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Research Article

Generalized Differentiable E-Invex Functions and Their Applications in Optimization

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The concept of E-convex function and its generalizations is studied with differentiability assumption. Generalized differentiable E-convexity and generalized differentiable E-invexity are used to derive the existence of optimal solution of a general optimization problem.

1. Introduction

E-convex function was introduced by Youness [1] and revised by Yang [2]. Chen [3] introduced Semi-E-convex function and studied some of its properties. Syau and Lee [4] defined E-quasi-convex function, strictly E-quasi-convex function and studied some basic properties. Fulga and Preda [5] introduced the class of E-preinvex and E-prequasi-invex functions. All the above E-convex and generalized E-convex functions are defined without differentiability assumptions. Since last few decades, generalized convex functions like quasiconvex, pseudoconvex, invex, B-vex, (p, r)-invex, and so forth, have been used in nonlinear programming to derive the sufficient optimality condition for the existence of local optimal point. Motivated by earlier works on convexity and E-convexity, we have introduced the concept of differentiable E-convex function and its generalizations to derive sufficient optimality condition for the existence of local optimal solution of a nonlinear programming problem. Some preliminary definitions and results regarding E-convex function are discussed below, which will be needed in the sequel. Throughout this paper, we consider functions $E : R^n \to R^n$, $f : M \to R$, and M are nonempty subset of R^n.

Definition 1.1 (see [1]). M is said to be E-convex set if $(1-\lambda)E(x) + \lambda E(y) \in M$ for $x, y \in M$, $\lambda \in [0, 1]$.

Definition 1.2 (see [1]). $f : M \to R$ is said to be E-convex on M if M is an E-convex set and for all $x, y \in M$ and $\lambda \in [0,1]$,

$$f\left((1-\lambda)E(x) + \lambda E(y)\right) \leq (1-\lambda)f(E(x)) + \lambda f\left(E(y)\right). \tag{1.1}$$

Definition 1.3 (see [3]). Let M be an E-convex set. f is said to be semi-E-convex on M if for $x, y \in M$ and $\lambda \in [0,1]$,

$$f\left(\lambda E(x) + (1-\lambda)E(y)\right) \leq \lambda f(x) + (1-\lambda)f(y). \tag{1.2}$$

Definition 1.4 (see [5]). M is said to be E-invex with respect to $\eta : R^n \times R^n \to R^n$ if for $x, y \in M$ and $\lambda \in [0,1]$, $E(y) + \lambda\eta(E(x), E(y)) \in M$.

Definition 1.5 (see [6]). Let M be an E-invex set with respect to $\eta : R^n \times R^n \to R^n$. Also $f : M \to R$ is said to be E-preinvex with respect to η on M if for $x, y \in M$ and $\lambda \in [0,1]$,

$$f\left(E(y) + \lambda\eta\left(E(x), E(y)\right)\right) \leq \lambda f(E(x)) + (1-\lambda)f\left(E(y)\right). \tag{1.3}$$

Definition 1.6 (see [7]). Let M be an E-invex set with respect to $\eta : R^n \times R^n \to R^n$. Also $f : M \to R$ is said to be semi-E-invex with respect to η at $y \in M$ if

$$f\left(E(y) + \lambda\eta\left(E(x), E(y)\right)\right) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1.4}$$

for all $x \in M$ and $\lambda \in [0,1]$.

Definition 1.7 (see [7]). Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, $E : R^n \to R^n$. Let $f : M \to R$ and $E(M)$ be an open set in R^n. Also f and E are differentiable on M. Then, f is said to be semi-E-quasiinvex at $y \in M$ if

$$f(x) \leq f(y) \quad \forall x \in M \Longrightarrow \left(\nabla(f \circ E)(y)\right)^T \eta\left(E(x), E(y)\right) \leq 0, \tag{1.5}$$

or

$$\left(\nabla(f \circ E)(y)\right)^T \eta\left(E(x), E(y)\right) > 0 \quad \forall x \in M \Longrightarrow f(x) > f(y). \tag{1.6}$$

Lemma 1.8 (see [1]). *If a set $M \subseteq R^n$ is E-convex, then $E(M) \subseteq M$.*

Lemma 1.9 (see [5]). *If M is E-invex, then $E(M) \subseteq M$.*

Lemma 1.10 (see [5]). *If $\{M_i\}_{i\in I}$ is a collection of E-invex sets and $M_i \subseteq R^n$, for all $i \in I$, then $\cap_{i\in I} M_i$ is E-invex.*

2. E-Convexity and Its Generalizations with Differentiability Assumption

E-convexity and convexity are different from each other in several contests. From the previous results on E-convex functions, as discussed by our predecessors, one can observe the following relations between E-convexity and convexity.

