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Research Article

The Exponential Dichotomy under Discretization on General Approximation Scheme

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This paper is devoted to the numerical analysis of abstract parabolic problem $u'(t) = Au(t)$; $u(0) = u^0$, with hyperbolic generator A. We are developing a general approach to establish a discrete dichotomy in a very general setting in case of discrete approximation in space and time. It is a well-known fact that the phase space in the neighborhood of the hyperbolic equilibrium can be split in a such way that the original initial value problem is reduced to initial value problems with exponential decaying solutions in opposite time direction. We use the theory of compact approximation principle and collectively condensing approximation to show that such a decomposition of the flow persists under rather general approximation schemes. The main assumption of our results is naturally satisfied, in particular, for operators with compact resolvents and condensing semigroups and can be verified for finite element as well as finite difference methods.

1. Introduction

Many problems like approximation of attractors, traveling waves, shadowing e.c. involve the notion of dichotomy. In numerical analysis of such problems, it is very important to know if they keep some kind of exponential estimates uniformly in discretization parameter.

Let $B(E)$ denote the Banach algebra of all bounded linear operators on a complex Banach space E. The set of all linear-closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. For $B \in \mathcal{C}(E)$, let $\sigma(B)$ be its spectrum and $\rho(B)$, its resolvent set. In the following let $A : D(A) \subseteq E \to E$ be a closed linear operator, such that

$$\left\| (\lambda I - A)^{-1} \right\|_{B(E)} \leq \frac{M}{1 + |\lambda|} \quad \text{for any } \operatorname{Re}\lambda \geq 0. \tag{1.1}$$

Under condition (1.1), the spectrum of A is on the left: $\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0$, so the fractional power operators $(-A)^{\alpha}$, $\alpha \in \mathbb{R}^{+}$, (see [1, 2]) associated to A and E^{α}, can be constructed and the corresponding fractional power spaces too, that is, $E^{\alpha} := D((-A)^{\alpha})$ endowed with the graph norm $\|x\|_{E^{\alpha}} = \|(-A)^{\alpha}x\|_{E}$. Define the ball $\mathcal{U}_{E^{\alpha}}(0;\rho)$ with the center at 0 of radius $\rho > 0$ in E^{α} space.

To show how the dichotomy problems appear in numerical analysis, we are considering for example the semilinear equation in Banach space E^{α}

$$\begin{aligned} u'(t) &= Au(t) + f(u(t)), \quad t \geq 0, \\ u(0) &= u^{0} \in E^{\alpha}, \end{aligned} \tag{1.2}$$

where $f(\cdot) : E^{\alpha} \subseteq E \to E$, $0 \leq \alpha < 1$, is assumed to be continuous, bounded, and continuously Fréchet differentiable function. More precisely, we assume that the following condition holds.

(F1) For any $\epsilon > 0$, there is $\delta > 0$ such that $\|f'(w) - f'(z)\|_{B(E^{\alpha},E)} \leq \epsilon$ as $\|w - z\|_{E^{\alpha}} \leq \delta$ for all $w, z \in \mathcal{U}_{E^{\alpha}}(u^{*};\rho)$, where u^{*} is a hyperbolic equilibrium point of (1.2).

By means of the change of variables $v(\cdot) = u(\cdot) - u^{*}$ in the problem (1.2), where u^{*} is the hyperbolic equilibrium, we obtain the problem

$$\begin{aligned} v'(t) &= \left(A + f'(u^{*})\right)v(t) + f(v(t) + u^{*}) - f(u^{*}) - f'(u^{*})v(t), \\ v(0) &= u^{0} - u^{*} = v^{0}. \end{aligned} \tag{1.3}$$

Such problem can be written in the form

$$v'(t) = A_{u^{*}}v(t) + F_{u^{*}}(v(t)), \quad v(0) = v^{0}, \ t \geq 0, \tag{1.4}$$

where $A_{u^{*}} = A + f'(u^{*})$, $F_{u^{*}}(v(t)) = f(v(t)+u^{*}) - f(u^{*}) - f'(u^{*})v(t)$. We note that from condition (F1) it follows that the function $F_{u^{*}}(v(t)) = f(v(t) + u^{*}) - f(u^{*}) - f'(u^{*})v(t)$ for small $\|v^{0}\|_{E^{\alpha}}$ is of order $o(\|v(t)\|_{E^{\alpha}})$. Since $f'(u^{*}) \in B(E^{\alpha}, E)$, $0 \leq \alpha < 1$, the operator $A_{u^{*}} = A + f'(u^{*})$ is the generator of an analytic C_0-semigroup [3]. It can happen that the spectrum of operator $A_{u^{*}}$ can be split into two parts σ^{+} and σ^{-}.

