EXISTENCE AND MULTIPLICITY OF WEAK SOLUTIONS FOR A CLASS OF DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

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Received 11 January 2005; Revised 4 July 2005; Accepted 17 July 2005

The goal of this paper is to study the existence and the multiplicity of non-trivial weak solutions for some degenerate nonlinear elliptic equations on the whole space $\mathbf{R}^N$. The solutions will be obtained in a subspace of the Sobolev space $W^{1,p}(\mathbf{R}^N)$. The proofs rely essentially on the Mountain Pass theorem and on Ekeland's Variational principle.

1. Introduction

The goal of this paper is to study a nonlinear elliptic equation in which the divergence form operator $-\operatorname{div}(a(x,\nabla u))$ is involved. Such operators appear in many nonlinear diffusion problems, in particular in the mathematical modeling of non-Newtonian fluids (see [5] for a discussion of some physical background). Particularly, the p-Laplacian operator $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a special case of the operator $-\operatorname{div}(a(x,\nabla u))$. Problems involving the p-Laplacian operator have been intensively studied in the last decades. We just remember the work on that topic of João Marcos B. do Ó [7], Pflüger [12], Rădulescu and Smets [14] and the references therein. In the case of more general types of operators we point out the papers of João Marcos B. do Ó [6] and Nápoli and Mariani [4]. On the other hand, when the operator $-\operatorname{div}(a(x,\nabla u))$ is of degenerate type we refer to Cîrstea and Rădulescu [15] and Motreanu and Rădulescu [11].

In this paper we study the existence and multiplicity of non-trivial weak solutions to equations of the type

$$-\operatorname{div}\left(a(x,\nabla u)\right) = \mathcal{F}(x,u), \quad x \in \mathbf{R}^N, \tag{1.1}$$

where the operator $\operatorname{div}(a(x,\nabla u))$ is nonlinear (and can be also degenerate), $N \geq 3$ and function $\mathcal{F}(x,u)$ satisfies several hypotheses. Our goal is to show how variational techniques based on the Mountain Pass theorem (see Ambrosetti and Rabinowitz [2]) and Ekeland's Variational principle (see Ekeland [8]) can be used in order to get existence of

Hindawi Publishing Corporation
Boundary Value Problems
Volume 2006, Article ID 41295, Pages 1–17
DOI 10.1155/BVP/2006/41295

one or two solutions for equations of type (1.1). Results regarding the multiplicity of solutions have been originally proven by Tarantello [16], but in the case of linear equations and in a different framework. More precisely, Tarantello proved that the equation

$$-\Delta u = |u|^{4/(N-2)}u + \Gamma(x) \tag{1.2}$$

has at least two distinct solutions, in a bounded domain of $\mathbf{R}^N$ ($N \geq 3$), provided that $\Gamma \not\equiv 0$ is sufficiently "small" in a suitable sense.

2. Main results

The starting point of our discussion is the equation

$$-\Delta v + b(x)v = f(x,v) \quad x \in \mathbf{R}^N \tag{2.1}$$

studied by Rabinowitz in [13]. Assuming that function $f(x,v)$ is subcritical and satisfies a condition of the Ambrosetti-Rabinowitz type (see [2]) and function $b(x)$ is sufficiently smooth and unbounded at infinity, it is showed in [13] that problem (2.1) has a nontrivial weak solution in the classical Sobolev space $W^{1,2}(\mathbf{R}^N)$.

In the case when $b(x)$ is continuous and nonnegative and $f(x,v) = h(x)v^\alpha + v^\beta$ is such that $h : \mathbf{R}^N \to \mathbf{R}$ is some integrable function and $1 < \alpha < 2 < \beta < (N+2)/(N-2)$, $N \geq 3$, Gonçalves and Miyagaki proved in [9] that problem (2.1) has at least two nonnegative solutions in a subspace of $W^{1,2}(\mathbf{R}^N)$. In a similar framework, when $f(x,v) = \lambda v^\alpha + v^{2^\star - 1}$ with $0 < \alpha < 1$ and $2^\star = (2N)/(N-2)$, $N \geq 3$ it is shown in [1] that problem (2.1) has a nonnegative solution for λ positive and small enough. Furthermore, in [1] it is also proved that in the case $N \geq 4$ and $\alpha = 1$ problem (2.1) has a nonnegative solution provided that λ is positive and small enough. For more information and connections on (2.1) the reader may consult the references in [9].

In this paper our aim is to study the problem

$$-\operatorname{div}\left(a(x, \nabla u)\right) + b(x)\,|\,u\,|^{p-2}u = f(x,u), \quad x \in \mathbf{R}^N, \tag{2.2}$$

where $N \geq 3$ and $2 \leq p < N$.

We point out the fact that in the case when $a(x,\nabla u) = |x|^\alpha \nabla u$, $\alpha \in (0,2)$ and $p = 2$ problem (2.2) was studied by Mihăilescu and Rădulescu in [10]. In that paper the authors present the connections between such equations and some Schrödinger equations with Hardy potential and show that (2.2) has a nontrivial weak solution. A discussion of some physical applications for equations of type (2.2) and a list of papers devoted with the study of such problems is also included in [10].

In the following we describe the framework in which we will study (2.2).

Consider $a : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$, $a = a(x,\xi)$, is the continuous derivative with respect to ξ of the continuous function $A : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$, $A = A(x,\xi)$, that is, $a(x,\xi) = (d/d\xi)A(x,\xi)$.

Suppose that a and A satisfy the hypotheses below:

(A1) $A(x,0)=0$ for all $x\in\mathbf{R}^N$;

(A2) $|a(x,\xi)|\leq c_1(\theta(x)+|\xi|^{p-1})$, for all $x,\xi\in\mathbf{R}^N$, with c_1 a positive constant and $\theta:\mathbf{R}^N\to\mathbf{R}$ is a function such that $\theta(x)\geq 0$ for all $x\in\mathbf{R}^N$ and $\theta\in L^\infty(\mathbf{R}^N)\cap L^{p/(p-1)}(\mathbf{R}^N)$;

(A3) there exists $k>0$ such that

$$A\left(x,\frac{\xi+\psi}{2}\right)\leq\frac{1}{2}A(x,\xi)+\frac{1}{2}A(x,\psi)-k|\xi-\psi|^p \tag{2.3}$$

for all $x,\xi,\psi\in\mathbf{R}^N$, that is, $A(x,\cdot)$ is p-uniformly convex;

(A4) $0\leq a(x,\xi)\cdot\xi\leq pA(x,\xi)$, for all $x,\xi\in\mathbf{R}^N$;

(A5) there exists a constant $\Lambda>0$ such that

$$A(x,\xi)\geq\Lambda|\xi|^p, \tag{2.4}$$

for all $x,\xi\in\mathbf{R}^N$.