(1) All convex functions are E-convex but all E-convex functions are not necessarily convex. (In particular, E-convex function reduces to convex function in case $E(x) = x$ for all x in the domain of E.)

(2) A real-valued function on R^n may not be convex on a subset of R^n, but E-convex on that set.

(3) An E-convex function may not be convex on a set M but E-convex on $E(M)$.

(4) It is not necessarily true that if M is an E-convex set then $E(M)$ is a convex set.

In this section we study E-convex and generalized E-convex functions with differentiability assumption.

2.1. Some New Results on E-Convexity with Differentiability

E-convexity at a point may be interpreted as follows.

Let M be a nonempty subset of R^n, $E : R^n \to R^n$. A function $f : M \to R$ is said to be E-convex at $\overline{x} \in M$ if M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(\overline{x})) \le \lambda(f \circ E)(x) + (1-\lambda)(f \circ E)(\overline{x}) \tag{2.1}$$

for all $x \in N_\delta(\overline{x})$ and $\lambda \in [0,1]$, where $N_\delta(\overline{x})$ is δ-neighborhood of $\overline{x}$, for small $\delta > 0$.

It may be observed that a function may not be convex at a point but E-convex at that point with a suitable mapping E.

Example 2.1. Consider $M = \{(x,y) \in R^2 \mid y \ge 0\}$. $E : R^2 \to R^2$ is $E(x,y) = (0,y)$ and $f(x,y) = x^3 + y^2$. Also f is not convex at $(-1,1)$. For all $(x,y) \in N_\delta(-1,1)$, $\delta > 0$, and $\lambda \in [0,1]$, $f(\lambda E(x,y)+(1-\lambda)E(-1,1))-\lambda(f\circ E)(x,y)-(1-\lambda)(f\circ E)(-1,1) = -\lambda(1-\lambda)(y-1)^2 \le 0$. Hence, f is E-convex at $(-1,1)$.

Proposition 2.2. *Let $M \subseteq R^n$, $E : R^n \to R^n$ be a homeomorphism. If $f : M \to R$ attains a local minimum point in the neighborhood of $E(\overline{x})$, then it is E-convex at $\overline{x}$.*

Proof. Suppose f has a local minimum point in a neighborhood $N_\epsilon(E(\overline{x}))$ of $E(\overline{x})$ for some $\overline{x} \in M$, $\epsilon > 0$. This implies f is convex on $N_\epsilon(E(\overline{x}))$. That is,

$$f(\lambda z + (1-\lambda)E(\overline{x})) \le \lambda f(z) + (1-\lambda)f(E(\overline{x})) \quad \forall z \in N_\epsilon(E(\overline{x})). \tag{2.2}$$

Since $E : R^n \to R^n$ is a homeomorphism, so inverse of the neighborhood $N_\epsilon(E(\overline{x}))$ is a neighborhood of $\overline{x}$ say $N_\delta(\overline{x})$ for some $\delta > 0$. Hence, there exists $x \in N_\delta(\overline{x})$ such that $E(x) = z$, $E(x) \in N_\epsilon(E(\overline{x}))$. Replacing z by $E(x)$ in the above inequality, we conclude that f is E-convex at $\overline{x}$. □

In the above discussion, it is clear that if a local minimum exists in a neighborhood of $E(\overline{x})$, then f is E-convex at $\overline{x}$. But it is not necessarily true that if f is E-convex at $\overline{x}$ then $E(\overline{x})$ is local minimum point. Consider the above example where f is E-convex at $(-1,1)$ but $E(-1,1)$ is not local minimum point of f.

Theorem 2.3. *Let M be an open E-convex subset of R^n, f and E are differentiable functions, and let E be a homeomorphism. Then, f is E-convex at $\overline{x} \in M$ if and only if*

$$(f \circ E)(x) \geq (f \circ E)(\overline{x}) + \left(\nabla (f \circ E)(\overline{x})\right)^T (E(x) - E(\overline{x})) \tag{2.3}$$

for all $E(x) \in N_\epsilon(E(\overline{x}))$ where $N_\epsilon(E(\overline{x}))$ is ϵ-neighborhood of $E(\overline{x})$, $\epsilon > 0$.

