Research Article

A Partial Backlogging Inventory Model for Deteriorating Items with Fluctuating Selling Price and Purchasing Cost

Hui-Ling Yang

Department of Computer Science and Information Engineering, Hung Kuang University, 34 Chung-Chie Road, Shalu, Taichung 43302, Taiwan

Correspondence should be addressed to Hui-Ling Yang, yanghl@ms19.hinet.net

In today's competitive markets, selling price and purchasing cost are usually fluctuating with economic conditions. Both selling price and purchasing cost are vital to the profitability of a firm. Therefore, in this paper, I extend the inventory model introduced by Teng and Yang (2004) to allow for not only the selling price but also the purchasing cost to change from one replenishment cycle to another during a finite time horizon. The objective is to find the optimal replenishment schedule and pricing policy to obtain the profit as maximum as possible. The conditions that lead to a maximizing solution guarantee that the existence, uniqueness, and global optimality are proposed. An efficient solution procedure and some theoretical results are presented. Finally, numerical examples for illustration and sensitivity analysis for managerial decision making are also performed.

1. Introduction

In today's time-based competitive market, the unit selling price of product may increase significantly while its demand increases such as fashionable or valuable goods. On the other hand, the selling prices of items may drop dramatically throughout their life cycles due to advances in technology, competition, and so forth. Thus, the selling price is fluctuating. From the other aspect, some products of the purchasing cost decreases as the demand increases such as the unit cost of a high-tech product declines significantly over its short product life cycle. For example, the cost of a personal computer drops constantly as shown in Lee et al. [1]. Furthermore, the purchasing cost as a percentage of sales is often substantial, which had been mentioned in Heizer and Render [2]. Therefore, from an integrated logistics management perspective, taking the varying selling price and purchasing cost into account is essential.

Moreover, in reality, for fashionable commodities and high-tech products with short life cycles, the backorder rate is diminishing with the length of waiting time. Customers who experience stock-out will be less likely to buy again from the suppliers, they may turn to another store to purchase the goods. The sales for the product may decline due to the introduction of more competitive product or the change in consumers' preferences. The longer the waiting time, the lower the backlogging rate is. This leads to a larger fraction of lost sales and a less profit. As a result, take the factor of partial backlogging into account is necessary. Abad [3] proposed an optimal pricing and lot-sizing policy under the conditions of perishability and partial backordering. Teng et al. [4] considered the partial backlogging inventory model with time varying demand and purchasing cost. Chang et al. [5] considered the inventory model with selling price and purchasing cost and then provided an optimal replenishment policy for a retailer to achieve its maximum profit. To study the effect of the factors (selling pricing and/or purchasing cost), several authors had developed various models with the related issue. For example, Teng and Yang [6] proposed the inventory lot-size models with time varying demand and purchasing cost under generalized holding cost. Abad [7] added the pricing strategy into consideration and provided the optimal price and lot-size for a retailer when the demand was a function of the selling price. Recently, Sana [8] provided the optimal pricing policy for partial backlogging inventory model with price-dependent demand. Das Roy et al. [9] proposed an optimal shipment strategy for imperfect items in a stock-out situation. Das Roy et al. [10] again provided an economic-order quantity model of imperfect quality items with partial backlogging. The major assumptions and objective used in the above research articles are summarized in Table 1.

Thus, in contrast to the above articles mentioned, the inventory model here is developed as introduced by Teng and Yang [11] to allow for not only selling price but also purchasing cost to be fluctuating, which are changed from one replenishment cycle to another during a finite time horizon. The objective is to find the optimal replenishment schedule and pricing policy to obtain the maximum profit rather than minimum cost. The total profit associated with the inventory system is a concave function of the number of replenishments, which simplifies the search for the optimal number of replenishments to find a local maximum. Moreover, an intuitively estimate for finding the optimal replenishment number is provided. Some numerical examples for illustration and sensitivity analysis for managerial decision making are performed. Finally, summary and suggestions for future research are provided.

2. Assumptions and Notation

The mathematical model of the inventory replenishment problem is based on the following assumptions:

(1) The planning horizon of the inventory problem here is finite and is taken as H time units. The initial and the final inventory levels are both zero during the time horizon H.

(2) Replenishment is instantaneous and lead time is zero.

(3) In reality, the item may deteriorate with time varying. For simplicity, we assume that the deterioration rate here is constant and there is no repair or replacement of the deteriorated items.

Table 1: Major characteristic of inventory models on selected articles.