We assume that the part σ^{+} of the spectrum of operator $A + f'(u^{*})$, which is located strictly to the right of the imaginary axis, consists of a finite number of eigenvalues with finite multiplicity. This assumption is satisfied, for instance, if the resolvent of operator A is compact. The conditions under which the operator $A_{u^{*}}$ has the dichotomy property were studied say in [4–6]. In case of hyperbolic equilibrium point u^{*}, there is no spectrum of $A_{u^{*}}$ on $i\mathbb{R}$. Let $U(\sigma^{+}) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ be an open connected neighborhood of σ^{+} which has a closed rectifiable curve $\partial U(\sigma^{+})$ as a boundary. We decompose E^{α} using the Riesz projection

$$P(\sigma^{+}) := P(\sigma^{+}, A_{u^{*}}) := \frac{1}{2\pi i}\int_{\partial U(\sigma^{+})} (\zeta I - A_{u^{*}})^{-1} d\zeta \tag{1.5}$$

defined by σ^+. Due to this definition and analyticity of the C_0-semigroup $e^{tA_{u^*}}, t \in \mathbb{R}_+$, we have positive constants $M_1, \beta > 0$, such that

$$\begin{aligned} \left\| e^{tA_{u^*}} z \right\|_{E^\alpha} &\le M_1 e^{-\beta t} \|z\|_{E^\alpha}, \quad t \ge 0, \\ \left\| e^{tA_{u^*}} v \right\|_{E^\alpha} &\le M_1 e^{\beta t} \|v\|_{E^\alpha}, \quad t \le 0, \end{aligned} \tag{1.6}$$

for all $v \in P(\sigma^+)E^\alpha$ and $z \in (I - P(\sigma^+))E^\alpha$. Since $F_{u^*}(v(t)) = o(\|v(t)\|_{E^\alpha})$ for small $v(\cdot)$, the estimates (1.6) are crucial to describe the behavior of solution of the problem (1.2) at the vicinity of the hyperbolic equilibrium point u^*.

If v^0 is close to 0, that is, say $v^0 \in \mathcal{U}_{E^\alpha}(0;\rho)$ with small $\rho > 0$, then the mild solution $v(t;v^0)$ of (1.4) can stay in the ball $\mathcal{U}_{E^\alpha}(0;\rho)$, for some time. We denote now the maximal time of staying $v(t;v^0)$ in $\mathcal{U}_{E^\alpha}(0;\rho)$ by $T = T(v^0) = \sup\{t \ge 0 : \|v(t;v^0)\|_{E^\alpha} \le \rho$ or $v(t;v^0) \in \mathcal{U}_{E^\alpha}(0;\rho)\}$. Now coming back to solution of (1.4) for any two $v^0, v^T \in \mathcal{U}_{E^\alpha}(0;\rho)$ we consider the boundary value problem

$$\begin{aligned} v'(t) &= A_{u^*} v(t) + F_{u^*}(v(t)), \quad 0 \le t \le T, \\ (I - P(\sigma^+))v(0) &= (I - P(\sigma^+))v^0, \quad P(\sigma^+)v(T) = P(\sigma^+)v^T. \end{aligned} \tag{1.7}$$

A mild solution of problem (1.7) as was shown in [7] satisfies the integral equation

$$\begin{aligned} v(t) &= e^{(t-T)A_{u^*}} P(\sigma^+)v^T + e^{tA_{u^*}}(I - P(\sigma^+))v^0 + \int_0^t e^{(t-s)A_{u^*}}(I - P(\sigma^+))F_{u^*}(v(s))ds \\ &\quad + \int_t^T e^{(t-s)A_{u^*}} P(\sigma^+)F_{u^*}(v(s))ds, \quad 0 \le t \le T. \end{aligned} \tag{1.8}$$

In case one would like to discretize the problem (1.4) in space and time variables it is very important to know what will happen to estimates like (1.6) for approximation solutions. If the estimates like (1.6) hold uniformly in parameter of discretization, then one can expect to get a similar behavior of approximated solutions of (1.8).

So, in this paper we are going to consider general approximation approach for keeping the dichotomy estimates (1.6) for approximations of trajectory $u(\cdot)$.

2. Preliminaries

Let $\mathbb{T}(r) = \{\lambda : \lambda \in \mathbf{C}, |\lambda| = \mathbf{r}\}, \mathbb{T} = \mathbb{T}(1)$.

Definition 2.1. A C_0-semigroup $e^{tA}, t \ge 0$, defined on a Banach space E is called hyperbolic if $\sigma(e^{tA}) \cap \mathbb{T} = \emptyset$ for all $t > 0$. The generator A is called hyperbolic if $\sigma(A) \cap i\mathbb{R} = \emptyset$.

Let us denote by $\Upsilon(\mathbb{R};E)$ any of the spaces $L^p(\mathbb{R};E),\ 1 \le p < \infty,\ C_0(\mathbb{R};E)$ or Stepanov's space $S^p(\mathbb{R};E),\ 1 \le p < \infty$. We consider in the Banach space $\Upsilon(\mathbb{R};E)$ (we call this space as

Palmer's space, see [8], where Fredholm property was mentioned for the first time) the linear differential operator

$$\mathcal{L} = -\frac{d}{dt} + A : D(\mathcal{L}) \subseteq \Upsilon(\mathbb{R};E) \longrightarrow \Upsilon(\mathbb{R};E), \tag{2.1}$$

where A generates C_0-semigroup, domain of $\mathcal{L}$ is assumed to consist of the functions $u(\cdot) \in \Upsilon(\mathbb{R};E)$ such that for some function $g(\cdot) \in \Upsilon(\mathbb{R};E)$ one has $u(t) = e^{(t-s)A}u(s) - \int_s^t e^{(t-\eta)A}g(\eta)d\eta$, $s \leq t, t \in \mathbb{R}$, and $\mathcal{L}u(\cdot) = g(\cdot)$. Let us note [9] that the operator $\mathcal{L}$ is the generator of C_0-semigroup $e^{t\mathcal{L}}$ on a Banach space $\Upsilon(\mathbb{R};E)$, which is defined for all $v(\cdot) \in \Upsilon(\mathbb{R};E)$ by formula

$$\left(e^{t\mathcal{L}}v\right)(s) = e^{tA}v(s-t) \quad \text{for any } s \in \mathbb{R},\ t \geq 0. \tag{2.2}$$