Proof. Since M is an E-convex set, by Lemma 1.8, $E(M) \subseteq M$. Also, $E(M)$ is an open set as E is a homeomorphism. Hence, there exists $\epsilon > 0$ such that $E(x) \in N_\epsilon(E(\overline{x}))$ for all $x \in N_\delta(\overline{x})$, $\delta > 0$, very small. So, f is differentiable on $E(M)$. Using expansion of f at $E(\overline{x})$ in the neighborhood $N_\epsilon(E(\overline{x}))$,

$$\begin{aligned} f(E(\overline{x}) + \lambda(z - E(\overline{x}))) = {} & f(E(\overline{x})) + \lambda \nabla f(E(\overline{x}))^T (z - E(\overline{x})) \\ & + \alpha[E(\overline{x}), \lambda(z - E(\overline{x}))] \lambda \|z - E(\overline{x})\|, \end{aligned} \tag{2.4}$$

where $z \in N_\epsilon(E(\overline{x}))$ and $\lim_{\lambda \to 0} \alpha[E(\overline{x}), \lambda(z - E(\overline{x}))] = 0$. Since f is E-convex at $\overline{x} \in M$, so for all $x \in N_\delta(\overline{x})$, $\lambda \in (0,1]$, $x \neq \overline{x}$,

$$\lambda\left((f \circ E)(x) - (f \circ E)(\overline{x})\right) \geq f(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) - (f \circ E)(\overline{x}). \tag{2.5}$$

Since E is a homeomorphism, there exists $x \in N_\delta(\overline{x})$ such that $E(x) = z$. Replacing z by $E(x)$ in (2.4) and using above inequality, we get

$$\begin{aligned} (f \circ E)(x) - (f \circ E)(\overline{x}) \geq {} & \left(\nabla (f \circ E)(\overline{x})\right)^T (E(x) - E(\overline{x}) \\ & + \alpha[E(\overline{x}), \lambda(E(x) - E(\overline{x}))] \|E(x) - E(\overline{x})\|), \end{aligned} \tag{2.6}$$

where $\lim_{\lambda \to 0} \alpha[E(\overline{x}), \lambda(E(x) - E(\overline{x}))] = 0$. Hence, (2.3) follows.

The converse part follows directly from (2.4). □

It is obvious that if $E(\overline{x})$ is a local minimum point of f, then $\nabla(f \circ E)(\overline{x}) = 0$. The following result proves the sufficient part for the existence of local optimal solution, proof of which is easy and straightforward. We leave this to the reader.

Corollary 2.4. *Let $M \subseteq R^n$ be an open E-convex set, and let f be a differentiable E-convex function at $\overline{x}$. If $E : R^n \to R^n$ is a homeomorphism and $\nabla(f \circ E)(\overline{x}) = 0$, then $E(\overline{x})$ is the local minimum of f.*

2.2. Some New Results on Generalized E-Convexity with Differentiability

Here, we introduce some generalizations of E-convex function like semi-E-convex, E-invex, semi-E-invex, E-pseudoinvex, E-quasi-invex and so forth, with differentiability assumption and discuss their properties.

2.2.1. Semi-E-Convex Function

Chen [3] introduced a new class of semi-E-convex functions without differentiability assumption. Semi-E-convexity at a point may be understood as follows:

$f : M \to R$ is said to be semi-E-convex at $\overline{x} \in M$ if M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(\overline{x})) \leq \lambda f(x) + (1-\lambda) f(\overline{x}) \tag{2.7}$$

for all $x \in N_\delta(\overline{x})$ and $\lambda \in [0,1]$, where $N_\delta(\overline{x})$ is δ-neighborhood of $\overline{x}$.

The following result proves the necessary and sufficient condition for the existence of a semi-E-convex function at a point.

Theorem 2.5. *Suppose $f : M \to R$ and $E : R^n \to R^n$ are differentiable functions. Let E be a homeomorphism and let $\overline{x}$ be a fixed point of E. Then, f is semi-E-convex at $\overline{x} \in M$ if and only if*

$$f(x) \geq f(\overline{x}) + \left(\nabla \left(f \circ E\right)(\overline{x})\right)^T (E(x) - E(\overline{x})) \tag{2.8}$$

for all $E(x) \in N_\epsilon(E(\overline{x}))$, very small $\epsilon > 0$.