Examples. (1) $A(x,\xi)=(1/p)|\xi|^p$, $a(x,\xi)=|\xi|^{p-2}\xi$, with $p\geq 2$ and we get the p-Laplacian operator

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{2.5}$$

(2) $A(x,\xi)=(1/p)|\xi|^p+\theta(x)[(1+|\xi|^2)^{1/2}-1]$, $a(x,\xi)=|\xi|^{p-2}\xi+\theta(x)(\xi/(1+|\xi|^2)^{1/2})$, with $p\geq 2$ and θ a function which verifies the conditions from (A2). We get the operator

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\operatorname{div}\left(\theta(x)\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) \tag{2.6}$$

which can be regarded as the sum between the p-Laplacian operator and a degenerate form of the mean curvature operator.

(3) $A(x,\xi)=(1/p)[(\theta(x)^{2/(p-1)}+|\xi|^2)^{p/2}-\theta(x)^{p/(p-1)}]$, $a(x,\xi)=(\theta(x)^{2/(p-1)}+|\xi|^2)^{(p-2)/2}\xi$, with $p\geq 2$ and θ a function which verifies the conditions from (A2). We get the operator

$$\operatorname{div}\left(\left(\theta(x)^{2/(p-1)}+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right) \tag{2.7}$$

which is a variant of the generalized mean curvature operator, $\operatorname{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u)$.

Assume that function $b:\mathbf{R}^N\to\mathbf{R}$ is continuous and verifies the hypotheses:

(B) There exists a positive constant $b_0>0$ such that

$$b(x)\geq b_0>0, \tag{2.8}$$

for all $x\in\mathbf{R}^N$.

In a first instance we assume that function $f:\mathbf{R}^N\times\mathbf{R}\to\mathbf{R}$ satisfies the hypotheses:

(F1) $f\in C^1(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$, $f=f(x,z)$ and $f(x,0)=0$ for all $x\in\mathbf{R}^N$;

(F2) there exist two functions $\tau_1, \tau_2 : \mathbf{R}^N \to \mathbf{R}$, $\tau_1(x), \tau_2(x) \geq 0$ for a.e. $x \in \mathbf{R}^N$ and two constants $r, s \in (p-1, (Np-N+p)/(N-p))$ such that

$$|f_z(x,z)| \leq \tau_1(x)|z|^{r-1} + \tau_2(x)|z|^{s-1}, \tag{2.9}$$

for all $x \in \mathbf{R}^N$ and all $z \in \mathbf{R}$, where $\tau_1 \in L^{r_0}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $\tau_2 \in L^{s_0}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, with $r_0 = Np/(Np-(r+1)(N-p))$ and $s_0 = Np/(Np-(s+1)(N-p))$;

(F3) there exists a constant $\mu > p$ such that

$$0 < \mu F(x,z) := \mu \int_0^z f(x,t)dt \leq z f(x,z), \tag{2.10}$$

for all $x \in \mathbf{R}^N$ and all $z \in \mathbf{R} \setminus \{0\}$.

Next, we study the problem

$$-\operatorname{div}(a(x,\nabla u)) + b(x)|u|^{p-2}u = h(x)|u|^{q-1}u + g(x)|u|^{s-1}u, \quad x \in \mathbf{R}^N \tag{2.11}$$

with $1 < q < p-1 < s < (Np-N+p)/(N-p)$ and $N \geq 3$.

Our basic assumptions on functions h and $g : \mathbf{R}^N \to \mathbf{R}$ are the following:

(H) $h(x) \geq 0$ for all $x \in \mathbf{R}^N$ and $h \in L^{q_0}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, where $q_0 = Np/(Np-(q+1)(N-p))$;

(G) $g(x) \geq 0$ for all $x \in \mathbf{R}^N$ and $g \in L^{s_0}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, where $s_0 = Np/(Np-(s+1)(N-p))$.

Let $W^{1,p}(\mathbf{R}^N)$ be the usual Sobolev space under the norm

$$\|u\|_1 = \left(\int_{\mathbf{R}^N} (|\nabla u|^p + |u|^p)\, dx \right)^{1/p} \tag{2.12}$$

and consider the subspace of $W^{1,p}(\mathbf{R}^N)$

$$E = \left\{ u \in W^{1,p}(\mathbf{R}^N);\ \int_{\mathbf{R}^N} (|\nabla u|^p + b(x)|u|^p)\, dx < \infty \right\}. \tag{2.13}$$

The Banach space E can be endowed with the norm

$$\|u\|^p = \int_{\mathbf{R}^N} (|\nabla u|^p + b(x)|u|^p)\, dx. \tag{2.14}$$

Moreover,

$$\|u\| \geq m_0^{1/p} \|u\|_1, \tag{2.15}$$

with $m_0 = \min\{1, b_0\}$. Thus the continuous embeddings

$$E \hookrightarrow W^{1,p}(\mathbf{R}^N) \hookrightarrow L^i(\mathbf{R}^N), \quad p \leq i \leq p^\star,\ p^\star = \frac{Np}{N-p} \tag{2.16}$$

hold true.

We say that $u \in E$ is a *weak solution* for problem (2.2) if

$$\int_{\mathbf{R}^N} a(x,\nabla u)\cdot\nabla\varphi\,dx+\int_{\mathbf{R}^N} b(x)|u|^{p-2}u\varphi\,dx-\int_{\mathbf{R}^N} f(x,u)\varphi\,dx=0, \tag{2.17}$$

for all $\varphi \in E$.

Similarly, we say that $u \in E$ is a *weak solution* for problem (2.11) if

$$\begin{aligned}\int_{\mathbf{R}^N} a(x,\nabla u)\cdot\nabla\varphi\,dx+\int_{\mathbf{R}^N} b(x)|u|^{p-2}u\varphi\,dx\\ -\int_{\mathbf{R}^N} h(x)|u|^{q-1}u\varphi\,dx-\int_{\mathbf{R}^N} g(x)|u|^{s-1}u\varphi\,dx=0,\end{aligned} \tag{2.18}$$

for all $\varphi \in E$.