Author(s) (published year)	Demand rate	Deterioration rate	Allow for shortages	With partial backlogging	Purchasing cost	Selling price	Objective
Abad (1996) [3]	Price dependent	Time varying	Yes	Yes	Constant	Variable	Profit maximization
Teng et al. (2002) [4]	Time dependent (logconcave)	Constant	Yes	Yes	Constant	X	Cost minimization
Teng and Yang (2004) [11]	Time dependent	Constant	Yes	Yes	Time varying	X	Cost minimization
Chang et al. (2006) [5]	Time and price dependent	Constant	Yes	Yes	Constant	Variable	Profit maximization
Teng and Yang (2007) [6]	Time dependent	X	Yes	X	Time varying	X	Cost minimization
Abad (2008) [7]	Price dependent	Time varying	Yes	Yes	Constant	Variable	Profit maximization
Sana (2010) [8]	Price dependent	Time varying	Yes	Yes	Constant	Variable	Profit maximization
Roy et al. (2011a) [9]	Constant	Uniform Distribution	Yes	Yes	Constant	Constant	Cost minimization/Profit maximization
Roy et al. (2011b) [10]	Constant	Uniform Distribution	Yes	Yes	Constant	Constant	Profit maximization
Present paper	Time dependent	Constant	Yes	Yes	Time varying	Time varying	Profit maximization

Note: "X" denote not considered.

(4) Shortages are allowed. Unsatisfied demand is backlogged, and the fraction of shortages backordered is a decreasing function of time t, denoted by $\beta(t)$, where t is the waiting time up to the next replenishment, and $0 \leq \beta(t) \leq 1$ with $\beta(0) = 1$. Note that if $\beta(t) = 1$ (or 0) for all t, then shortages are completely backlogged (or lost).

(5) In the lost-sales case, the opportunity cost due to lost sale is the sum of the revenue loss and the cost of lost goodwill. Hence, the opportunity cost due to lost sale here is greater than the unit purchasing cost. For details, see Teng et al. [4].

(6) In today's global high competition environment, we here assume that the selling price and purchasing cost are fluctuating with time and changed from one replenishment cycle to another during a finite time horizon.

For convenience, the following notation is used throughout this paper:

H: the time horizon under consideration,

$f(t)$: the demand rate at time t, without loss of generality, we here assume that $f(t)$ is increase, positive, differentiable in $[0, H]$,

$c_v(t)$: the purchasing cost per unit at time t, which is positive, differentiable in $[0, H]$,

$p(t)$: the selling price per unit at time t, which is positive, differentiable in $[0, H]$,

θ: the deterioration rate,

c_f: the fixed ordering cost per order,

c_h: the inventory holding cost per unit per unit time,

c_b: the backlogging cost per unit per unit time, if the shortage is backlogged,

c_l: the unit opportunity cost of lost sales, if the shortage is lost. We assume without loss of generality that $c_l > c_v(t)$,

n: the number of replenishments over $[0, H]$ (a decision variable),

t_i: the ith replenishment time (a decision variable), $i = 1, 2, \ldots, n$,

s_i: the time at which the inventory level reaches zero in the ith replenishment cycle (a decision variable), $i = 1, 2, \ldots, n$.

3. Mathematical Model

For simplicity, we use the same inventory model as in Teng and Yang [11], which is shown in Figure 1.

As a result, we obtain the time-weighted inventory during the ith cycle as

$$I_i = \frac{1}{\theta} \int_{t_i}^{s_i} \left[e^{\theta(t - t_i)} - 1 \right] f(t) dt, \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Similarly, the time-weighted backorders due to shortages during the ith cycle is

$$B_i = \int_{s_{i-1}}^{t_i} (t_i - t) \beta(t_i - t) f(t) dt, \quad i = 1, 2, \ldots, n, \tag{3.2}$$
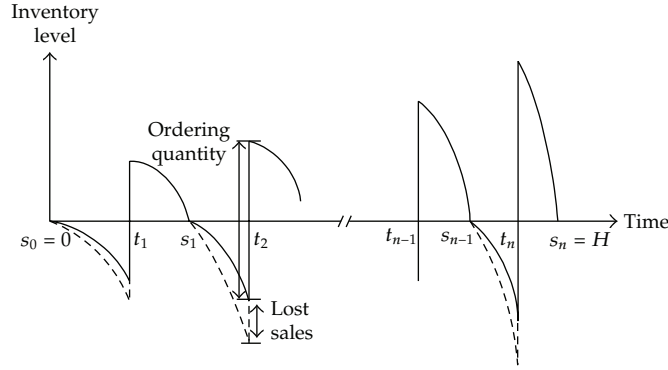
Figure 1: Graphical representation of inventory model.

and the total number of lost sales due to shortages during the ith cycle is

$$L_i = \int_{s_{i-1}}^{t_i} \left[1 - \beta(t_i - t)\right] f(t) dt, \quad i = 1, 2, \ldots, n. \tag{3.3}$$

The order quantity and unit sold at t_i in the ith replenishment cycle is

$$Q_i = \int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t) dt + \int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t) dt, \quad i = 1, 2, \ldots, n, \tag{3.4}$$

and unit sold at t_i in the ith replenishment cycle is

$$S_i = \int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t) dt + \int_{t_i}^{s_i} f(t) dt, \quad i = 1, 2, \ldots, n. \tag{3.5}$$