Proof. Proceeding as in Theorem 2.3, we get the following relation from the expansion of f at $E(\overline{x})$ in the neighborhood $N_\epsilon(E(\overline{x}))$, where $\overline{x}$ is the fixed point of E. (Since E is a homeomorphism, there exists $\epsilon > 0$ such that $E(x) \in N_\epsilon(E(\overline{x}))$ for all $x \in N_\delta(\overline{x})$, very small $\delta > 0$):

$$\begin{aligned} f(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) = {} & \left(f \circ E\right)(\overline{x}) + \lambda\left(\nabla\left(f \circ E\right)(\overline{x})\right)^T (E(x) - E(\overline{x})) \\ & + \alpha[E(\overline{x}), \lambda(E(x) - E(\overline{x}))]\lambda \|E(x) - E(\overline{x})\|, \end{aligned} \tag{2.9}$$

where $E(x) \in N_\epsilon(E(\overline{x}))$, $\lim_{\lambda \to 0} \alpha[E(\overline{x}), \lambda(E(x) - E(\overline{x}))] = 0$. Since f is semi-E-convex at $\overline{x} \in M$, and $\overline{x}$ is a fixed point of E, so, for all $x \in N_\delta(\overline{x})$, $\lambda \in (0,1]$, $x \neq \overline{x}$,

$$\lambda\left(f(x) - f(\overline{x})\right) \geq f(E(\overline{x}) + \lambda(E(x) - E(\overline{x}))) - \left(f \circ E\right)(\overline{x}). \tag{2.10}$$

Using (2.9), the above inequality reduces to

$$\begin{aligned} f(x) - f(\overline{x}) \geq {} & \left(\nabla\left(f \circ E\right)(\overline{x})\right)^T (E(x) - E(\overline{x}) \\ & + \alpha[E(\overline{x}), \lambda(E(x) - E(\overline{x}))] \|E(x) - E(\overline{x})\|), \end{aligned} \tag{2.11}$$

where $\lim_{\lambda \to 0} \alpha[E(\overline{x}), \lambda(E(x) - E(\overline{x}))] = 0$. Hence Inequality (2.8) follows for all $E(x) \in N_\epsilon(E(\overline{x}))$.

Conversely, suppose Inequality (2.8) holds at the fixed point $\overline{x}$ of E for all $E(x) \in N_\epsilon(E(\overline{x}))$. Using (2.9) and $E(\overline{x}) = \overline{x}$ in (2.8), we can conclude that f is semi-E-convex at $\overline{x} \in M$. □

2.2.2. Generalized E-Invex Function

The class of preinvex functions defined by Ben-Israel and Mond is not necessarily differentiable. Preinvexity, for the differential case, is a sufficient condition for invexity. Indeed, the converse is not generally true. Fulga and Preda [5] defined E-invex set, E-preinvex function, and E-prequasiinvex function where differentiability is not required (Section 1). Chen [3] introduced semi-E-convex, semi-E-quasiconvex, and semi-E-pseudoconvex functions without differentiability assumption. Jaiswal and Panda [7] studied some generalized E-invex functions and applied these concepts to study primal dual relations. Here, we define some more generalized E-invex functions with and without differentiability assumption, which will be needed in next section. First, we see the following lemma.

Lemma 2.6. *Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$. Also $f : M \to R$ are differentiable on M. $E(M)$ is an open set in R^n. If f is E-preinvex on M then $(f \circ E)(x) \geq (f \circ E)(y) + (\nabla(f \circ E)(y))^T \eta(E(x), E(y))$ for all $x, y \in M$.*

Proof. If $E(M)$ is an open set, f and E are differentiable on M, then $f \circ E$ is differentiable on M. From Taylor's expansion of f at $E(y)$ for some $y \in M$ and $\lambda > 0$,

$$\begin{aligned} f(E(y) + \lambda\eta(E(x), E(y))) = (f \circ E)(y) + \lambda(\nabla(f \circ E)(y))^T(\eta(E(x), E(y))) \\ + \lambda\|\eta(E(x), E(y))\|\alpha(E(y), \lambda\eta(E(x), E(y))), \end{aligned} \tag{2.12}$$

where $E(x) \neq E(y)$, $\lim_{\lambda \to 0} \alpha(E(y), \lambda\eta(E(x), E(y))) = 0$.