Our main results are given by the following two theorems.

Theorem 2.1. *Assuming hypotheses (A1)–(A5), (B) and (F1)–(F3) are fulfilled then problem (2.2) has at least one non-trivial weak solution.*

Theorem 2.2. *Assume $1 < q < p-1 < s < (Np-N+p)/(N-p)$ and conditions (A1)–(A5), (B), (H) and (G) are fulfilled. Then problem (2.11) has at least two non-trivial weak solutions provided that the product $\|h\|_{L^{q_0}(\mathbf{R}^N)}^{(s+1-p)/(s-q)}\cdot\|g\|_{L^{s_0}(\mathbf{R}^N)}^{(p-q-1)/(s-q)}$ is small enough.*

3. Auxiliary results

In this section we study certain properties of functional $T : E \to \mathbf{R}$ defined by

$$T(u)=\int_{\mathbf{R}^N} A(x,\nabla u)dx+\frac{1}{p}\int_{\mathbf{R}^N} b(x)|u|^p dx, \tag{3.1}$$

for all $u \in E$. It is easy to remark that $T \in C^1(E,\mathbf{R})$ and

$$\langle T'(u),v\rangle=\int_{\mathbf{R}^N} a(x,\nabla u)\cdot\nabla v\,dx+\int_{\mathbf{R}^N} b(x)|u|^{p-2}uv\,dx, \tag{3.2}$$

for all u, $v \in E$.

Proposition 3.1. *Functional T is weakly lower semicontinuous.*

Proof. Let $u \in E$ and $\epsilon > 0$ be fixed. Using the properties of lower semicontinuous functions (see [3, Section I.3]) is enough to prove that there exists $\delta > 0$ such that

$$T(v)\geq T(u)-\epsilon,\quad \forall v\in E \text{ with } \|u-v\|<\delta. \tag{3.3}$$

We remember Clarkson's inequality (see [3, page 59])

$$\left|\frac{\alpha+\beta}{2}\right|^p+\left|\frac{\alpha-\beta}{2}\right|^p\leq\frac{1}{2}\left(|\alpha|^p+|\beta|^p\right),\quad \forall\alpha,\beta\in\mathbf{R}. \tag{3.4}$$

Thus we deduce that

$$\begin{aligned}&\int_{\mathbf{R}^N} b(x)\left|\frac{u+v}{2}\right|^p dx+\int_{\mathbf{R}^N} b(x)\left|\frac{u-v}{2}\right|^p dx\\&\qquad\le \frac{1}{2}\int_{\mathbf{R}^N} b(x)|u|^p dx+\frac{1}{2}\int_{\mathbf{R}^N} b(x)|v|^p dx,\quad \forall u,v\in E.\end{aligned}\tag{3.5}$$

The above inequality and condition (A3) imply that there exists a positive constant $k_1>0$ such that

$$T\left(\frac{u+v}{2}\right)\le \frac{1}{2}T(u)+\frac{1}{2}T(v)-k_1\|u-v\|^p,\quad \forall u,v\in E,\tag{3.6}$$

that is, T is p-uniformly convex.

Since T is convex we have

$$T(v)\ge T(u)+\langle T^{'}(u),v-u\rangle,\quad \forall v\in E.\tag{3.7}$$

Using condition (A2) and Hölder's inequality we deduce that there exists a positive constant $C>0$ such that

$$\begin{aligned}T(v)&\ge T(u)-\int_{\mathbf{R}^N}\left|a(x,\nabla u)\right|\cdot|\nabla v-\nabla u|dx-\int_{\mathbf{R}^N} b(x)|u|^{p-1}|u-v|dx\\&\ge T(u)-\int_{\mathbf{R}^N} c_1\left(\theta(x)+|\nabla u|^{p-1}\right)|\nabla v-\nabla u|dx\\&\quad-\int_{\mathbf{R}^N} b(x)^{(p-1)/p}|u|^{p-1}b(x)^{1/p}|u-v|dx\\&\ge T(u)-c_1\cdot\left(\|\theta\|_{L^{p/(p-1)}(\mathbf{R}^N)}+\|\nabla u\|^{p-1}_{L^p(\mathbf{R}^N)}\right)\cdot\left(\int_{\mathbf{R}^N}|\nabla v-\nabla u|^p dx\right)^{1/p}\\&\quad-\left(\int_{\mathbf{R}^N} b(x)|u|^p dx\right)^{(p-1)/p}\cdot\left(\int_{\mathbf{R}^N} b(x)|v-u|^p dx\right)^{1/p}\\&\ge T(u)-C\|u-v\|,\quad \forall v\in E.\end{aligned}\tag{3.8}$$

It is clear that taking $\delta=\epsilon/C$ relation (3.3) holds true for all $v\in E$ with $\|v-u\|<\delta$. Thus we have proved that T is strongly lower semicontinuous. Taking into account the fact that T is convex then by [3, Corollary III.8] we conclude that T is weakly lower semicontinuous and the proof of Proposition 3.1 is complete. □

Proposition 3.2. *Assume $\{u_n\}$ is a subsequence from E which is weakly convergent to $u\in E$ and*

$$\limsup_{n\to\infty}\langle T^{'}(u_n),u_n-u\rangle\le 0.\tag{3.9}$$

Then $\{u_n\}$ converges strongly to u in E.

Proof. Since $\{u_n\}$ is weakly convergent to u in E it follows that $\{u_n\}$ is bounded in E.