Therefore, the purchasing cost during the ith replenishment cycle is

$$\begin{aligned} P_i &= c_f + c_v(t_i) Q_i \\ &= c_f + c_v(t_i) \left[\int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t) dt + \int_{t_i}^{s_i} e^{\theta(t-t_i)} f(t) dt \right], \end{aligned} \tag{3.6}$$

and the revenue realized in the ith replenishment cycle is

$$R_i = p(t_i) \left[\int_{s_{i-1}}^{t_i} \beta(t_i - t) f(t) dt + \int_{t_i}^{s_i} f(t) dt \right], \quad i = 1, 2, \ldots, n. \tag{3.7}$$

Hence, if n replenishment orders are placed in $[0, H]$, then the total realized profit of the inventory system during the planning horizon H is

$$
\begin{aligned}
TP(n, \{s_i\}, \{t_i\}) &= \sum_{i=1}^{n}(R_i - P_i - c_h I_i - c_b B_i - c_l L_i) \\
&= \sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}\left\{\left[p(t_i) - c_v(t_i) - c_b(t_i - t) + c_l\right]\beta(t_i - t) - c_l\right\}f(t)dt - nc_f \\
&\quad + \sum_{i=1}^{n}\int_{t_i}^{s_i}\left[p(t_i) - c_v(t_i) - \left(\frac{c_h}{\theta} + c_v(t_i)\right)\left(e^{\theta(t-t_i)} - 1\right)\right]f(t)dt,
\end{aligned}
\tag{3.8}
$$

with $0 = s_0 < t_1$ and $s_n = H$. The problem is to determine n, $\{s_i\}$, and $\{t_i\}$ such that $TP(n, \{s_i\}, \{t_i\})$ in (3.8) is maximized.

4. Theoretical Results and Solution

For a fixed value of n, the necessary conditions for $TP(n, \{s_i\}, \{t_i\})$ to be maximized are: $\partial TP(n, \{s_i\}, \{t_i\})/\partial t_i = 0$ for $i = 1, 2, \dots, n$, and $\partial TP(n, \{s_i\}, \{t_i\})/\partial s_i = 0$, for. $i = 1, 2, \dots, n-1$. Consequently, we obtain

$$
\begin{aligned}
\int_{t_i}^{s_i}&\left\{p'(t_i) + \left[c_h + \theta c_v(t_i) - c_v'(t_i)\right]e^{\theta(t-t_i)}\right\}f(t)dt \\
&= -\int_{s_{i-1}}^{t_i}\left\{\left[p'(t_i) - c_v'(t_i) - c_b\right]\beta(t_i - t) + \left[p(t_i) + c_l - c_v(t_i) - c_b(t_i - t)\right]\beta'(t_i - t)\right\}f(t)dt,
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\left[p(t_{i+1}) - c_v(t_{i+1}) - c_b(t_{i+1} - s_i)\right]&\beta(t_{i+1} - s_i) - c_l\left[1 - \beta(t_{i+1} - s_i)\right] \\
&= p(t_i) - c_v(t_i) - \left[\frac{c_h}{\theta} + c_v(t_i)\right]\left(e^{\theta(s_i - t_i)} - 1\right),
\end{aligned}
\tag{4.2}
$$

respectively. Note that (4.1) and (4.2) are coincident with the following articles'

    (1) Equations (12) and (11) in Teng and Yang [11], if $p(t) = 0$,

    (2) Equations (15) and (14) in Teng et al. [4], if $p(t) = 0$ and $c_v(t) = c_v$,

    (3) Equations (11) and (10) in Chang et al. [5], if $p(t) = p$ and $c_v(t) = c_v$.

    Thus, the model here proposed is a generalization of the above three mentioned models. For simplicity, from (4.2), let the marginal resultant profit per unit during no-shortage and shortage period be

$$
R(t, u) = p(t) - c_v(t) - \left[\frac{c_h}{\theta} + c_v(t)\right]\left(e^{\theta(u-t)} - 1\right),
\tag{4.3}
$$

with $t \leq u$, and

$$
P(s, t) = \left[p(t) - c_v(t) - c_b(t - s)\right]\beta(t - s) - c_l\left[1 - \beta(t - s)\right],
\tag{4.4}
$$

with $s \leq t$, respectively. Taking the partial derivative of $R(t, u)$ and $P(s, t)$ with respect to t respectively, we obtained the following results:

$$R_t(t, u) = p'(t) + [c_h + \theta c_v(t) - c'_v(t)] e^{\theta(u-t)}, \tag{4.5}$$

$$P_t(s, t) = [p'(t) - c'_v(t) - c_b] \beta(t - s) + [p(t) + c_l - c_v(t) - c_b(t - s)] \beta'(t - s). \tag{4.6}$$

Note that the longer the waiting time, the lower the marginal resultant profit. Consequently, $P(s, t)$ is a decreasing function of t. Thus, we may assume without loss of generality that $P_t(s, t) < 0$, for all $t > s$. Then, we obtain the following result.