If f is E-preinvex on M with respect to η (Definition 1.5), then as $\lambda \to 0^+$, the above inequality reduces to $(f \circ E)(x) \geq (f \circ E)(y) + (\nabla(f \circ E)(y))^T \eta(E(x), E(y))$ for all $x, y \in M$. □

As a consequence of the above lemma, we may define E-invexity with differentiability assumption as follows.

Definition 2.7. Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$. Also $f : M \to R$ are differentiable on M. $E(M)$ is an open set in R^n. Then, f is E-invex on M if $(f \circ E)(x) \geq (f \circ E)(y) + (\nabla(f \circ E)(y))^T \eta(E(x), E(y))$ for all $x, y \in M$.

From the above discussions on E-invexity and E-preinvexity, it is true that E-preinvexity with differentiability is a sufficient condition for E-invexity. Also a function which is not E-convex may be E-invex with respect to some η. This may be verified in the following example.

Example 2.8. $M = \{(x, y) \in R^2 \mid x, y > 0\}$, $E : R^2 \to R^2$ is $E(x, y) = (0, y)$ and $f : M \to R$ is defined by $f(x, y) = -x^2 - y^2$, and

$$\eta((x_1, y_1), (x_2, y_2)) = \begin{cases} \left(\dfrac{x_1^2}{2x_2}, \dfrac{y_1^2}{2y_2}\right), & x_2 \neq 0,\ y_2 \neq 0, \\ \left(0, \dfrac{y_1^2}{2y_2}\right), & x_2 = 0,\ y_2 \neq 0, \\ \left(\dfrac{x_1^2}{2x_2}, 0\right), & x_2 \neq 0,\ y_2 = 0, \\ (0, 0), & \text{otherwise}. \end{cases} \tag{2.13}$$

Definition 2.9. Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, let $E(M)$ be an open set in R^n. Suppose f and E are differentiable on M. Then, f is said to be E-quasiinvex on M if

$$(f \circ E)(x) \leq (f \circ E)(y) \quad \forall x, y \in M \Longrightarrow (\nabla (f \circ E)(y))^T \eta(E(x), E(y)) \leq 0 \tag{2.14}$$

or

$$(\nabla (f \circ E)(y))^T \eta(E(x), E(y)) > 0 \Longrightarrow (f \circ E)(x) > (f \circ E)(y). \tag{2.15}$$

A function may not be E-invex with respect to some η but E-quasiinvex with respect to same η. This may be justified in the following example.

Example 2.10. Consider $M = \{(x, y) \in R^2 \mid x, y < 0\}$, $E : R^2 \to R^2$ is $E(x, y) = (0, y)$, and $f : M \to R$ is $f(x, y) = x^3 + y^3$, $\eta : R^2 \times R^2 \to R^2$ is $\eta((x_1, y_1), (x_2, y_2)) = (x_1 - x_2, y_1 - y_2)$. Now for all $(x_1, y_1), (x_2, y_2) \in M$, $(f \circ E)(x_1, y_1) - (f \circ E)(x_2, y_2) - \nabla (f \circ E)(x_2, y_2)^T \eta(E(x_1, y_1), E(x_2, y_2)) = y_1^3 + y_2^3 - 3y_2^2(y_1 - y_2)$, which is not always positive. Hence, f is not E-invex with respect to η on M.

If we assume that $(f \circ E)(x_1, y_1) \leq (f \circ E)(x_2, y_2)$ for all $(x_1, y_1), (x_2, y_2) \in M$, then $(\nabla (f \circ E)(x_2, y_2))^T \eta(E(x_1, y_1), E(x_2, y_2)) = 3y_2^2(y_1 - y_2) \leq 0$. Hence, f is E-quasiinvex with respect to same η on M.

Definition 2.11. Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$, let $E(M)$ be an open set in R^n. Suppose f and E are differentiable on M. Then, f is said to be E-pseudoinvex on M if

$$(\nabla (f \circ E)(y))^T \eta(E(x), E(y)) \geq 0 \quad \forall x, y \in M \Longrightarrow (f \circ E)(x) \geq (f \circ E)(y) \tag{2.16}$$

or

$$(f \circ E)(x) < (f \circ E)(y) \quad \forall x, y \in M \Longrightarrow (\nabla (f \circ E)(y))^T \eta(E(x), E(y)) < 0. \tag{2.17}$$

A function may not be E-invex with respect to some η but E-pseudoinvex with respect to same η. This can be verified in the following example.