Definition 2.2. A C_0-semigroup e^{tA}, $t \geq 0$, has an exponential dichotomy on $\mathbb{R}$ with exponential dichotomy data $(M \geq 1, \beta > 0)$ if there exists projector $P : E \to E$ such that

(i) $e^{tA}P = Pe^{tA}$ for all $t \geq 0$,

(ii) the restriction $e^{tA}|_{\mathcal{R}(P)}, t \geq 0$, is invertible on $P(E)$ and

$$\begin{aligned} \left\| e^{-tA}Px \right\| &\leq Me^{-\beta t}\|Px\|, \quad t \geq 0,\ x \in E, \\ \left\| e^{tA}(I-P)x \right\| &\leq Me^{-\beta t}\|(I-P)x\|, \quad t \geq 0,\ x \in E. \end{aligned} \tag{2.3}$$

Theorem 2.3 (see [10]). *The operator $\mathcal{L}$ in the Banach space $\Upsilon(\mathbb{R};E)$ is invertible if and only if the condition*

$$\sigma\left(e^{1A}\right) \cap \mathbb{T} = \emptyset \tag{2.4}$$

holds. If condition (2.4) *is satisfied, then*

$$\left(\mathcal{L}^{-1}f\right)(t) = \int_{-\infty}^{\infty} G(t-s)f(s)ds, \quad t \in \mathbb{R},\ f(\cdot) \in \Upsilon(\mathbb{R};E), \tag{2.5}$$

where the Green's function

$$G(\eta) = \begin{cases} -e^{\eta A}P_-, & \eta \geq 0, \\ e^{\eta A}P_+, & \eta < 0, \end{cases} \tag{2.6}$$

$$\|G(\eta)\| \leq \begin{cases} M_+e^{-\gamma_+\eta}, & \eta \geq 0, \\ M_-e^{\gamma_-\eta}, & \eta < 0, \end{cases} \tag{2.7}$$

where

$$M_+ = 2M\text{æ}(\mathcal{L})\left(1+\frac{1}{2\text{æ}(\mathcal{L})}\right)^2, \qquad M_- = 2M\text{æ}(\mathcal{L})\left(1-\frac{1}{2\text{æ}(\mathcal{L})}\right)^2,$$
$$\gamma_+ = \ln\left(1+\frac{1}{2\text{æ}(\mathcal{L})}\right), \qquad \gamma_- = -\ln\left(1-\frac{1}{2\text{æ}(\mathcal{L})}\right), \tag{2.8}$$

and $\text{æ}(\mathcal{L}) = 1 + C(\Upsilon)(M + M^2\|\mathcal{L}^{-1}\|)$.

We note that the constant $C(\Upsilon)$ is defined as $C(\Upsilon) = 1$ if $\Upsilon(\mathbb{R};E) = L^\infty(\mathbb{R},E)$ or $\Upsilon(\mathbb{R};E) = C_0(\mathbb{R},E)$ and $C(\Upsilon) = 2^{1-1/p}$ if $\Upsilon(\mathbb{R};E) = L^p(\mathbb{R},E)$ or $\Upsilon(\mathbb{R};E) = S^p(\mathbb{R},E), p \in [1,\infty)$.

Denote by $\Upsilon(\mathbb{Z};E)$ the Banach space of E-valued sequences with corresponding discrete norm which is consistent with the norm of $\Upsilon(\mathbb{R};E)$. For any $u(\cdot) \in \Upsilon(\mathbb{Z};E)$, that is, $\{u(k)\}_{k\in\mathbb{Z}}$, and $B = e^{1A} \in B(E)$, we define an operator $\mathcal{B} : D(\mathcal{B}) \subseteq l^p(\mathbb{Z};E) \to l^p(\mathbb{Z};E)$ by formula

$$(\mathcal{B}u)(k) = Bu(k-1), \quad k \in \mathbb{Z},\ u(\cdot) \in l^p(\mathbb{Z};E). \tag{2.9}$$

Now we define an operator $\mathfrak{D} = I - \mathcal{B} : D(\mathfrak{D}) = D(\mathcal{B}) \subseteq l^p(\mathbb{Z};E) \to l^p(\mathbb{Z};E)$ as

$$(\mathfrak{D}u)(k) = u(k) - Bu(k-1), \quad u(\cdot) \in D(\mathcal{B}),\ k \in \mathbb{Z}. \tag{2.10}$$