Lemma 4.1. *For any given n, if $R_{t_i}(t_i, t) \leq 0$, with $t \geq t_i$, and $P_{t_i}(t, t_i) < 0$, with $t \leq t_i$, $i = 1, 2, \ldots, n$, then the optimal solution is $n^* = 1$ and $t_1^* = 0$ (i.e., purchase at the beginning).*

Proof. See Appendix A.　　　　　　　　　　　　　　　　　　　□

The results in Lemma 4.1 can be interpreted as follows. The condition $R_{t_i}(t_i, t) \leq 0$ implies that $p'(t) + [c_h + \theta c_v(t)] e^{\theta(u-t)} \leq c'_v(t) e^{\theta(u-t)}$. This means that the increasing rate of the unit purchasing cost is higher than or equal to the sum of the marginal selling price and marginal inventory carrying cost per unit (which includes inventory and deterioration costs). Therefore, buying and storing a unit and then selling now are more profitable than buying and selling it later.

Theorem 4.2. *For any given n, if $R_{t_i}(t_i, t) > 0$, with $t \geq t_i$, and $P_{t_i}(t, t_i) < 0$, with $t \leq t_i$, $i = 1, 2, \ldots, n$, then the solution that satisfies the system of (4.1) and (4.2) exists uniquely and $0 \leq s_{i-1} < t_i < s_i$, for $i = 1, 2, \ldots, n$.*

Proof. See Appendix B.　　　　　　　　　　　　　　　　　　　□

The result in Theorem 4.2 reduces the $2n$-dimensional problem of finding $\{s_i^*\}$ and $\{t_i^*\}$ to a one-dimensional problem. Since $s_0 = 0$, we only need to find t_1^* to generate s_1^* by (4.1), t_2^* by (4.2), and then the rest of $\{s_i^*\}$ and $\{t_i^*\}$ uniquely by repeatedly using (4.1) and (4.2). For any chosen t_1^*, if $s_n^* = H$, then t_1^* is chosen correctly. Otherwise, we can easily find the optimal t_1^* by standard search techniques.

Having calculated the second partial derivatives of the function $TP(n, \{s_i\}, \{t_i\})$ shows that the Hessian matrix is negative definite if

$$\frac{\partial^2 TP}{\partial t_i^2} \leq -\left[\frac{\partial^2 TP}{\partial s_i \partial t_i} + \frac{\partial^2 TP}{\partial t_i \partial s_{i-1}} \right] < 0, \quad \text{for } i = 1, 2, \ldots, n, \tag{4.7}$$

$$\frac{\partial^2 TP}{\partial s_i^2} \leq -\left[\frac{\partial^2 TP}{\partial s_i \partial t_i} + \frac{\partial^2 TP}{\partial t_{i+1} \partial s_i} \right] < 0, \quad \text{for } i = 1, 2, \ldots, n. \tag{4.8}$$

Theorem 4.3. *For any given n, if $R_{t_i}(t_i, t) > 0$, with $t \geq t_i$, and $P_{t_i}(t, t_i) < 0$, with $t \leq t_i$, $i = 1, 2, \ldots, n$, under conditions (4.7)-(4.8), then the solution that satisfies the system of (4.1) and (4.2) is a global maximum solution.*

Proof. See Appendix C.　　　　　　　　　　　　　　　　　　　□

Next, we show that the total profit $TP(n, \{s_i^*\}, \{t_i^*\})$ is a concave function of the number of replenishments. As a result, the search for the optimal replenishment number, n^*, is reduced to find a local maximum. For simplicity, let

$$TP(n) = P(n, \{s_i^*\}, \{t_i^*\}). \tag{4.9}$$

By applying Bellman's principle of optimality [12], we have the following theorem:

Theorem 4.4. $TP(n)$ *is concave in* n.

Proof. The proof is similar to that of Teng and Yang [11], the reader can easily prove it.   □

By a similar discussion as in Teng and Yang [11], I here use the average backlogging rate $\overline{\beta}$, unit purchasing cost $\overline{c_v}$ and average unit selling price $\overline{p}$ to replace $\beta(t_{i+1} - s_i)$, $c_v(t_i)$ and $p(t_i)$, respectively. the estimate of the number of replenishments is obtained as

$$n_1 = \text{ rounded integer of } \left[\frac{(c_h + \theta\overline{c_v})\left[c_b\overline{\beta} + (c_l - \overline{c_v} + \overline{p})\left(1 - \overline{\beta}\right)\right]HQ(H)}{2c_f\left[c_h + \theta\overline{c_v} + c_b\overline{\beta} + (c_l - \overline{c_v} + \overline{p})\left(1 - \overline{\beta}\right)\right]} \right]^{1/2}, \tag{4.10}$$

where $Q(H) = \int_0^H f(t)dt$. It is obvious that searching for n^* by starting with n in (4.10) will speed the computational efficiency significantly, comparing to starting with $n = 1$. The algorithm for determining the optimal number of replenishments n^* and schedule is summarized as follows.

Algorithm for Finding Optimal Number and Schedule

Step 1. Choose two initial trial values of n^*, say n as in (4.10) and $n - 1$. Use a standard search method to obtain $\{t_i^*\}$ and $\{s_i^*\}$, and compute the corresponding $TP(n)$ and $TP(n-1)$, respectively.

Step 2. If $TP(n) \geq TP(n-1)$, then compute $TP(n+1), TP(n+2), \ldots,$ until we find $TP(k) > TP(k+1)$. Set $n^* = k$ and stop.