Example 2.12. Consider $M = \{(x,y) \in R^2 \mid x,y > 0\}$. $E : R^2 \to R^2$ is $E(x,y) = (0,y)$ and $f : M \to R$ is $f(x,y) = -x^2 - y^2$. For $\eta : R^2 \times R^2 \to R^2$, defined by $\eta((x_1,y_1),(x_2,y_2)) = (x_1 - x_2, y_1 - y_2)$, and for all $(x_1,y_1),(x_2,y_2) \in M$, $y_1 \neq y_2$, $(f \circ E)(x_1,y_1) - (f \circ E)(x_2,y_2) - \nabla(f \circ E)(x_2,y_2)^T \eta(E(x_1,y_1),E(x_2,y_2)) = -(y_2 - y_1)^2 < 0$. Hence, f is not E-invex with respect to η on M. If $\nabla(f \circ E)(x_2,y_2))^T \eta(E(x_1,y_1),E(x_2,y_2)) \geq 0$, then $(f \circ E)(x_1,y_1) - (f \circ E)(x_2,y_2) - f(0,y_1) - f(0,y_2) = (y_2 + y_1)(y_2 - y_1) \geq 0$. Hence, f is E-pseudoinvex with respect to η on M.

If a function $f : M \to R$ is semi-E-invex with respect to η at each point of an E-invex set M, then f is said to be semi-E-invex with respect to η on M. Semi-E-invex functions and some of its generalizations are studied in [7]. Here, we discuss some more results on generalized semi-E-invex functions.

Proposition 2.13. *If $f : M \to R$ is semi-E-invex on an E-invex set M, then $f(E(y)) \leq f(y)$ for each $y \in M$.*

Proof. Since f is semi-E-invex on $M \subseteq R^n$ and M is an E-invex set so for $x,y \in M$ and $\lambda \in [0,1]$, we have $E(y) + \lambda\eta(E(x),E(y)) \in M$ and $f(E(y) + \lambda\eta(E(x),E(y))) \leq \lambda f(x) + (1-\lambda)f(y)$. In particular, for $\lambda = 0$, $f(E(y)) \leq f(y)$ for each $y \in M$. □

An E-invex function with respect to some η may not be semi-E-invex with respect to same η may be verified in the following example.

Example 2.14. Consider the previous example where $M = \{(x,y) \in R^2 \mid x,y < 0\}$, $E : R^2 \to R^2$ is $E(x,y) = (0,y)$ and $f : M \to R$ is $f(x,y) = x^3 + y^3$,$\eta : R^2 \times R^2 \to R^2$ is $\eta((x_1,y_1),(x_2,y_2)) = (x_1 - x_2, y_1 - y_2)$. It is verified that f is E-invex with respect to η on M. But $f(E(2,0)) > f(2,0)$. From Proposition 2.13 it can be concluded that f is not semi-E-invex with respect to same η. Also, using Definition 1.6, for all $(x_1,y_1),(x_2,y_2) \in M$, $\lambda \in [0,1]$,$f(E(x_2,y_2) + \lambda\eta(E(x_1,y_1),E(x_2,y_2))) - \lambda f(x_1,y_1) - (1-\lambda)f(x_2,y_2) = -(y_2 + \lambda y_1^2/2y_2)^2 + \lambda(x_1^2 + y_1^2) + (1-\lambda)(x_2^2 + y_2^2)$, which is not always negative. Hence, f is not semi-E-invex with respect to η on M.

3. Application in Optimization Problem

In this section, the results of previous section are used to derive the sufficient optimality condition for the existence of solution of a general nonlinear programming problem. Consider a nonlinear programming problem

$$\begin{aligned} (P)\ \min \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0, \end{aligned} \tag{3.1}$$

where $f : M \to R$, $g_i : M \to R^m$, $M \subseteq R^n$, $g = (g_1, g_2, \ldots, g_m)^T$. $M' = \{x \in M : g_i(x) \leq 0,\ i = 1, \ldots, m\}$ is the set of feasible solutions.

Theorem 3.1 (sufficient optimality condition). *Let M be a nonempty open E-convex subset of R^n,$f : M \to R$, $g : M \to R^m$, and $E : R^n \to R^n$ are differentiable functions. Let E be* *a homeomorphism and let $\overline{x}$ be a fixed point of E. If f and g are semi-E-convex at $\overline{x} \in M'$ and $(\overline{x}, \overline{y}) \in M' \times R^m$ satisfies*

$$\nabla\left[(f \circ E)(x) + \langle y, (g \circ E)(x)\rangle\right] = 0,$$
$$\langle y, (g \circ E)(x)\rangle = 0, \ y \geq 0, \tag{3.2}$$

then $\overline{x}$ is local optimal solution of (P).