By conditions (A2) and (A3) we have

$$0 \le A(x,\xi) = \int_0^1 \frac{d}{dt} A(x,t\xi)dt = \int_0^1 a(x,t\xi)\cdot\xi\,dt$$
$$\le c_1 \int_0^1 (\theta(x) + |\xi|^{p-1}t^{p-1})dt \tag{3.10}$$
$$\le c_1\left(\theta(x)|\xi| + \frac{1}{p}|\xi|^p\right), \quad \forall x,\xi \in \mathbf{R}^N.$$

Thus, there exists a constant $c_2 > 0$ such that

$$|A(x,\xi)| \le c_2(\theta(x)|\xi| + |\xi|^p), \quad \forall x,\xi \in \mathbf{R}^N. \tag{3.11}$$

Relation (3.11) and Hölder's inequality imply

$$\int_{\mathbf{R}^N} A(x,\nabla u_n)dx \le c_2\left(\int_{\mathbf{R}^N} \theta(x)|\nabla u_n|dx + \int_{\mathbf{R}^N} |\nabla u_n|^p dx\right)$$
$$\le c_2\cdot\left(\|\theta\|_{L^{p/(p-1)}(\mathbf{R}^N)}\cdot\|u_n\| + \|u_n\|^p\right). \tag{3.12}$$

The above inequality and the fact that $\{u_n\}$ is bounded in E show that there exists $M_1 > 0$ such that $T(u_n) \le M_1$ for all n. Then we may assume that $T(u_n) \to \gamma$. Using Proposition 3.1 we find

$$T(u) \le \liminf_{n\to\infty} T(u_n) = \gamma. \tag{3.13}$$

Since T is convex the following inequality holds true

$$T(u) \ge T(u_n) + \langle T'(u_n), u_n - u\rangle, \quad \forall n. \tag{3.14}$$

Relation (3.9) and the above inequality imply $T(u) \ge \gamma$ and thus $T(u) = \gamma$.

We also have $(u_n + u)/2$ converges weakly to u in E. Using again Proposition 3.1 we deduce

$$\gamma = T(u) \le \liminf_{n\to\infty} T\left(\frac{u_n + u}{2}\right). \tag{3.15}$$

If we assume by contradiction that $\|u_n - u\|$ does not converge to 0 then there exists $\epsilon > 0$ such that passing to a subsequence $\{u_{nm}\}$ we have $\|u_{nm} - u\| \ge \epsilon$. That fact and relation (3.6) imply

$$\frac{1}{2}T(u) + \frac{1}{2}T(u_{nm}) - T\left(\frac{u + u_{nm}}{2}\right) \ge k_1\|u - u_{nm}\|^p \ge k_1\epsilon^p. \tag{3.16}$$

Letting $m \to \infty$ we find

$$\limsup_{m\to\infty} T\left(\frac{u + u_{nm}}{2}\right) \le \gamma - k_1\epsilon^p \tag{3.17}$$

and that is a contradiction with (3.15). Thus we have

$$||u_n - u|| \longrightarrow 0. \tag{3.18}$$

The proof of Proposition 3.2 is complete. □

4. Proof of Theorem 2.1

In order to prove Theorem 2.1 we define the functional

$$J(u) = \int_{\mathbf{R}^N} A(x,\nabla u)dx + \frac{1}{p}\int_{\mathbf{R}^N} b(x)|u|^p dx - \int_{\mathbf{R}^N} F(x,u)dx. \tag{4.1}$$

$J : E \to \mathbf{R}$ is well defined and of class C^1 with the derivative given by

$$\langle J'(u),\varphi\rangle = \int_{\mathbf{R}^N} a(x,\nabla u)\cdot\nabla\varphi\, dx + \int_{\mathbf{R}^N} b(x)|u|^{p-2}u\varphi\, dx - \int_{\mathbf{R}^N} f(x,u)\varphi\, dx, \tag{4.2}$$

for all u, $\varphi \in E$. We have denoted by $\langle , \rangle$ the duality pairing between E and $E^\star$, where $E^\star$ is the dual of E.

We remark that the critical points of the functional J correspond to the weak solutions of (2.2). Thus, our idea is to apply the Mountain Pass theorem (see [2]) in order to obtain a non-trivial critical point and thus a non-trivial weak solution.

First, we prove a lemma which shows that functional J has a mountain-pass geometry.

Lemma 4.1. (1) *There exist $\rho > 0$ and $\varrho > 0$ such that*

$$J(u) \geq \varrho > 0, \quad \forall u \in E \text{ with } \|u\| = \rho. \tag{4.3}$$

(2) *There exists $u_0 \in E$ such that*

$$\lim_{t\to\infty} J(tu_0) = -\infty. \tag{4.4}$$

Proof. (1) By (F2) there exist A_1, $A_2 > 0$ two constants such that

$$0 \leq F(x,z) \leq A_1|z|^{r+1} + A_2|z|^{s+1}. \tag{4.5}$$

Then we deduce that

$$\lim_{|z|\to 0}\frac{F(x,z)}{|z|^p} = 0, \qquad \lim_{|z|\to\infty}\frac{F(x,z)}{|z|^{p^\star}} = 0. \tag{4.6}$$

Then, for a $\epsilon > 0$ there exist two constants δ_1 and δ_2 such that

$$\begin{aligned} F(x,z) &< \epsilon|z|^p \quad \forall z \text{ with } |z| < \delta_1, \\ F(x,z) &< \epsilon|z|^{p^\star} \quad \forall z \text{ with } |z| > \delta_2. \end{aligned} \tag{4.7}$$

Relation (4.5) implies that for all z with $|z| \in [\delta_1,\delta_2]$ there exists a positive constant $C > 0$ such that

$$F(x,z) < C. \tag{4.8}$$

We obtain that for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$F(x,z) \le \epsilon |z|^p + C_\epsilon |z|^{p^\star}. \tag{4.9}$$

Relation (4.9), conditions (A5) and (b1) and the Sobolev embedding imply

$$\begin{aligned} J(u) &= \int_{\mathbf{R}^N} A(x,\nabla u)dx + \frac{1}{p}\int_{\mathbf{R}^N} b(x)|u|^p dx - \int_{\mathbf{R}^N} F(x,u)dx \\ &\ge \Lambda \int_{\mathbf{R}^N} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbf{R}^N} b(x)|u|^p dx - \epsilon \int_{\mathbf{R}^N} |u|^p dx - C_\epsilon \int_{\mathbf{R}^N} |u|^{p^\star} dx \\ &\ge \min\left\{\Lambda, \frac{1}{p}\right\} \cdot \|u\|^p - \frac{\epsilon}{b_0}\int_{\mathbf{R}^N} b(x)|u|^p dx - C_\epsilon \int_{\mathbf{R}^N} |u|^{p^\star} dx \\ &\ge \|u\|^p \cdot \left[\left(\min\left\{\Lambda, \frac{1}{p}\right\} - \frac{\epsilon}{b_0}\right) - C_\epsilon^{'} \cdot \|u\|^{p^\star - p}\right]. \end{aligned} \tag{4.10}$$

Letting $\epsilon \in (0, \min\{\Lambda, 1/p\} \cdot b_0)$ be fixed, we obtain that the first part of Lemma 4.1 holds true.