Proposition 2.4 (see [10]). *Let an operator* $\mathcal{L} = -d/dt + A : D(\mathcal{L}) \subseteq \Upsilon(\mathbb{R};E) \to \Upsilon(\mathbb{R};E)$ *be invertible. Then, an operator* $\mathfrak{D} : D(\mathfrak{D}) \subseteq \Upsilon(\mathbb{Z};E) \to \Upsilon(\mathbb{Z};E)$ *is invertible too and*

$$\left\|\mathfrak{D}^{-1}\right\| \le 1 + C(\Upsilon)\left(M + M^2\left\|\mathcal{L}^{-1}\right\|\right). \tag{2.11}$$

Conversely, if $\mathfrak{D}$ *is invertible, then* $\mathcal{L} : D(\mathcal{L}) \subseteq \Upsilon(\mathbb{R};E) \to \Upsilon(\mathbb{R};E)$ *is invertible and*

$$\left\|\mathcal{L}^{-1}\right\| \le C(\Upsilon)\left(M + M^2\left\|\mathfrak{D}^{-1}\right\|\right). \tag{2.12}$$

Theorem 2.5 (see [11]). *The difference operator*

$$(\mathfrak{D}u)(k) = u(k) - Bu(k-1), \quad k \in \mathbb{N},\ u(\cdot) \in l^p(\mathbb{R};E),\ 1 \le p \le \infty, \tag{2.13}$$

is invertible if and only if

$$\sigma(B) \cap \mathbb{T} = \emptyset. \tag{2.14}$$

If condition (2.14) *is satisfied, then the inverse operator has the form*

$$\left(\mathfrak{D}^{-1}v\right)(k) = \Sigma_{m\in\mathbb{Z}}\Gamma(k-m)v(m), \tag{2.15}$$

where $v(\cdot) \in l^p(\mathbb{Z}; E), 1 \le p \le \infty$, *and the function* $\Gamma(\cdot) : \mathbb{Z} \to B(E)$ *is defined by*

$$\Gamma(k) = \begin{cases} B^k(I-P), & k \ge 0, \\ -B_0^{-k}P, & k \le -1, \end{cases} \tag{2.16}$$

where B_0 *is the restriction of* B *on* $\mathcal{R}(P)$.

Definition 2.6. The operator $B \in B(E)$ has an exponential discrete dichotomy with data (M, r, P) if $P \in B(E)$ is a projector in E and M, r are constants with $0 \le r < 1$ such that the following properties hold

(i) $B^k P = PB^k$ for all $k \in \mathbb{N}$,

(ii) $\|B^k(I-P)\| \le Mr^k$ for all $k \in \mathbb{N}$,

(iii) $\widehat{B} := B|_{\mathcal{R}(P)} : \mathcal{R}(P) \mapsto \mathcal{R}(P)$ is a homeomorphism that satisfies $\|\widehat{B}^{-k}P\| \le Mr^k,\ k \in \mathbb{N}$.

The following result relates the constants in an exponential discrete dichotomy to a bound on the resolvent $(\lambda I - B)^{-1}$ for $\lambda \in \mathbb{T}$.

Theorem 2.7. *For* $B \in B(E)$, *the following conditions are equivalent:*

(i) $\lambda \in \rho(B)$ *for all* $\lambda \in \mathbb{T}$ *and* $\|(\lambda I - B)^{-1}\| \le \beta < \infty\ \forall \lambda \in \mathbb{T}$;

(ii) B *has an exponential dichotomy with data* (M, r, P).

More precisely, one shows that (i) implies (ii) with $M = 2\beta^2/(\beta - 1)$ *and* $r = 1 - 1/2\beta$. *Conversely, (ii) implies (i) with* $\beta = M((1+r)/(1-r))$.

Proof. First assume (i) and without loss of generality let $\beta > 1$. For $z \in \mathbf{C}, \mathbf{z} \neq \mathbf{0}$, we have

$$\left| z - \frac{z}{|z|} \right| = |1 - |z||. \tag{2.17}$$

Hence, if $|1 - |z||\beta < 1$, the classical perturbation estimate shows that $z \in \rho(B)$ and

$$\left\| (zI - B)^{-1} \right\| \le \frac{\beta}{1 - \beta|1 - |z||}. \tag{2.18}$$

We define P as the Riesz projector defined by the formula

$$I - P = \frac{1}{2\pi i} \int_{|z|=1} (zI - B)^{-1} dz. \tag{2.19}$$

Since B commutes with the resolvent, condition (i) of Definition 2.6 holds. Further, using (2.18) and Cauchy's theorem, we can shift the contour

$$I - P = \frac{1}{2\pi i} \int_{|z|=r} (zI - B)^{-1} dz, \quad \text{if } |1 - r| < \frac{1}{\beta}. \tag{2.20}$$

Now we claim

$$B^k(I-P) = \frac{1}{2\pi i}\int_{|z|=r} z^k(zI-B)^{-1}dz \quad \text{if } |1-r| < \frac{1}{\beta}. \tag{2.21}$$