Proof. Since f and g are semi-E-convex at $\overline{x} \in M$ by Theorem 2.5,

$$f(x) - f(\overline{x}) \geq (\nabla(f \circ E)(\overline{x}))^T (E(x) - E(\overline{x})) \quad \forall E(x) \in N_\epsilon(E(\overline{x})),$$
$$g(x) - g(\overline{x}) \geq (\nabla(g \circ E)(\overline{x}))^T (E(x) - E(\overline{x})) \quad \forall E(x) \in N_\epsilon(E(\overline{x})). \tag{3.3}$$

Adding the above two inequalities, we have

$$[f(x) - f(\overline{x})] + \overline{y}^T [g(x) - g(\overline{x})] \geq \nabla\left[(f \circ E)(\overline{x}) + \overline{y}^T((g \circ E)(\overline{x}))\right]^T (E(x) - E(\overline{x})). \tag{3.4}$$

If (3.2) hold, then $f(x) - f(\overline{x}) + \overline{y}^T g(x) \geq 0$ for all $E(x) \in N_\epsilon(E(\overline{x}))$. Since $g(x) \leq 0$ for all $x \in M'$ and $\overline{y} \geq 0$, so $\overline{y}^T g(x) \leq 0$. Hence, $f(x) - f(\overline{x}) \geq 0$ for all $E(x) \in N_\epsilon(E(\overline{x}))$. Since E is a homeomorphism, there exists $\delta > 0$ such that $x \in N_\delta(\overline{x})$ for all $E(x) \in N_\epsilon(E(\overline{x}))$, which means $f(x) \geq f(\overline{x})$ for all $x \in N_\delta(\overline{x})$. Hence, $\overline{x}$ is a local optimal solution of (P). □

Also we see that a fixed point of E is a local optimal solution of (P) under generalized E-invexity assumptions.

Lemma 3.2. *Let M be a nonempty E-invex subset of R^n with respect to some $\eta : R^n \times R^n \to R^n$. Let $g_i : M \to R$, $i = 1, \ldots, m$ be semi-E-quasiinvex functions with respect to η on M. Then, M' is an E-invex set.*

Proof. Let $M_i = \{x \in M : g_i(x) \leq 0\}$, $i = 1, \ldots, m$. $M' = \cap_{i=1}^m M_i$ and $M' \subseteq M$. Since g_i, $i = 1, \ldots, m$ are semi-E-quasiinvex function on M, so for all $x, y \in M_i$ and $\lambda \in [0,1]$, $g_i(E(y) + \lambda\eta(E(x), E(y))) \leq \max\{g_i(x),\ g_i(y)\} \leq 0$. Hence, $E(y) + \lambda\eta(E(x), E(y)) \in M_i$ for all $x, y \in M_i$. So M_i is E-invex with respect to same η. From Lemma 1.10, $M' = \cap_{i=1}^m M_i$ is E-invex with respect to same η. □

Corollary 3.3. *Let M be a nonempty E-invex subset of R^n with respect to some $\eta : R^n \times R^n \to R^n$. Let $g_i : M \to R$, $i = 1, \ldots, m$, be semi-E-quasiinvex functions with respect to η on M. If x is a feasible solution of (P), then E(x) is also a feasible solution of (P).*

Proof. Since x is a feasible solution of (P), so $x \in M' \Rightarrow E(x) \in E(M')$. Since each g_i, $i = 1, \ldots, m$ is semi-E-quasiinvex function on M, from Lemma 3.2, M' is an E-invex set. Also $E(M') \subseteq M'$. Hence, $E(x) \in M'$. That is, $E(x)$ is a feasible solution of (P). □

Theorem 3.4 (sufficient optimality condition). *Let M be a nonempty E-invex subset of R^n with respect to $\eta : R^n \times R^n \to R^n$. Let E(M) be an open set in R^n. Suppose $f : R^n \to R$, $g : R^n \to R^m$*

and E are differentiable functions on M. If f is E-pseudoinvex function with respect to η and for $u \geq 0$, $u^T g$ is semi-E-quasiinvex function with respect to the same η at $x \in M'$, where x is a fixed point of the map E and $(x,u) \in M' \times R^m$, $u \geq 0$ satisfies the following system:

$$\nabla\left[(f \circ E)(x) + \langle u, (g \circ E)(x)\rangle\right] = 0, \tag{3.5}$$

$$\langle u, (g \circ E)(x)\rangle = 0, \tag{3.6}$$

then x is a local optimal solution of (P).