(2) To prove the second part of the lemma, first, we remark that by condition (F3) we have

$$F(x,z) \ge \lambda |z|^\mu, \quad \forall |z| \ge \eta,\ x \in \mathbf{R}^N, \tag{4.11}$$

where λ and η are two positive constants.

On the other hand we claim that

$$A(x,z\xi) \le A(x,\xi)z^p, \quad \forall z \ge 1,\ x,\xi \in \mathbf{R}^N. \tag{4.12}$$

Indeed, if we put $\alpha(t) = A(x,t\xi)$ then by (A1) and (A4) we have

$$\alpha^{'}(t) = a(x,t\xi) \cdot \xi = \frac{1}{t}a(x,t\xi) \cdot (t\xi) \le \frac{p}{t}A(x,t\xi) = \frac{p}{t}\alpha(t). \tag{4.13}$$

Hence

$$\frac{\alpha^{'}(t)}{\alpha(t)} \le \frac{p}{t} \tag{4.14}$$

or

$$\log(\alpha(t)) - \log(\alpha(1)) \le p\log(t). \tag{4.15}$$

We deduce that $\alpha(t)/\alpha(1) \le t^p$ and thus (4.12) holds true.

Let now $u_0 \in E$ be such that $\text{meas}(\{x \in \mathbf{R}^N;\ |u_0(x)| \geq \eta\}) > 0$. Using relations (4.11) and (4.12) we obtain

$$\begin{aligned} J(tu_0) &= \int_{\mathbf{R}^N} \left[A(x, t\nabla u_0) + \frac{1}{p} b(x) t^p |u_0|^p\right] dx - \int_{\mathbf{R}^N} F(x, tu_0)\, dx \\ &\leq t^p \int_{\mathbf{R}^N} \left[A(x, \nabla u_0) + \frac{1}{p} b(x) |u_0|^p\right] dx - \int_{\{x\in\mathbf{R}^N; |u_0(x)|\geq\eta\}} F(x, tu_0)\, dx \\ &\quad - \int_{\{x\in\mathbf{R}^N; |u_0(x)|\leq\eta\}} F(x, tu_0)\, dx \\ &\leq t^p \int_{\mathbf{R}^N} \left[A(x, \nabla u_0) + \frac{1}{p} b(x) |u_0|^p\right] dx - t^\mu \lambda \int_{\{x\in\mathbf{R}^N; |u_0(x)|\geq\eta\}} |u_0|^\mu dx. \end{aligned} \tag{4.16}$$

Since $\mu > p$ the right-hand side of the above inequality converges to $-\infty$ as $t \to \infty$.

The lemma is completely proved. □

Proof of Theorem 2.1. Using Lemma 4.1 we may apply the Mountain Pass theorem (see [2]) to functional J. We obtain that there exists a sequence $\{u_n\}$ in E such that

$$J(u_n) \longrightarrow c > 0, \qquad J'(u_n) \longrightarrow 0 \quad \text{in } E^\star. \tag{4.17}$$

We prove that $\{u_n\}$ is bounded in E. We assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Then, using relation (4.17) and conditions (A4), (A5) and (F3) we deduce that for n large enough the following inequalities hold

$$\begin{aligned} c + 1 + \|u_n\| &\geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\ &= \int_{\mathbf{R}^N} \left[A(x, \nabla u_n) - \frac{1}{\mu} a(x, \nabla u_n) \cdot \nabla u_n\right] dx \\ &\quad + \int_{\mathbf{R}^N} \left[\frac{1}{p} b(x) |u_n|^p - \frac{1}{\mu} b(x) |u_n|^p\right] dx \\ &\quad + \int_{\mathbf{R}^N} \left[\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n)\right] dx \\ &\geq \left(1 - \frac{p}{\mu}\right) \int_{\mathbf{R}^N} A(x, \nabla u_n)\, dx + \left(\frac{1}{p} - \frac{1}{\mu}\right) \int_{\mathbf{R}^N} b(x) |u_n|^p dx \\ &\geq \left(1 - \frac{p}{\mu}\right) \Lambda \int_{\mathbf{R}^N} |\nabla u_n|^p dx + \left(\frac{1}{p} - \frac{1}{\mu}\right) \int_{\mathbf{R}^N} b(x) |u_n|^p dx \\ &\geq \min\left\{\left(1 - \frac{p}{\mu}\right)\Lambda, \frac{1}{p} - \frac{1}{\mu}\right\} \cdot \|u_n\|^p. \end{aligned} \tag{4.18}$$

Dividing by $\|u_n\|$ and letting $n \to \infty$ we obtain a contradiction. Therefore $\{u_n\}$ is bounded in E by a positive constant denoted by M. It follows that there exists $u \in E$ such that, passing to a subsequence still denoted by $\{u_n\}$, it converges weakly to u in E and $u_n(x) \to u(x)$ a.e. $x \in \mathbf{R}^N$. Since E is continuously embedded in $L^{p^\star}(\mathbf{R}^N)$ by [17, Theorem 10.36] we deduce that u_n converges weakly to u in $L^{p^\star}(\mathbf{R}^N)$. Then it is clear that $|u_n|^{r-1}u_n$ converges weakly to $|u|^{r-1}u$ in $L^{p^\star/r}(\mathbf{R}^N)$.