For $k=0$ this follows from (2.20). If (2.21) holds for some k, then we obtain

$$\begin{aligned} B^{k+1}P &= \frac{1}{2\pi i}\int_{|z|=r}(B-zI+zI)z^k(zI-B)^{-1}dz \\ &= \frac{1}{2\pi i}\int_{|z|=r} z^{k+1}(zI-B)^{-1}dz - \frac{1}{2\pi i}\int_{|z|=r} z^k dz, \end{aligned} \tag{2.22}$$

and thus the assertion holds for $k+1$. Equations (2.21) and (2.18) immediately lead to the first dichotomy estimate for $1-1/\beta < r \le 1$

$$\left\|B^kP\right\| \le \frac{1}{2\pi}2\pi r r^k\frac{\beta}{1-\beta(1-r)} = \frac{\beta r^{k+1}}{1-\beta(1-r)} \quad \text{for } k \ge 0. \tag{2.23}$$

For the second dichotomy estimate, we use the resolvent equation

$$(zI-B)^{-1} = \frac{1}{z}I + z^{-1}B(zI-B)^{-1}. \tag{2.24}$$

For $|1-r| < 1/\beta$, (2.20) and (2.24) lead to

$$\begin{aligned} I-P &= \frac{1}{2\pi i}\int_{|z|=1}\left(\frac{1}{z}I-(zI-B)^{-1}\right)dz \\ &= -\frac{1}{2\pi i}\int_{|z|=r}\frac{1}{z}B(zI-B)^{-1}dz. \end{aligned} \tag{2.25}$$

This shows that the following equality holds for $k=1$

$$I-P = -B^k\frac{1}{2\pi i}\int_{|z|=r} z^{-k}(zI-B)^{-1}dz, \quad \text{for } k \ge 1. \tag{2.26}$$

If (2.26) is known for some k, then use (2.24) and find

$$\begin{aligned} I-P &= -B^k\frac{1}{2\pi i}\int_{|z|=r}\left(z^{-(k+1)} + z^{-(k+1)}B(zI-B)^{-1}\right)dz \\ &= -B^{k+1}\frac{1}{2\pi i}\int_{|z|=r} z^{-(k+1)}(zI-B)^{-1}dz. \end{aligned} \tag{2.27}$$

We apply (2.26) to $x \in E$, using that $I - P$ commutes with B as well as the estimate (2.18)

$$\begin{aligned} \|(I-P)x\| &= \left\| \frac{1}{2\pi i} \int_{|z|=r} z^{-k}(zI-B)^{-1} dz B^k (I-P)x \right\| \\ &\leq \frac{r^{-k}}{2\pi} 2\pi r \frac{\beta}{1-\beta(r-1)} \left\| B^k (I-P)x \right\|. \end{aligned} \tag{2.28}$$

Summarizing, we have shown for all $k \geq 1, 1 \leq r < 1 + 1/\beta, u \in E$

$$\|(I-P)u\| \leq \frac{\beta r^{-k+1}}{1-\beta(r-1)} \left\| B^k (I-P)u \right\|. \tag{2.29}$$

For $k = 1$, this estimate shows that $\widehat{B} = B_{|\mathcal{N}(P)} : \mathcal{N}(P) \mapsto \mathcal{N}(P)$ is one-to-one with a bound for the inverse. To show that $\widehat{B}$ is onto, we take $f \in \mathcal{N}(P)$ and set

$$v = -\frac{1}{2\pi i} \int_{|z|=1} z^{-1}(zI-B)^{-1} f \, dz. \tag{2.30}$$

From this equation, we have $(I-P)v = 0$, and using (2.24), we find

$$Bv = \frac{1}{2\pi i} \int_{|z|=1} \left(z^{-1} I - (zI-B)^{-1} \right) f \, dz = (I-P)f = f. \tag{2.31}$$

Therefore, $\widehat{B}$ is a linear homeomorphism on $\mathcal{N}(P)$ satisfying

$$\left\| \widehat{B}^{-k}(I-P)u \right\| \leq \frac{\beta r^{-k+1}}{1-\beta(r-1)} \|(I-P)u\| \quad \text{for } 1 \leq r < 1 + \frac{1}{\beta}. \tag{2.32}$$

This proves exponential dichotomy.

To arrive at the specific constants, we choose $r = 1 - 1/2\beta$ in (2.23) and obtain the bound $(2\beta - 1)r^k$. In order to have the same rate in the opposite direction, we apply (2.32), with $r = (1-1/2\beta)^{-1} < 1+1/\beta$. In (2.32) we then find the upper bound Mr^{-k} with the constant $M = 2\beta^2/(\beta - 1)$. Since $M > 2\beta - 1$, our assertion follows.

Now we assume exponential dichotomy and prove condition (i). For $|\lambda| = 1$ the equation $(\lambda I - B)u = f$ is equivalent to the system

$$\begin{aligned} (\lambda I - BP)Pu &= Pf, \\ \left(\lambda I - \widehat{B}\right)(I-P)u &= (I-P)f, \end{aligned} \tag{2.33}$$

which we can rewrite as

$$\begin{aligned} \left(I - \lambda^{-1} BP\right)Pu &= \lambda^{-1} Pf, \\ \left(I - \lambda \widehat{B}^{-1}\right)(I-P)u &= -\widehat{B}^{-1}(I-P)f. \end{aligned} \tag{2.34}$$

Both equations have a unique solution given by a geometric series

$$Pu = \sum_{k=0}^{\infty} \lambda^{-(k+1)} B^k P f,$$
$$(I-P)u = -\sum_{k=0}^{\infty} \widehat{B}^{-(k+1)} \lambda^k (I-P) f. \tag{2.35}$$