Proof. Suppose $(x,u) \in M' \times R^m$ satisfies (3.5) and (3.6). For all $y \in M'$, $g(y) \leq 0$. Also, $u \geq 0$. Hence, $u^T g(y) \leq 0$ for all $y \in M'$. From (3.6), $\langle u, (g \circ E)(x)\rangle = 0$, that is, $u^T g(E(x)) = 0$. x is a fixed point of E that is $E(x) = x$. So $u^T g(x) = 0$. Hence,

$$u^T g(y) \leq u^T g(x) \quad \forall y \in M'. \tag{3.7}$$

Since $u \geq 0$ and $u^T g$ is semi-E-quasiinvex function with respect to η at x, so the above inequality implies

$$\nabla u^T g(E(x))\eta(E(y), E(x)) \leq 0. \tag{3.8}$$

From (3.5), $\nabla f(E(x)) = -\nabla u^T g(E(x))$. Putting this value in the above inequality, we have $\nabla f(E(x))\eta(E(x), E(y)) \geq 0$.

f is E-pseudoinvex at x with respect to η. Hence, $\nabla f(E(x))\eta(E(x), E(y)) \geq 0$ implies

$$f(E(y)) \geq f(E(x)) = f(x) \quad \forall y \in M'. \tag{3.9}$$

Hence, x is the optimal solution (P) on $E(M')$. □

The following example justifies the above theorem.

Example 3.5. Consider the optimization problem,

$$\begin{array}{ll} (P)\ \min & -x^2 - y^2 \\ \text{subject to} & x^2 + y^2 - 4 \leq 0, \end{array} \tag{3.10}$$

where $M = \{(x,y) \in R^2 \mid x, y > 0\}$. $E : R^2 \to R^2$ is $E(x,y) = (0,y)$. This is not a convex programming problem. Consider $\eta : R^2 \times R^2 \to R^2$ defined by $\eta((x_1,y_1),(x_2,y_2)) = (x_1 - x_2, y_1 - y_2)$. Here, $M' = \{(x,y) \in M : x^2 + y^2 - 4 \leq 0\}$, and $E(M') = \{(0,y) : y \geq 0,\ y^2 - 4 \leq 0\}$. The sufficient conditions (7-8) reduce to

$$-2y + u2y = 0, \quad u\left(y^2 - 4\right) = 0, \quad u \geq 0, \tag{3.11}$$

whose solution is $y = 2, u = 1$ and $E(0,2) = (0,2)$.

In Example 2.12, we have already proved that $f(x, y) = -x^2 - y^2$ is E-pseudoinvex function with respect to η. Using Definition 1.7, one can verify that $ug(x, y)$ is semi-E-quasi-invex with respect to same η at $(0, 2) \in M'$, where $ug(x, y) = x^2 + y^2 - 4$. So $(0, 2)$ is the optimal solution of (P) on $E(M')$.

4. Conclusion

E-convexity and its generalizations are studied by many authors earlier without differentiability assumption. Here, we have studied the the properties of E-convexity, E-invexity, and their generalizations with differentiable assumption. From the developments of this paper, we conclude the following interesting properties.

(1) A function may not be convex at a point but E-convex at that point with a suitable mapping E, and if a local minimum of f exists in a neighborhood of $E(x)$, then f is E-convex at x. But it is not necessarily true that if f is E-convex at x then $E(x)$ is local minimum point.

(2) From the relation between E-invexity and its generalizations, one may observe that a function which is not E-convex may be E-invex with respect to some η and E-preinvexity with differentiability is a sufficient condition for E-invexity. Moreover, a function may not be E-invex with respect to some η but E-quasi-invex with respect to same η, a function may not E-invex with respect to some η but E-pseudoinvex with respect to the same η and an E-invex function with respect to some η may not be semi-E-invex with respect to same η.

Here, we have studied E-convexity for first-order differentiable functions. Higher-order differentiable E-convex functions may be studied in a similar manner to derive the necessary and sufficient optimality conditions for a general nonlinear programming problems.

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