Define the operator $U : L^{p^\star/r}(\mathbf{R}^N) \to \mathbf{R}$ by

$$\langle U, w\rangle = \int_{\mathbf{R}^N} \tau_1(x) u w \, dx. \tag{4.19}$$

We remark that U is linear and continuous provided that $\tau_1 \in L^{r_0}(\mathbf{R}^N)$, $u \in L^{p^\star}(\mathbf{R}^N)$ and $1/p^\star + r/p^\star + 1/r_0 = 1$. All the above pieces of information imply

$$\langle U, |u_n|^{r-1} u_n\rangle \longrightarrow \langle U, |u|^{r-1} u\rangle, \tag{4.20}$$

that is,

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \tau_1(x) |u_n|^{r-1} u_n u \, dx = \int_{\mathbf{R}^N} \tau_1(x) |u|^{r+1} dx. \tag{4.21}$$

With the same arguments we can show that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \tau_2(x) |u_n|^{s-1} u_n u \, dx = \int_{\mathbf{R}^N} \tau_2(x) |u|^{s+1} dx, \tag{4.22}$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \tau_1(x) |u_n|^{r+1} dx = \int_{\mathbf{R}^N} \tau_1(x) |u|^{r+1} dx, \tag{4.23}$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \tau_2(x) |u_n|^{s+1} dx = \int_{\mathbf{R}^N} \tau_2(x) |u|^{s+1} dx. \tag{4.24}$$

Relations (4.21), (4.23) and the fact that

$$\begin{aligned}\int_{\mathbf{R}^N} \tau_1(x) |u_n|^{r-1} u_n (u_n - u) dx = {} & \int_{\mathbf{R}^N} \tau_1(x) |u_n|^{r+1} dx - \int_{\mathbf{R}^N} \tau_1(x) |u|^{r+1} dx \\ & + \int_{\mathbf{R}^N} \tau_1(x) |u|^{r+1} dx - \int_{\mathbf{R}^N} \tau_1(x) |u_n|^{q-1} u_n u \, dx\end{aligned} \tag{4.25}$$

yield

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \tau_1(x) |u_n|^{r-1} u_n (u_n - u) dx = 0. \tag{4.26}$$

Similarly we obtain

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \tau_2(x) |u_n|^{s-1} u_n (u_n - u) dx = 0. \tag{4.27}$$

By (4.26), (4.27) and condition (F2) we get

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} f(x, u_n)(u_n - u) dx = 0. \tag{4.28}$$

On the other hand we have

$$\begin{aligned}\int_{\mathbf{R}^N} a(x, \nabla u_n) \cdot \nabla u_n \, dx + \int_{\mathbf{R}^N} b(x) |u_n|^{p-2} u_n (u_n - u) dx \\ = \langle J'(u_n), u_n - u\rangle + \int_{\mathbf{R}^N} f(x, u_n)(u_n - u) dx.\end{aligned} \tag{4.29}$$

Relations (4.28) and (4.29) imply

$$\lim_{n\to\infty}\left(\int_{\mathbf{R}^N} a(x,\nabla u_n)\cdot\nabla(u_n-u)\,dx+\int_{\mathbf{R}^N} b(x)\,|\,u_n\,|^{p-2}(u_n-u)\,dx\right)=0, \tag{4.30}$$

that is,

$$\lim_{n\to\infty}\langle T'(u_n),u_n-u\rangle=0, \tag{4.31}$$

where T is the functional defined in the above section. Then applying Proposition 3.2 we deduce that $\{u_n\}$ converges strongly to u in E. Since $J\in C^1(E,\mathbf{R})$ by (4.17) we deduce that $\langle J'(u),\varphi\rangle=0$ for all $\varphi\in E$, that is, u is a weak solution of problem (2.2). Relation (4.17) also implies that $J(u)=c>0$ and that shows that u is non-trivial.

The proof of Theorem 2.1 is complete. □

5. Proof of Theorem 2.2

We remark that the weak solutions of (2.11) correspond to the critical points of the energy functional $I:E\to\mathbf{R}$ defined as follows

$$\begin{aligned} I(u)=&\int_{\mathbf{R}^N}A(x,\nabla u)\,dx+\frac{1}{p}\int_{\mathbf{R}^N}b(x)|u|^p\,dx-\frac{1}{q+1}\int_{\mathbf{R}^N}h(x)|u|^{q+1}\,dx\\ &-\frac{1}{s+1}\int_{\mathbf{R}^N}g(x)|u|^{s+1}\,dx,\quad \forall u\in E. \end{aligned}\tag{5.1}$$

A simple calculation shows that I is well defined on E and $I\in C^1(E,\mathbf{R})$ with

$$\begin{aligned} \langle I'(u),\varphi\rangle=&\int_{\mathbf{R}^N}a(x,\nabla u)\cdot\nabla\varphi\,dx+\int_{\mathbf{R}^N}b(x)|u|^{p-2}u\varphi\,dx\\ &-\int_{\mathbf{R}^N}h(x)|u|^{q-1}u\varphi\,dx-\int_{\mathbf{R}^N}g(x)|u|^{s-1}u\varphi\,dx, \end{aligned}\tag{5.2}$$

for all u and $\varphi\in E$.

Lemma 5.1. *The following assertions hold.*

(i) *There exist $\rho>0$ and $\varrho>0$ such that*

$$I(u)\ge\varrho>0,\quad \forall u\in E\ \text{with}\ \|u\|=\rho. \tag{5.3}$$

(ii) *There exists $\psi\in E$ such that*

$$\lim_{t\to\infty}I(t\psi)=-\infty. \tag{5.4}$$

(iii) *There exists $\varphi\in E$ such that $\varphi\ge0$, $\varphi\ne0$ and*

$$I(t\varphi)<0 \tag{5.5}$$

for $t>0$ small enough.

Proof. (i) First, let $\mathcal{S}$ be the best Sobolev constant of the embedding $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^{p^\star}(\mathbf{R}^N)$, that is,

$$\mathcal{S} = \inf_{u \in W^{1,p}(\mathbf{R}^N)\setminus\{0\}} \frac{\int_{\mathbf{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbf{R}^N} |u|^{p^\star} dx\right)^{p/p^\star}}. \tag{5.6}$$

Thus we obtain

$$\mathcal{S}^{1/p}\|v\|_{L^{p^\star}(\mathbf{R}^N)} \leq \|v\|, \quad \forall v \in E. \tag{5.7}$$

By Hölder's inequality and relation (5.7) we deduce

$$\begin{aligned}
\int_{\mathbf{R}^N} h(x)|u|^{q+1}dx &\leq \|h\|_{L^{q_0}(\mathbf{R}^N)} \cdot \|u\|^{q+1}_{L^{p^\star}(\mathbf{R}^N)} \\
&\leq \|h\|_{L^{q_0}(\mathbf{R}^N)} \cdot \frac{1}{\mathcal{S}^{(q+1)/p}} \cdot \left(\mathcal{S}^{1/p} \cdot \|u\|_{L^{p^\star}(\mathbf{R}^N)}\right)^{q+1} \\
&\leq \|h\|_{L^{q_0}(\mathbf{R}^N)} \cdot \frac{1}{\mathcal{S}^{(q+1)/p}} \cdot \|u\|^{q+1} \\
&\leq (q+1)\mu\|u\|^{q+1},
\end{aligned} \tag{5.8}$$

where $\mu = \|h\|_{L^{q_0}(\mathbf{R}^N)}/[(q+1)\mathcal{S}^{(q+1)/p}]$. With similar arguments we have

$$\int_{\mathbf{R}^N} g(x)|u|^{s+1}dx \leq (p+1)\nu\|u\|^{s+1}, \tag{5.9}$$

where $\nu = \|g\|_{L^{s_0}(\mathbf{R}^N)}/[(p+1)\mathcal{S}^{(s+1)/p}]$.