The exponential dichotomy then implies the estimates

$$\|Pu\| \leq \frac{M}{1-r}\|f\|,$$
$$\|(I-P)u\| \leq M\frac{r}{1-r}\|f\|. \tag{2.36}$$

By the triangle inequality, we obtain condition (i) with $\beta = M((1+r)/(1-r))$. □

3. Discretization of Operators and Semigroups

In the papers [12–15], a general framework was developed that allows to analyze convergence properties of numerical discretizations in a unifying way. This approach is able to cover such seemingly different methods as (conforming and nonconforming) finite elements, finite differences, or collocation methods. It is the purpose of this paper to show that it is also possible to handle dichotomy properties with discretization in space and time on general approximation scheme. Moreover, we also consider the case when resolvent of operator A is not necessarily compact.

3.1. General Approximation Scheme

Let E_n and E be Banach spaces and $\{p_n\}$ a sequence of linear bounded operators $p_n : E \to E_n$, $p_n \in B(E, E_n)$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, with the property:

$$\|p_n x\|_{E_n} \longrightarrow \|x\|_E \quad \text{as } n \to \infty \text{ for any } x \in E. \tag{3.1}$$

Definition 3.1. The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-convergent to $x \in E$ if and only if $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$, and we write this $x_n \xrightarrow{\mathcal{P}} x$.

Definition 3.2. The sequence of bounded linear operators $B_n \in B(E_n)$, $n \in \mathbb{N}$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n$, $n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We write then $B_n \xrightarrow{\mathcal{PP}} B$.

In the case of unbounded operators as it occurs for general infinitesimal generators of PDE's, the notion of *compatibility* turns out to be useful.

Definition 3.3. The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n)$, $n \in \mathbb{N}$, is called compatible with a closed linear operator $A \in \mathcal{C}(E)$ if and only if for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n,\ n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax$. We write that (A_n, A) are compatible.

For analytic C_0-semigroups, the following ABC Theorem holds.

Theorem 3.4 (see [16]). *Let operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).*

(A) *Compatibility. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$.*

(B_1) *Stability. There are some constants $M_1 \geq 1$ and $\omega_1 \in \mathbb{R}$ such that*

$$\left\|(\lambda I_n - A_n)^{-1}\right\| \leq \frac{M_1}{|\lambda - \omega_1|}, \quad \operatorname{Re}\lambda > \omega_1,\ n \in \mathbb{N}. \tag{3.2}$$

(C_1) *Convergence. For any finite $\mu > 0$ and some $0 < \theta < \pi/2$, one has*

$$\max_{\eta \in \Sigma(\theta,\mu)} \left\| e^{\eta A_n} u_n^0 - p_n e^{\eta A} u^0 \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \text{ whenever } u_n^0 \xrightarrow{\mathcal{P}} u^0. \tag{3.3}$$

Here one used the sector of angle 2θ and radius ρ given by $\Sigma(\theta,\mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$, and $\Sigma(\theta) = \{z \in \mathbb{C} : |\arg\ z| \leq \theta\}$.

It is natural to assume in semidiscretization that conditions like (A) and (B_1) are satisfied.

Definition 3.5. The region of stability $\Delta_s = \Delta_s(\{A_n\}),\ A_n \in \mathcal{C}(B_n)$, is defined as the set of all $\lambda \in \mathbf{C}$ such that $\lambda \in \rho(A_n)$ for almost all n and such that the sequence $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n\in\mathbb{N}}$ is bounded. The region of convergence $\Delta_c = \Delta_c(\{A_n\}),\ A_n \in \mathcal{C}(E_n)$, is defined as the set of all $\lambda \in \mathbf{C}$ such that $\lambda \in \Delta_s(\{A_n\})$ and such that the sequence of operators $\{(\lambda I_n - A_n)^{-1}\}_{n\in\mathbb{N}}$ is $\mathcal{PP}$-convergent to some operator $S(\lambda) \in B(E)$.

Definition 3.6. A sequence of operators $\{B_n\}, B_n \in B(E_n),\ n \in \mathbb{N}$, is said to be stably convergent to an operator $B \in B(E)$ if and only if $B_n \xrightarrow{\mathcal{PP}} B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1),\ n \to \infty$. We will write this as: $B_n \xrightarrow{\mathcal{PP}} B$ stably.

Definition 3.7. A sequence of operators $\{B_n\}, B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ if and only if $B_n \xrightarrow{\mathcal{PP}} B$ and the following implication holds

$$\|x_n\|_{E_n} = O(1), \quad \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact} \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact}. \tag{3.4}$$

We write this as: $B_n \xrightarrow{\mathcal{PP}} B$ regularly.

