5$) there exists $0 < r_1 < \varrho_1$, such that if $\|\varphi\| \geq r_1$, then

$$f(\varphi(n),\varphi_n) \geq (Br_1)^{p-1}; \tag{2.16}$$

(H$_6$) there exists $R_1 > \varrho_2$, such that if $\|\varphi\| \leqslant R_1 + \varrho_0$, then

$$f(\varphi(n),\varphi_n) \leqslant (BR_1)^{p-1}, \tag{2.17}$$

where

$$\varrho_0 = \frac{\|h\|_\tau}{\alpha_0}, \qquad b = \frac{1}{(T+1)\Phi_q\left(\sum_{n=0}^{T} r(n)\right)}, \qquad B = \frac{1}{\Phi_q\left(\sum_{n=0}^{T} r(n)\right)}. \tag{2.18}$$

THEOREM 2.4. *Assume that (H$_1$), (H$_2$) hold. Then BVP (1.6) has at least one positive solution if one of the following conditions is satisfied:*
(H$_7$) $\limsup_{\|\varphi_n\|_\tau \to 0}(f(\varphi(n),\varphi_n)/\|\varphi_n\|_\tau^{p-1}) < m^{p-1}$, $\liminf_{\|\varphi_n\|_\tau \to \infty}(f(\varphi(n),\varphi_n)/\|\varphi_n\|_\tau^{p-1}) > M^{p-1}$, $h(\vartheta) = 0, \vartheta \in [-\tau, 0]$;
(H$_8$) $\liminf_{\|\varphi_n\|_\tau \to 0}(f(\varphi(n),\varphi_n)/\|\varphi_n\|_\tau^{p-1}) > M^{p-1}$, $\limsup_{\|\varphi_n\|_\tau \to \infty}(f(\varphi(n),\varphi_n)/\|\varphi_n\|_\tau^{p-1}) < m^{p-1}$,

where

$$m=\frac{1}{(T+1)\Phi_q\left(\sum_{n=0}^{T}r(n)\right)},\qquad M=\frac{\beta_1(T+1)}{(\beta_1-\beta_0)\Phi_q\left(\sum_{n=\tau+1}^{T}r(n)\right)}. \tag{2.19}$$

THEOREM 2.5. *Assume that* (H_1), (H_2) *hold. Then BVP (1.6) has at least two positive solutions if the conditions* (H_3)–(H_5) *or* (H_3), (H_4), *and* (H_6) *hold.*

THEOREM 2.6. *Assume that* (H_1), (H_2) *hold. Then BVP (1.6) has at least three positive solutions if the conditions* (H_3)–(H_6) *hold.*

3. Proofs of the theorems

Proof of Theorem 2.3. Assume that (H_3) and (H_4) hold.

For every $y\in K\cap\partial\Omega_{\varrho_1}$, $\|y\|=\varrho_1$, $\|y+\overline{x}\|\leqslant\|y\|+\|\overline{x}\|\leqslant\varrho_1+\varrho_0$, then by (2.10) and ($H_3$),

$$\begin{aligned}\|\Psi y\|&=\sum_{m=0}^{T}\Phi_q\left(\sum_{n=m}^{T}r(n)f\left(y(n)+\overline{x}(n),y_n+\overline{x}_n\right)\right)\\&\leqslant\sum_{m=0}^{T}\Phi_q\left(\sum_{n=m}^{T}r(n)(b\varrho_1)^{p-1}\right)\leqslant b\varrho_1(T+1)\Phi_q\left(\sum_{n=0}^{T}r(n)\right)=\varrho_1=\|y\|.\end{aligned} \tag{3.1}$$

For every $y\in K\cap\partial\Omega_{\varrho_2}$, $\|y\|=\varrho_2$, $\|y+\overline{x}\|=\max\{\varrho_2,\varrho_0\}\geq\varrho_2$, then by (2.10) and ($H_4$),

$$\begin{aligned}\|\Psi y\|&=\sum_{m=0}^{T}\Phi_q\left(\sum_{n=m}^{T}r(n)f\left(y(n)+\overline{x}(n),y_n+\overline{x}_n\right)\right)\\&\geq\sum_{m=0}^{T}\Phi_q\left(\sum_{n=m}^{T}r(n)(B\varrho_2)^{p-1}\right)\\&\geq B\varrho_2\Phi_q\left(\sum_{n=0}^{T}r(n)\right)=\varrho_2=\|y\|.\end{aligned} \tag{3.2}$$

So by (3.1), (3.2) and Lemma 1.1, there exists one positive fixed point y_1 of operator Ψ with $y_1\in K\cap(\overline{\Omega}_{\varrho_2}\setminus\Omega_{\varrho_1})$.