Step 3. If $TP(n) < TP(n-1)$, then compute $TP(n-2), TP(n-3), \ldots,$ until we find $TP(k) > TP(k-1)$. Set $n^* = k$ and stop.

5. Numerical Examples

Example 5.1. Let $f(t) = 200 + 20t$, $H = 3$, $p(t) = 200 + 30t$, $c_v(t) = 150 + 10t$, $c_f = 250$, $c_h = 40$, $c_b = 50$, $c_l = 200$, $\theta = 0.08$, $\beta(t) = e^{-0.4t}$ in appropriate units. After calculation, we have $\overline{p} = 245$, $\overline{c_v} = 165$, and $\overline{\beta} = 0.582$. By (4.10), we obtain the estimate number of replenishments $n_1 = 12$. From computational results, we have $TP(13) = 49021.79, TP(14) = 49044.31$, and $TP(15) = 49030.61$. Therefore, the optimal number of replenishments is 14, and the optimal profit is 49044.31. The optimal replenishment schedule is shown in Table 2.

Table 2: The optimal replenishment schedule for Example 5.1.

i	1	2	3	4	5	6	7
t_i	0.0849	0.3160	0.5437	0.7683	0.9899	1.2086	1.4245
s_i	0.2330	0.4626	0.6889	0.9121	1.1323	1.3497	1.5644
$p(t_i)$	202.55	209.48	216.31	223.05	229.70	236.26	242.73
$c_v(t_i)$	150.85	153.16	155.44	157.68	159.90	162.09	164.24
i	8	9	10	11	12	13	14
t_i	1.6378	1.8485	2.0569	2.2629	2.4666	2.6682	2.8676
s_i	1.7765	1.9861	2.1933	2.3982	2.6009	2.8015	3.0000
$p(t_i)$	249.13	255.46	261.71	267.89	274.00	280.05	286.03
$c_v(t_i)$	166.38	168.49	170.57	172.63	174.67	176.68	178.68

Table 3: Sensitivity analysis on parameters changed for Example 5.2.

Parameter	Parameter value	% change in parameter	Estimated n_1	Optimal n^*	$TP^*(n^*)$	% change in TP^* (n^*)
c_f	200	−20	14	16	49787.47	1.52
	300	+20	11	13	48371.79	−1.37
c_h	30	−25	12	13	49379.80	0.68
	50	+25	13	15	48748.20	−0.60
c_b	40	−20	12	14	49135.56	0.19
	60	+20	13	14	48961.05	−0.17
c_l	150	−25	12	14	49238.33	0.40
	250	+25	13	14	48883.42	−0.33
θ	0.06	−25	12	14	49152.16	0.22
	0.1	+25	13	14	48939.71	−0.21
$\beta(t)$	$e^{-0.6t}$	−50	13	15	48673.26	−0.76
	$e^{-0.2t}$	+50	12	13	49719.47	1.38

Example 5.2. To understand the effect of changes in parameters c_f, c_h, c_b, c_l, θ, $\beta(t)$ on the optimal solution, the sensitivity analysis is performed by changing one parameter at a time and keeps the others unchanged. The parameter values are the same as in Example 5.1. The results obtained are shown as in Table 3.

From Table 3, the following phenomena can be obtained.

(1) The optimal maximum profit decreases as c_f, c_h, c_b, c_l, θ increases, however, it increases as the backlogging rate $\beta(t)$ increases.

(2) The optimal maximum profit is more sensitive on parameters c_f than others.

(3) The optimal replenishment number is very slightly sensitive to the change of these parameters except c_f and $\beta(t)$.

(4) The estimated number n_1 is very close to the optimal replenishment number n^*, no matter what magnitude of the parameters changed.

Example 5.3. Using the same numerical values as in Example 5.1, we consider the influence of changes of the rate of change of selling price $p(t)$ and purchasing cost $c_v(t)$ on the total profit. The results are obtained as shown in Table 4.

Table 4: Sensitivity analysis on rate of change for **Example 5.3**.

$p(t)$, $c_v(t)$	% change in rate of change	Estimated n_1	Optimal n^*	TP^* (n^*)	% change in TP^* (n^*)
$p(t) = 200 + 30t$	—	12	14	49044.31	0.00
$c_v(t) = 150 + 10t$	—				
$p(t) = 200 + 45t$	+50	13	15	64887.92	32.30
$c_v(t) = 150 + 10t$	—				
$p(t) = 200 + 15t$	−50	12	14	33314.34	−32.07
$c_v(t) = 150 + 10t$	—				
$p(t) = 200 + 30t$	—	12	14	43768.8	−10.76
$c_v(t) = 150 + 15t$	+50				
$p(t) = 200 + 30t$	—	12	14	54329.55	10.78
$c_v(t) = 150 + 5t$	−50				

From Table 4, it is obviously that the phenomena are obtained.

(1) The percentage change in total maximum profit is significantly sensitive on the variation of rate of change.