Thus, we obtain

$$\begin{aligned}
I(u) &\geq \min\left\{\Lambda, \frac{1}{p}\right\} \cdot ||u_n||^p - \mu \cdot \|u\|^{q+1} - \nu \cdot \|u\|^{s+1} \\
&= \left(\lambda - \mu \cdot \|u\|^{q+1-p} - \nu \cdot \|u\|^{s+1-p}\right) \cdot \|u\|^p, \quad \forall u \in E,
\end{aligned} \tag{5.10}$$

where $\lambda = \min\{\Lambda, 1/p\} > 0$. We show that there exists $t_0 > 0$ such that

$$\mu \cdot t_0^{q+1-p} + \nu \cdot t_0^{s+1-p} < \lambda. \tag{5.11}$$

To do that we define the function

$$Q(t) = \mu \cdot t^{q+1-p} + \nu \cdot t^{s+1-p}, \quad t > 0. \tag{5.12}$$

Since $\lim_{t\to 0} Q(t) = \lim_{t\to\infty} Q(t) = \infty$ it follows that Q possesses a positive minimum, say $t_0 > 0$. In order to find t_0 we have to solve equation $Q'(t_0) = 0$, where $Q'(t) = (q+1-p) \cdot \mu \cdot t^{q-p} + (s+1-p) \cdot \nu \cdot t^{s-p}$. A simple computation yields $t_0 = [((p-q-1)/(s+1-p)) \cdot (\mu/\nu)]^{1/(s-q)}$. Thus relation (5.11) holds provided that

$$\mu \cdot \left[\frac{p-q-1}{s+1-p} \cdot \frac{\mu}{\nu}\right]^{(q+1-p)/(s-q)} + \nu \cdot \left[\frac{p-q-1}{s+1-p} \cdot \frac{\mu}{\nu}\right]^{(s+1-p)/(s-q)} < \lambda. \tag{5.13}$$

Since $\mu = C_1 \cdot \|h\|_{L^{q_0}(\mathbf{R}^N)}$ and $\nu = C_2 \cdot \|g\|_{L^{s_0}(\mathbf{R}^N)}$ with C_1, C_2 positive constants, we deduce that (5.13) holds true if and only if the following inequality holds

$$C_3 \cdot \|h\|_{L^{q_0}(\mathbf{R}^N)}^{(s+1-p)/(s-q)} \cdot \|g\|_{L^{s_0}(\mathbf{R}^N)}^{(p-q-1)/(s-q)} < \lambda, \tag{5.14}$$

where C_3 is a positive constant. But inequality (5.14) holds provided that product $\|h\|_{L^{q_0}(\mathbf{R}^N)}^{(s+1-p)/(s-q)} \cdot \|g\|_{L^{s_0}(\mathbf{R}^N)}^{(p-q-1)/(s-q)}$ is small enough.

(ii) Let $\psi \in C_0^\infty(\mathbf{R}^N)$, $\psi \geq 0$, $\psi \neq 0$. Then using relation (4.12) we have

$$\begin{aligned} I(t\psi) &= \int_{\mathbf{R}^N} A(x, t\nabla\psi)dx + \frac{t^p}{p}\int_{\mathbf{R}^N} b(x)|\psi|^p dx \\ &\quad - \frac{t^{q+1}}{q+1}\int_{\mathbf{R}^N} h(x)|\psi|^{q+1}dx - \frac{t^{s+1}}{s+1}\int_{\mathbf{R}^N} g(x)|\psi|^{s+1}dx \\ &\leq t^p \int_{\mathbf{R}^N} A(x,\nabla\psi)dx + \frac{t^p}{p}\int_{\mathbf{R}^N} b(x)|\psi|^p dx - \frac{t^{s+1}}{s+1}\int_{\mathbf{R}^N} g(x)|\psi|^{s+1}dx. \end{aligned} \tag{5.15}$$

Thus $I(t\psi) \to -\infty$ as $t \to \infty$ and (ii) is proved.

(iii) Let $\varphi \in C_0^\infty(\mathbf{R}^N)$, $\varphi \geq 0$, $\varphi \neq 0$ and $t > 0$. Then the above inequality implies

$$I(t\varphi) \leq t^p \int_{\mathbf{R}^N} A(x,\nabla\varphi)dx + \frac{t^p}{p}\int_{\mathbf{R}^N} b(x)|\varphi|^p dx - \frac{t^{q+1}}{q+1}\int_{\mathbf{R}^N} h(x)|\varphi|^{q+1}dx < 0 \tag{5.16}$$

for $t < \delta^{1/(p-q-1)}$ with

$$\delta = \frac{(1/(q+1))\int_{\mathbf{R}^N} h(x)|\varphi|^{q+1}dx}{\left[\int_{\mathbf{R}^N} A(x,\nabla\varphi)dx + (1/p)\int_{\mathbf{R}^N} b(x)|\varphi|^p dx\right]}. \tag{5.17}$$

It follows that (iii) holds true.