Assume that (H_5) and (H_6) hold. Similar to the above proof, we have that for every $y\in K\cap\partial\Omega_{r_1}$,

$$\|\Psi y\|\geq\|y\|, \tag{3.3}$$

and for every $y\in K\cap\partial\Omega_{R_1}$,

$$\|\Psi y\|\leqslant\|y\|. \tag{3.4}$$

So by (3.3) and (3.4), there exists one positive fixed point y_2 of operator Ψ with $y_2\in K\cap(\overline{\Omega}_{R_1}\setminus\Omega_{r_1})$. Consequently, $x_1=y_1+\overline{x}$ or $x_2=y_2+\overline{x}$ is a positive solution of BVP (1.6). □

Proof of Theorem 2.4. Assume that (H_7) holds. By $h(\vartheta) = 0$, $\vartheta \in [-\tau, 0]$, we have $\overline{x}(n) = 0$ for $n \in [-\tau, T+1]$.

From

$$\limsup_{\|\varphi_n\|_\tau \to 0} \frac{f(\varphi(n), \varphi_n)}{\|\varphi_n\|_\tau^{p-1}} < m^{p-1}, \tag{3.5}$$

there exists a constant $\varrho_1 > 0$, such that for $\|\varphi_n\|_\tau < \varrho_1$,

$$f(\varphi(n), \varphi_n) \leqslant (m\|\varphi_n\|_\tau)^{p-1}. \tag{3.6}$$

Let $\Omega_\varrho = \{y \in K \mid \|y\| < \varrho\}$.

For every $y \in K \cap \partial\Omega_{\varrho_1}$, $\|y_n\|_\tau \leqslant \|y\| \leqslant \varrho_1$, then by (2.10) and (3.6),

$$\begin{aligned} \|\Psi y\| &= \sum_{m=0}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) f(y(n), y_n)\right) \leqslant \sum_{m=0}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) m^{p-1} \|y_n\|_\tau^{p-1}\right) \\ &\leqslant \sum_{m=0}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) m^{p-1} \|y\|^{p-1}\right) \leqslant m(T+1)\|y\| \Phi_q\left(\sum_{n=0}^{T} r(n)\right) = \|y\|. \end{aligned} \tag{3.7}$$

Furthermore, by

$$\liminf_{\|\varphi_n\|_\tau \to \infty} \frac{f(\varphi(n), \varphi_n)}{\|\varphi_n\|_\tau^{p-1}} > M^{p-1}, \tag{3.8}$$

there exists a positive constant $\varrho_2 > \varrho_1$, such that for $\|\varphi_n\|_\tau \geq ((\beta_1 - \beta_0)/\beta_1(T+1))\varrho_2$,

$$f(\varphi(n), \varphi_n) \geq (M\|\varphi_n\|_\tau)^{p-1}. \tag{3.9}$$

For $y \in K$, we have $y(t) \geq ((\beta_1 - \beta_0)/\beta_1(T+1))\|y\|$ for $t \in [1, T+2]$. So, if $n \in [\tau+1, T+1]$, then

$$\|y_n\|_\tau \geq \frac{\beta_1 - \beta_0}{\beta_1(T+1)}\|y\| = \frac{\beta_1 - \beta_0}{\beta_1(T+1)}\varrho_2. \tag{3.10}$$

For $y \in K \cap \partial\Omega_{\varrho_2}$, by (2.10) and (3.9),

$$\begin{aligned} \|\Psi y\| &= \sum_{m=0}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) f(y(n), y_n)\right) \geq \sum_{m=\tau+1}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) f(y(n), y_n)\right) \\ &\geq \sum_{m=\tau+1}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) (M\|y_n\|_\tau)^{p-1}\right) \geq \Phi_q\left(\sum_{n=\tau+1}^{T} r(n)\left(\frac{M(\beta_1 - \beta_0)}{\beta_1(T+1)}\|y\|\right)^{p-1}\right) \\ &= \frac{M(\beta_1 - \beta_0)}{\beta_1(T+1)}\|y\| \Phi_q\left(\sum_{n=\tau+1}^{T} r(n)\right) = \|y\|. \end{aligned} \tag{3.11}$$