Theorem 3.8 (see [15]). *Let $C_n, Q_n \in B(E_n)$, $C, Q \in B(E)$ and $\mathcal{R}(Q) = E$. Assume also that $C_n \xrightarrow{\mathcal{PP}} C$ compactly and $Q_n \xrightarrow{\mathcal{PP}} Q$ stably. Then, $Q_n + C_n \xrightarrow{\mathcal{PP}} Q + C$ converge regularly.*

Theorem 3.9 (see [15]). *For $Q_n \in B(E_n)$ and $Q \in B(E)$, the following conditions are equivalent:*

(i) $Q_n \xrightarrow{\mathcal{PP}} Q$ *regularly,* Q_n *are Fredholm operators of index 0 and* $\mathcal{N}(Q) = \{0\}$;

(ii) $Q_n \xrightarrow{\mathcal{PP}} Q$ *stably and* $\mathcal{R}(Q) = E$;

(iii) $Q_n \xrightarrow{\mathcal{PP}} Q$ *stably and regularly;*

(iv) *if one of conditions (i)–(iii) holds, then there exist* $Q_n^{-1} \in B(E_n)$, $Q^{-1} \in B(E)$, *and* $Q_n^{-1} \xrightarrow{\mathcal{PP}} Q^{-1}$ *regularly and stably.*

Theorem 3.10. *Let operators* $\lambda I_n - B_n \in B(E_n)$ *be Fredholm operators with* $ind\ (\lambda I_n - B_n) = 0$ *for any* $\lambda \in \mathbb{T},\ n \in \mathbb{N}$. *Assume also that* $B \in B(E)$ *has the property* $\mathbb{T} \cap \sigma(B) = \emptyset$ *and* $\lambda I_n - B_n \xrightarrow{\mathcal{PP}} \lambda I - B$ *regularly for any* $\lambda \in \mathbb{T}$. *Then* $\lambda I_n - B_n \xrightarrow{\mathcal{PP}} \lambda I - B$ *stably for any* $\lambda \in \mathbb{T}$ *and* $\sup_{\lambda\in\mathbb{T}}\|(\lambda I_n - B_n)^{-1}\| < \infty$.

Proof. Assume that there are some sequences $\{\lambda_n\}, \lambda_n \in \mathbb{T}$, and $\{x_n\}, x_n \in E_n$, such that $\|x_n\| = 1$ and $(\lambda_n I_n - B_n)x_n \xrightarrow{\mathcal{P}} 0$ as $n \to \infty$. Since $\mathbb{T}$ is compact, one can find $\mathbb{N}' \subset \mathbb{N}$ such that $\lambda_n \to \lambda_0 \in \mathbb{T}$ as $n \in \mathbb{N}'$. In the meantime, $\lambda_0 I_n - B_n \xrightarrow{\mathcal{PP}} \lambda_0 I - B$ regularly for such $\lambda_0 \in \mathbb{T}$, and $(\lambda_0 I_n - B_n)x_n = (\lambda_0 I_n - \lambda_n I_n)x_n + (\lambda_n I_n - B_n)x_n \xrightarrow{\mathcal{P}} 0$ for $n \in \mathbb{N}'$. Therefore, there is $\mathbb{N}'' \subset \mathbb{N}'$ such that $x_n \xrightarrow{\mathcal{P}} x_0 \neq 0$ as $n \in \mathbb{N}''$. But in such case $(\lambda_0 I_n - B_n)x_n \xrightarrow{\mathcal{PP}} (\lambda_0 I - B)x_0 = 0$ as $n \in \mathbb{N}''$, which contradicts our assumption $\mathbb{T} \cap \sigma(B) = \emptyset$. □

Definition 3.11. The operator $B_n \in B(E_n)$ has a uniform exponential discrete dichotomy with data (M, r, P_n) if $P_n \in B(E_n)$ is a projector in E_n and M, r are constants with $0 \le r < 1$ such that the following properties hold

(i) $B_n^k P_n = P_n B_n^k$ and $\|P_n\| \le constant$ for all $k, n \in \mathbb{N}$;

(ii) $\|B_n^k(I_n - P_n)\| \le Mr^k$ for all $k, n \in \mathbb{N}$;

(iii) $\widehat{B}_n := B_n|_{\mathcal{R}(P_n)} : \mathcal{R}(P_n) \mapsto \mathcal{R}(P_n)$ is a homeomorphism that satisfies

$$\left\|\widehat{B_n}^{-k} P_n\right\| \le Mr^k, \quad k, n \in \mathbb{N}. \tag{3.5}$$

Theorem 3.12. *The following conditions are equivalent:*

(i) $\lambda I_n - B_n \xrightarrow{\mathcal{PP}} \lambda I - B$ *stably and* $\lambda \in \rho(B)$ *for any* $\lambda \in \mathbb{T}$;

(ii) *operator* $\mathfrak{D} = I - \mathcal{B}$ *is invertible and* $\mathfrak{D}_n \xrightarrow{\mathcal{PP}} \mathfrak{D}$ *stably, where* $(\mathcal{B}u)(k) = Bu(k-1), k \in \mathbb{N}$;

(iii) $B_n \xrightarrow{\mathcal{PP}} B$ *and* $\lambda I - B$ *be invertible for any* $\lambda \in \mathbb{T}$ *and the operators* B_n *have an exponential discrete dichotomy with data* (M, r, P_n) *uniformly in* $n \in \mathbb{N}$.

Proof. The equivalence (i)⇔(iii) follows from Theorem 2.7. Indeed, by formula (2.19), one gets from (i) that $P_n \xrightarrow{\mathcal{PP}} P$ and $\|P_n\| \le constant$. By Theorem 3.10, one has $\sup_{\lambda\in\mathbb{T}}\|(\lambda I_n - B_n)^{-1}\| < \infty$, and by Theorem 2.7 (ii) we have (iii). Conversely, from condition (iii), it follows by Theorem 2.7 (i) that $\lambda I_n - B_n \xrightarrow{\mathcal{PP}} \lambda I - B$ stably for all $\lambda \in \mathbb{T}$.