(2) The total profit increases as the rate of change of selling price increases, while decreases as the rate of change of purchasing cost increases.

(3) The estimated number n_1 is also close to the optimal replenishment number n^*.

(4) The optimal replenishment number is slightly sensitive to the change of rate of change.

6. Conclusions

In this paper, a partial-backlogging inventory lot-size model for deteriorating items with time dependent demand, fluctuating selling price and purchasing cost is considered. We show that the optimal replenishment schedule exists uniquely and the total profit associated with the inventory system is a concave function of the number of replenishments. An intuitively estimate for finding the optimal replenishment number is provided. From the sensitivity analysis, the results indicate that the effect of variation of rate of change of the selling price and purchasing cost on the system behavior is significant. Thus, to incorporate the selling price and purchasing cost into the inventory model is vital, especially in the present changeable market.

With more practical features, the model developed here can be further extended by incorporating with other kind of functions or parameters, such as considering the demand as a function of selling price or stock dependent, or time varying deterioration rate, or to develop the model by employing other factors, such as inflation and price discount.

Appendices

A. Proof of Lemma 4.1

Let

$$
\begin{aligned}
TP_i(s_{i-1}, t_i, s_i) &= \int_{s_{i-1}}^{t_i} \big\{ [p(t_i) - c_v(t_i) - c_b(t_i - t) + c_l]\beta(t_i - t) - c_l \big\} f(t)dt \\
&\quad + \int_{t_i}^{s_i} \left\{ p(t_i) - c_v(t_i) - \left[\frac{c_h}{\theta} + c_v(t_i)\right] \left(e^{\theta(t-t_i)} - 1\right) \right\} f(t)dt \\
&= \int_{s_{i-1}}^{t_i} P(t, t_i)f(t)dt + \int_{t_i}^{s_i} R(t_i, t)f(t)dt.
\end{aligned}
\tag{A.1}
$$

We then have

$$
\frac{\partial TP_i}{\partial t_i} = \int_{s_{i-1}}^{t_i} P_{t_i}(t, t_i)f(t)dt + \int_{t_i}^{s_i} R_{t_i}(t_i, t)f(t)dt.
\tag{A.2}
$$

If $R_{t_i}(t_i, t) \leq 0$, then we know from (A.2) that $\partial TP_i / \partial t_i \leq 0$. Therefore, for any given i, TP_i is decreasing with t_i. This implies that $TP_i(s_{i-1}, s_{i-1}, s_i) \geq TP_i(s_{i-1}, t_i, s_i)$ for any fixed i. Consequently, we obtain

$$
\begin{aligned}
TP(n, \{s_i\}, \{t_i\}) &= \sum_{i=1}^{n} TP_i(s_{i-1}, t_i, s_i) - nc_f \\
&\leq \sum_{i=1}^{n} TP_i(s_{i-1}, s_{i-1}, s_i) - nc_f = \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} R(s_{i-1}, t)f(t)dt - nc_f \\
&\leq \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} R(0, t)f(t)dt - nc_f \quad (\text{since } R_{t_i}(t_i, t) \leq 0) \\
&\leq \int_{0}^{H} R(0, t)f(t)dt - c_f.
\end{aligned}
\tag{A.3}
$$

This completes the proof.

B. Proof of Theorem 4.2

For any given s_{i-1} and t_i, from (4.1), we set

$$
\begin{aligned}
F(x) &= \int_{t_i}^{x} \left\{ p'(t_i) + [c_h + \theta c_v(t_i) - c_v'(t_i)]e^{\theta(t-t_i)} \right\} f(t)dt \\
&\quad + \int_{s_{i-1}}^{t_i} \big\{ [p'(t_i) - c_v'(t_i) - c_b]\beta(t_i - t) + [p(t_i) + c_l - c_v(t_i) - c_b(t_i - t)]\beta'(t_i - t) \big\} f(t)dt \\
&= \int_{t_i}^{x} R_{t_i}(t_i, t)f(t)dt + \int_{s_{i-1}}^{t_i} P_{t_i}(t, t_i)f(t)dt, \quad \text{with } x \geq t_i \geq s_{i-1}.
\end{aligned}
\tag{B.1}
$$

We then have

$$F(t_i) = \int_{s_{i-1}}^{t_i} P_{t_i}(t, t_i) f(t) dt < 0, \tag{B.2}$$

and $\lim_{x \to \infty} F(x) > 0$. Taking the first derivatives of $F(x)$ with respect to x, we obtain

$$F'(x) = R_{t_i}(t_i, x) f(x) > 0. \tag{B.3}$$

As a result, we know that there exists a unique s_i ($> t_i$) such that $F(s_i) = 0$. Thus, the solution to (4.1) uniquely exists. Similarly, from (4.2), we set

$$G(x) = \left[p(x) - c_v(x) - c_b(x - s_i) + c_l \right] \beta(x - s_i) - \left[p(t_i) - c_v(t_i) + c_l \right]$$
$$+ \left[\frac{c_h}{\theta} + c_v(t_i) \right] \left(e^{\theta(s_i - t_i)} - 1 \right) = P(s_i, x) - R(t_i, s_i), \quad \text{with } x \geq s_i \geq t_i. \tag{B.4}$$