So, by (3.7), (3.11), and Lemma 1.1, there exists a positive fixed point y_3 of operator Ψ with $y_3 \in K \cap (\overline{\Omega}_{\varrho_2} \setminus \Omega_{\varrho_1})$, such that

$$0 < \varrho_1 \leqslant \|y\| \leqslant \varrho_2. \tag{3.12}$$

Assume that (H_8) holds. From

$$\liminf_{\|\varphi_n\|_\tau \to 0} \frac{f(\varphi(n), \varphi_n)}{\|\varphi_n\|_\tau^{p-1}} > M^{p-1}, \tag{3.13}$$

there exists a constant $\varrho_1 > 0$, such that for $\|\varphi_n\|_\tau < \varrho_1$,

$$f(\varphi(n), \varphi_n) \geq (M\|\varphi_n\|_\tau)^{p-1}. \tag{3.14}$$

For every $y \in K \cap \partial\Omega_{\varrho_1}$, $\|y_n\|_\tau \leqslant \|y\| \leqslant \varrho_1$, then by (2.10), (3.10), and (3.14),

$$\begin{aligned}
\|\Psi y\| &= \sum_{m=0}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) f(y(n)+\overline{x}(n), y_n+\overline{x}_n)\right) \geq \sum_{m=\tau+1}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) f(y(n), y_n)\right) \\
&\geq \sum_{m=\tau+1}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) (M\|y\|_\tau)^{p-1}\right) \geq \sum_{m=\tau+1}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) \left(\frac{M(\beta_1-\beta_0)}{\beta_1(T+1)}\|y\|\right)^{p-1}\right) \\
&\geq \frac{M(\beta_1-\beta_0)}{\beta_1(T+1)}\|y\| \Phi_q\left(\sum_{n=\tau+1}^{T} r(n)\right) = \|y\|.
\end{aligned} \tag{3.15}$$

Furthermore, by

$$\limsup_{\|\varphi_n\|_\tau \to \infty} \frac{f(\varphi(n), \varphi_n)}{\|\varphi_n\|_\tau^{p-1}} < m^{p-1}, \tag{3.16}$$

there exists a positive constant $N > \max\{\varrho_1, \|h\|_\tau\}$, such that for $\|\varphi_n\|_\tau \geq N$,

$$f(\varphi(n), \varphi_n) \leqslant (m\|\varphi_n\|_\tau)^{p-1}. \tag{3.17}$$

Let

$$\begin{aligned}
\varrho_2 &= N + 2\frac{\|h\|_\tau}{\alpha_0} \\
&\quad + m^{-1} \max\left\{m\left(\varrho_2 + \frac{\|h\|_\tau}{\alpha_0}\right), \Phi_q\left(\max\left\{f(\varphi(n), \varphi_n) : \|\varphi_n\|_\tau \leqslant \varrho_2 + \frac{\|h\|_\tau}{\alpha_0}\right\}\right)\right\}.
\end{aligned} \tag{3.18}$$

For $y \in K \cap \partial\Omega_{\varrho_2}$, by (2.10), (3.17),

$$
\begin{aligned}
\|\Psi y\| &= \sum_{m=0}^{T} \Phi_q\left(\sum_{n=m}^{T} r(n) f(y(n)+\overline{x}(n), y_n+\overline{x}_n)\right) \\
&\leqslant (T+1)\Phi_q\left(\sum_{n=0}^{T} r(n) f(y(n)+\overline{x}(n), y_n+\overline{x}_n)\right) \\
&\leqslant (T+1)\Phi_q\left[\left(\sum_{\|y_n\|_\tau > N+\|h\|_\tau/\alpha_0} + \sum_{\|y_n\|_\tau \leqslant N+\|h\|_\tau/\alpha_0}\right) r(n) f(y(n)+\overline{x}(n), y_n+\overline{x}_n)\right] \\
&\leqslant (T+1)\Phi_q\left(\sum_{n=0}^{T} r(n)\right) \\
&\quad \times \max\left\{ m\left(\varrho_2 + \frac{\|h\|_\tau}{\alpha_0}\right), \Phi_q\left(\max\left\{ f(\varphi(n),\varphi_n) : \|\varphi_n\|_\tau \leqslant \varrho_2 + \frac{\|h\|_\tau}{\alpha_0}\right\}\right)\right\} \\
&\leqslant \varrho_2 = \|y\|.
\end{aligned}
\tag{3.19}
$$

So, by (3.15), (3.19), and Lemma 1.1, there exists a positive fixed point y_4 of operator Ψ with $y_4 \in K \cap (\overline{\Omega}_{\varrho_2} \setminus \Omega_{\varrho_1})$, such that

$$
0 < \varrho_1 \leqslant \|y\| \leqslant \varrho_2. \tag{3.20}
$$

Hence, $x_3(t) = y_3(t) + \overline{x}(t)$ or $x_4(t) = y_4(t) + \overline{x}(t)$ is a positive solution of BVP (1.6).