The proof of Lemma 5.1 is complete. □

Proof of Theorem 2.2. Using Lemma 5.1 and the Mountain Pass theorem we deduce the existence of a sequence $\{u_n\}$ in E such that

$$I(u_n) \longrightarrow \overline{c} > 0, \qquad I'(u_n) \longrightarrow 0 \quad \text{in } E^\star. \tag{5.18}$$

We prove that $\{u_n\}$ is bounded in E. We assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Using relation (5.18) and conditions (A4) and (A5) we deduce that for n large enough we obtain

$$\begin{aligned} \overline{c} + 1 + ||u_n|| &\geq I(u_n) - \frac{1}{s+1}\langle I'(u_n), u_n\rangle \\ &= \int_{\mathbf{R}^N}\left(A(x,\nabla u_n) - \frac{1}{s+1}a(x,\nabla u_n)\cdot\nabla u_n\right)dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{s+1}\right)\int_{\mathbf{R}^N} b(x)|u_n|^{q+1}dx \\ &\quad - \frac{s-q}{(q+1)(s+1)}\int_{\mathbf{R}^N} h(x)|u_n|^{q+1}dx \end{aligned} \tag{5.19}$$

or

$$\begin{aligned}\overline{c}+1+\|u_n\|+\frac{s-q}{(q+1)(s+1)}\int_{\mathbf{R}^N}h(x)\,|\,u_n\,|^{q+1}dx\\ \geq\left(1-\frac{p}{s+1}\right)\Lambda\int_{\mathbf{R}^N}|\,\nabla u_n\,|^{p}dx\\ +\left(\frac{1}{p}-\frac{1}{s+1}\right)\int_{\mathbf{R}^N}b(x)\,|\,u_n\,|^{p}dx\\ \geq\min\left\{\left(1-\frac{p}{s+1}\right)\Lambda,\left(\frac{1}{p}-\frac{1}{s+1}\right)\right\}\cdot\|u_n\|^{p}.\end{aligned}\tag{5.20}$$

By relation (5.8) and the above inequality we obtain

$$\begin{aligned}\overline{c}+1+\|u_n\|+\frac{s-q}{(q+1)(s+1)}\cdot\|h\|_{L^{q_0}(\mathbf{R}^N)}\cdot\frac{1}{\mathcal{S}^{(q+1)/p}}\cdot\|u_n\|^{q+1}\\ \geq\min\left\{\left(1-\frac{p}{s+1}\right)\Lambda,\left(\frac{1}{p}-\frac{1}{s+1}\right)\right\}\cdot\|u_n\|^{p}.\end{aligned}\tag{5.21}$$

Since $1<q<p-1$ and $\|u_n\|\to\infty$, dividing the above inequality by $\|u_n\|^p$ and passing to the limit as $n\to\infty$ we obtain a contradiction. Thus $\{u_n\}$ is bounded in E. It follows that there exists $u_1\in E$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u_1 in E and $u_n(x)\to u_1(x)$ a.e. $x\in\mathbf{R}^N$. With the same arguments as those used in the proof of relation (4.29) we can show that

$$\lim_{n\to\infty}\langle T^{'}(u_n),u_n-u_1\rangle=0,\tag{5.22}$$

where T is the functional defined in the third section.

Then applying Proposition 3.2 we deduce that $\{u_n\}$ converges strongly to u_1 in E. Since $I\in C^1(E,\mathbf{R})$ relation (5.18) implies $\langle I^{'}(u_1),\varphi\rangle=0$ for all $\varphi\in E$, that is, u_1 is a weak solution of problem (2.11). Relation (5.18) also yields $I(u_1)=\overline{c}>0$ and thus u_1 is non-trivial.

We prove now that there exists a second weak solution $u_2\in E$ such that $u_2\neq u_1$. By Lemma 5.1(i) it follows that there exists a ball centered at the origin $B\subset E$, such that

$$\inf_{\partial B}I>0.\tag{5.23}$$

On the other hand, by Lemma 5.1(iii) there exists $\phi\in E$ such that $I(t\phi)<0$, for all $t>0$ small enough. Recalling that relation (5.10) holds for all $u\in E$, that is,

$$I(u)\geq\lambda\cdot\|u\|^{p}-\mu\cdot\|u\|^{q+1}-\nu\cdot\|u\|^{s+1}\tag{5.24}$$

we get that

$$-\infty<\underline{c}:=\inf_{\overline{B}}I<0.\tag{5.25}$$

We let now $0 < \epsilon < \inf_{\partial B} I - \inf_B I$. Applying Ekeland's Variational principle for functional $I : \overline{B} \to \mathbf{R}$, (see [8]), there exists $u_\epsilon \in \overline{B}$ such that

$$I(u_\epsilon) < \inf_{\overline{B}} I + \epsilon \\ I(u_\epsilon) < I(u) + \epsilon \cdot ||u - u_\epsilon||, \quad u \neq u_\epsilon. \tag{5.26}$$

Since

$$I(u_\epsilon) \leq \inf_{\overline{B}} I + \epsilon \leq \inf_B I + \epsilon < \inf_{\partial B} I \tag{5.27}$$

it follows that $u_\epsilon \in B$. Now, we define $\mathcal{M} : \overline{B} \to \mathbf{R}$ by $\mathcal{M}(u) = I(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of $\mathcal{M}$ and thus

$$\frac{\mathcal{M}(u_\epsilon + \zeta \cdot v) - \mathcal{M}(u_\epsilon)}{\zeta} \geq 0 \tag{5.28}$$

for a small $\zeta > 0$ and v in the unit sphere of E. The above relation yields

$$\frac{I(u_\epsilon + \zeta \cdot v) - I(u_\epsilon)}{\zeta} + \epsilon \cdot \|v\| \geq 0. \tag{5.29}$$

Letting $\zeta \to 0$ it follows that $\langle I'(u_\epsilon), v\rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|I'(u_\epsilon)\| \leq \epsilon$. We deduce that there exists $\{u_n\} \subset B$ such that $I(u_n) \to \underline{c}$ and $I'(u_n) \to 0$. Using the same arguments as in the case of solution u_1 we can prove that $\{u_n\}$ converges strongly to u_2 in E. Moreover, that fact yields that $I'(u_2) = 0$. Thus, u_2 is a weak solution for (2.11) and since $0 > \underline{c} = I(u_2)$ it follows that u_2 is non-trivial.

Finally, we point out the fact that $u_1 \neq u_2$ since

$$I(u_1) = \overline{c} > 0 > \underline{c} = I(u_2). \tag{5.30}$$

The proof of Theorem 2.2 is complete. □

Acknowledgment

The author would like to thank Professor V. Rădulescu for proposing these problems and for numerous valuable discussions.

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