We then have

$$G(s_i) = p(s_i) - c_v(s_i) - R(t_i, s_i) = R(s_i, s_i) - R(t_i, s_i) > 0, \tag{B.5}$$

since $R_{t_i}(t_i, t) > 0$, and $\lim_{x \to \infty} G(x) = -c_l < 0$. By taking the first derivatives of $G(x)$ with respect to x, we obtain

$$G'(x) = P_x(s_i, x) < 0. \tag{B.6}$$

Consequently, there exists a unique t_{i+1} ($> s_i$) such that $G(t_{i+1}) = 0$, which implies that solution to (4.2) uniquely exists. Therefore, we complete the proof.

C. Proof of Theorem 4.3

Taking the second derivatives with respect to t_i and s_i on $TP(n, \{s_i\}, \{t_i\})$, we have

$$\frac{\partial^2 TP}{\partial t_i^2} = \left[(p(t_i) + c_l - c_v(t_i)) \beta'(0) - c_h - \theta c_v(t_i) - c_b \right] f(t_i)$$

$$+ \int_{s_{i-1}}^{t_i} \left\{ \left[p''(t_i) - c_v''(t_i) \right] \beta(t_i - t) + 2 \left[p'(t_i) - c_v'(t_i) - c_b \right] \beta'(t_i - t) \right.$$

$$\left. + \left[p(t_i) + c_l - c_v(t_i) - c_b(t_i - t) \right] \beta''(t_i - t) \right\} f(t) dt$$

$$+ \int_{t_i}^{s_i} \left\{ p''(t_i) + [\theta c_v'(t_i) - c_v''(t_i) - \theta(c_h + \theta c_v(t_i) - c_v'(t_i))]e^{\theta(t-t_i)} \right\} f(t)dt,$$

$$\frac{\partial^2 TP}{\partial s_{i-1} \partial t_i} = -P_{t_i}(s_{i-1}, t_i) f(s_{i-1}) > 0,$$

$$\frac{\partial^2 TP}{\partial t_i \partial s_i} = R_{t_i}(t_i, s_i) f(s_i) > 0,$$

$$\frac{\partial^2 TP}{\partial s_i^2} = \left\{ [p(t_{i+1}) + c_l - c_v(t_{i+1}) - c_b(t_{i+1} - s_i)]\beta'(t_{i+1} - s_i) - c_b \beta(t_{i+1} - s_i) \right.$$

$$\left. - [c_h + \theta c_v(t_i)]e^{\theta(s_i - t_i)} \right\} f(s_i).$$

$$(C.1)$$

Let Δ_k be the principal minor of order k, then, under condition (4.7)-(4.8), it is clear that

$$\Delta_1 = \frac{\partial^2 TP}{\partial t_1^2} \leq -\frac{\partial^2 TP}{\partial s_1 \partial t_1} = -R_{t_1}(t_1, s_1) f(s_1) < 0, \qquad (C.2)$$

which implies that $\Delta_1 + (\partial^2 TP / \partial s_1 \partial t_1) < 0$:

$$\Delta_2 = \frac{\partial^2 TP}{\partial t_1^2} \frac{\partial^2 TP}{\partial s_1^2} - \frac{\partial^2 TP}{\partial s_1 \partial t_1} \frac{\partial^2 TP}{\partial t_1 \partial s_1} \geq -\frac{\partial^2 TP}{\partial t_2 \partial s_1} \Delta_1 - \frac{\partial^2 TP}{\partial t_1 \partial s_1} \left(\Delta_1 + \frac{\partial^2 TP}{\partial t_1 \partial s_1} \right) > 0, \qquad (C.3)$$

which implies that $\Delta_2 + (\partial^2 TP / \partial t_2 \partial s_1)\Delta_1 > 0$. For principal minor of higher order, $i = 2, 3, \ldots,$ it is not difficult to show that they satisfy the following recursive relation:

$$\Delta_{2i-1} = \frac{\partial^2 TP}{\partial t_i^2} \Delta_{2i-2} - \left[\frac{\partial^2 TP}{\partial t_i \partial s_{i-1}} \right]^2 \Delta_{2i-3},$$

$$\Delta_{2i} = \frac{\partial^2 TP}{\partial s_i^2} \Delta_{2i-1} - \left[\frac{\partial^2 TP}{\partial s_i \partial t_i} \right]^2 \Delta_{2i-2},$$

$$(C.4)$$

with the initial $\Delta_0 = 1$. From (4.7)-(4.8) and the relation between second-order partial derivatives, we have

$$\Delta_{2i-1} \leq -\frac{\partial^2 TP}{\partial s_i \partial t_i} \Delta_{2i-2} - \frac{\partial^2 TP}{\partial t_i \partial s_{i-1}} \left(\Delta_{2i-2} + \frac{\partial^2 TP}{\partial t_i \partial s_{i-1}} \Delta_{2i-3} \right),$$