If $h(0) \neq 0$, then by the transformation

$$
z = x - \frac{h(0)}{\alpha_0}, \tag{3.21}
$$

the BVP (1.6) is reduced to the following BVP:

$$
\begin{gathered}
\Delta\Phi_p(\Delta z(t)) + r(t) f\left(z(t) + \frac{h(0)}{\alpha_0}, z_t + \frac{h(0)}{\alpha_0}\right) = 0, \quad t \in [1,T] \\
\alpha_0 z_0 - \alpha_1 \Delta z(0) = \overline{h} = h - h(0), \quad t \in [-\tau, 0] \\
\beta_0 x(T+1) + \beta_1 \Delta x(T+1) = A + \frac{\beta_0 h(0)}{\alpha_0},
\end{gathered}
\tag{3.22}
$$

where obviously $\overline{h}(0) = 0$.

Similar to the above proof, we can prove that BVP (3.22) has at least one positive solution. Consequently, BVP (1.6) has at least one positive solution. □

Proof of Theorem 2.5. By (3.1)–(3.3) and Lemma 1.1, or by (3.1), (3.2), (3.4), and Lemma 1.1, it is easy to see that BVP (1.6) has two positive solutions. □

Proof of Theorem 2.6. By (3.1)–(3.4) and Lemma 1.1, it is easy to see that BVP (1.6) has three positive solutions. □

4. An example

Consider BVP

$$\begin{gathered}\Delta\Phi_{3/2}(\Delta x(t)) + tf(x(t),x_t) = 0, \quad t \in [1,4], \\ x_0 - \Delta x(0) = h, \quad t \in [-2,0], \\ \Delta x(5) = 1,\end{gathered} \tag{4.1}$$

where $h(t) = -t$, for $(\varphi(t),\varphi_t) \in \mathbb{R}^+ \times \mathbb{C}_\tau^+$,

$$f(\varphi(t),\varphi_t) = \begin{cases} 10^{-2}, & 0 < s \leqslant 3, \\ \dfrac{44\times 10^{-4}}{49}(s-3)^2 + 10^{-2}, & 3 < s \leqslant 8, \\ 7956\times 10^{-4}(s-8), & 8 < s \leqslant 9, \\ 10^{-2}[100 - 19(s-52)^2], & 9 < s \leqslant 52, \\ 1, & 52 < s, \end{cases} \tag{4.2}$$

where $s = \|\varphi\|$.

In BVP (4.1), $p = 3/2$, $q = 3$, $T = 4$, $\tau = 2$, $r(t) = t$, $\alpha_0 = 1$, $\alpha_1 = 1$, $\beta_0 = 0$, $\beta_1 = 1$, $A = 1$, $\varrho_0 = 2$, $b = 0.02$, $B = 0.1$.

Let $r_1 = 1$, $\varrho_1 = 6$, $\varrho_2 = 9$, $R_1 = 50$. Then by simple computation, we can show that

$$f(\varphi(t),\varphi_t) \begin{cases} \geq (Br_1)^{p-1} = 0.01 & \text{if } s \geq r_1 = 1, \\ \leqslant (b\varrho_1)^{p-1} = 1.44\times 10^{-2} & \text{if } s \leqslant \varrho_1 + \varrho_0 = 8, \\ \geq (B\varrho_2)^{p-1} = 0.81 & \text{if } s \geq \varrho_2 = 9, \\ \leqslant (Br_1)^{p-1} = 1 & \text{if } s \leqslant R_1 + \varrho_0 = 52, \end{cases}$$

$$\overline{x}(t) = \begin{cases} 0 & \text{if } t \in [0, T+1], \\ -t & \text{if } t \in [-\tau, 0), \\ 1 & \text{if } t = T+2. \end{cases} \tag{4.3}$$

By Theorem 2.6, BVP (4.1) has three positive solutions

$$x_1 = y_1 + \overline{x}, \qquad x_2 = y_2 + \overline{x}, \qquad x_3 = y_3 + \overline{x}, \tag{4.4}$$

with

$$y_1 \in K \cap (\overline{\Omega}_{\varrho_1} \setminus \Omega_{r_1}), \qquad y_2 \in K \cap (\overline{\Omega}_{\varrho_2} \setminus \Omega_{\varrho_1}), \qquad y_3 \in K \cap (\overline{\Omega}_{R_1} \setminus \Omega_{\varrho_2}). \tag{4.5}$$

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