$$\Delta_{2i} \geq -\frac{\partial^2 TP}{\partial t_{i+1} \partial s_i} \Delta_{2i-1} - \frac{\partial^2 TP}{\partial s_i \partial t_i} \left(\Delta_{2i-1} + \frac{\partial^2 TP}{\partial s_i \partial t_i} \Delta_{2i-2} \right).$$

$$(C.5)$$

For $i = 2$ in (C.5), we obtain

$$
\begin{aligned}
\Delta_3 + \frac{\partial^2 TP}{\partial s_2 \partial t_2} \Delta_2 \leq -\frac{\partial^2 TP}{\partial t_2 \partial s_1} \left(\Delta_2 + \frac{\partial^2 TP}{\partial t_2 \partial s_1} \Delta_1 \right) < 0, \\
\Delta_4 + \frac{\partial^2 TP}{\partial t_3 \partial s_2} \Delta_3 \geq -\frac{\partial^2 TP}{\partial s_2 \partial t_2} \left(\Delta_3 + \frac{\partial^2 TP}{\partial s_2 \partial t_2} \Delta_2 \right) > 0.
\end{aligned}
\tag{C.6}
$$

Thus,

$$
\Delta_3 < -\frac{\partial^2 TP}{\partial s_2 \partial t_2} \Delta_2 < 0, \qquad \Delta_4 > -\frac{\partial^2 TP}{\partial t_3 \partial s_2} \Delta_3 > 0.
\tag{C.7}
$$

Proceeding inductively, we have

$$
\Delta_{2i-1} + \frac{\partial^2 TP}{\partial s_i \partial t_i} \Delta_{2i-2} < 0, \qquad \Delta_{2i} + \frac{\partial^2 TP}{\partial t_{i+1} \partial s_i} \Delta_{2i-1} > 0.
\tag{C.8}
$$

Therefore, $\Delta_{2i-1} < 0$ and $\Delta_{2i} > 0$, for $i = 2, 3, \ldots$. This completes the proof.

Acknowledgments

References

[1] H. L. Lee, V. Padmanabhan, T. A. Taylor, and S. Whang, "Price protection in the personal computer industry," *Management Science*, vol. 46, no. 4, pp. 467–482, 2000.

[2] J. Heizer and B. Render, *Operations Management*, Prentice-Hall, Upper Saddle River, NJ, USA, 6th edition, 2000.

[3] P. L. Abad, "Optimal pricing and lot-sizing under conditions of perishability and partial backordering," *Management Science*, vol. 42, no. 8, pp. 1093–1104, 1996.

[4] J.-T. Teng, H.-J. Chang, C.-Y. Dye, and C.-H. Hung, "An optimal replenishment policy for deteriorating items with time-varying demand and partial backlogging," *Operations Research Letters*, vol. 30, no. 6, pp. 387–393, 2002.

[5] H.-J. Chang, J.-T. Teng, L.-Y. Ouyang, and C.-Y. Dye, "Retailer's optimal pricing and lot-sizing policies for deteriorating items with partial backlogging," *European Journal of Operational Research*, vol. 168, no. 1, pp. 51–64, 2006.

[6] J.-T. Teng and H.-L. Yang, "Deterministic inventory lot-size models with time-varying demand and cost under generalized holding costs," *International Journal of Information and Management Sciences*, vol. 18, no. 2, pp. 113–125, 2007.

[7] P. L. Abad, "Optimal price and order size under partial backordering incorporating shortage, backorder and lost sale costs," *International Journal of Production Economics*, vol. 114, no. 1, pp. 179–186, 2008.

[8] S. S. Sana, "Optimal selling price and lotsize with time varying deterioration and partial backlogging," *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 185–194, 2010.

[9] M. Das Roy, S. S. Sana, and K. Chaudhuri, "An optimal shipment strategy for imperfect items in a stock-out situation," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2528–2543, 2011.

[10] M. Das Roy, S. S. Sana, and K. Chaudhuri, "An economic order quantity model of imperfect quality items with partial backlogging," *International Journal of Systems Science. Principles and Applications of Systems and Integration*, vol. 42, no. 8, pp. 1409–1419, 2011.

[11] J. T. Teng and H. L. Yang, "Deterministic economic order quantity models with partial backlogging when demand and cost are fluctuating with time," *Journal of the Operational Research Society*, vol. 55, no. 5, pp. 495–503, 2004.

[12] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ, USA, 1957.

Advances in
Operations Research

Advances in
Decision Sciences

Mathematical Problems
in Engineering

Algebra

Journal of
Probability and Statistics

The Scientific
World Journal

International Journal of
Differential Equations

International Journal of
Combinatorics

Advances in
Mathematical Physics

Hindawi

Submit your manuscripts at
http://www.hindawi.com

Journal of
Complex Analysis

Journal of
Mathematics

International Journal of
Stochastic Analysis

Abstract and
Applied Analysis

Discrete Dynamics in
Nature and Society

International
Journal of
Mathematics and
Mathematical
Sciences

Journal of
Discrete Mathematics

Journal of
Function Spaces

Journal of
Applied Mathematics

Journal of
Optimization