To prove (ii)⇒(i), we note that (ii) means that for any $u(\cdot) \in l^p(\mathbb{Z};E)$ one has $\Sigma_{k=-\infty}^{\infty}\|p_n u(k) - B_n p_n u(k-1) - p_n u(k) + p_n Bu(k-1)\|_{E_n}^p \to 0$ as $n \to \infty$, that is, $B_n \xrightarrow{\mathcal{PP}} B$. Assume now that $I_n - B_n$ is not uniformly invertible in $l^\infty(\mathbb{Z};E_n)$, that is, for some sequence $\|x_n\| = 1$, one has $(\lambda_0 I_n - B_n)x_n \xrightarrow{\mathcal{P}} 0$ as $n \to \infty$ for some $\lambda_0 = 1 \in \mathbb{T}$. This means that for stationary sequence $u_n(k) = x_n,\ k \in \mathbb{Z},\ n \in \mathbb{T}$, one has $(\mathfrak{D}_n u_n)(k) = u_n(k) - B_n u_n(k-1) = x_n - B_n x_n \xrightarrow{\mathcal{P}} 0$ for any $k \in \mathbb{N}$ and $n \to \infty$. But, $\mathfrak{D}_n \xrightarrow{\mathcal{PP}} \mathfrak{D}$ stably, that is, $\|\mathfrak{D}_n u_n\|_{l^\infty(\mathbb{Z};E_n)} \geq \gamma\|u_n\|_{l^\infty(\mathbb{Z};E_n)}$ which contradicts $(\lambda_0 I_n - B_n)x_n \xrightarrow{\mathcal{P}} 0$ as $n \to \infty$. Now we show that $\mathcal{R}(\lambda_0 I_n - B_n) = E_n$. For any $y_n \in E_n, \|y_n\| = 1$, consider $v_n(k) = y_n,\ k \in \mathbb{Z},\ n \in \mathbb{N}$. The solution of $\mathfrak{D}_n u_n = v_n$ is a sequence $u_n(k)$, which is stationary too, that is, $(\lambda_0 I_n - B_n)x_n = y_n$, where $x_n = u_n(k),\ k \in \mathbb{Z},\ n \in \mathbb{N}$. To show (i)⇒(ii), we note that $\|\mathfrak{D}_n\|_{B(l^p(\mathbb{Z};E_n))} \leq constant,\ n \in \mathbb{N}$. Now for any $u(\cdot) \in l^p(\mathbb{Z};E)$ and for any $\epsilon > 0$, one can find $K \in \mathbb{N}$ such that $(\Sigma_{k=K}^{\infty} + \Sigma_{k=-K}^{-\infty})\|u(k)\|^p \leq \epsilon$. In the meantime, $\Sigma_{k=-K}^{K}\|p_n u(k) - B_n p_n u(k-1) - p_n u(k) + p_n Bu(k-1)\|_{E_n}^p \to 0$ as $n \to \infty$, since $B_n \xrightarrow{\mathcal{PP}} B$. So we have $\mathfrak{D}_n \xrightarrow{\mathcal{PP}} \mathfrak{D}$. The convergence $\mathfrak{D}_n^{-1} \xrightarrow{\mathcal{PP}} \mathfrak{D}^{-1}$ follows from formula (2.15). The theorem is proved. □

3.2. *Dichotomy for Compact Resolvents in Semidiscretization*

In the case of operators which have compact resolvent, it is natural to consider approximate operators that "preserve" the property of compactness.

Definition 3.13. A sequence of operators $\{B_n\}, B_n : E_n \to E_n,\ n \in \mathbb{N}$, converges compactly to an operator $B : E \to E$ if $B_n \xrightarrow{\mathcal{PP}} B$ and the following compactness condition holds

$$\|x_n\|_{E_n} = O(1) \Longrightarrow \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact}. \tag{3.6}$$

Definition 3.14. The region of compact convergence of resolvents, $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.

Proposition 3.15. *Assume that operators B, B_n are compact, $\mathbb{T} \subset \rho(B)$ and $\Delta_{cc}(B_n, B) \neq \emptyset$. Then, $\lambda I_n - B_n \xrightarrow{\mathcal{PP}} \lambda I - B$ stably for any $\lambda \in \mathbb{T}$ and $\sup_{\lambda\in\mathbb{T}}\|(\lambda I_n - B_n)^{-1}\| < \infty$.*

Proof. The proof follows from Theorems 3.8, 3.9, and 3.10.

Here we continue considering an example of discretization of problem (1.2). To this end we will also use the operators $p_n^\alpha = (-A_n)^{-\alpha} p_n (-A)^\alpha \in B(E^\alpha, E_n^\alpha)$ which satisfy property (3.1), but for the spaces E^α, E_n^α. The operators A_n and A are supposed to be related by condition (1.1), and conditions (A) and (B_1) of Theorem 3.4 are assumed to hold. So we say that $x_n \xrightarrow{\mathcal{P}^\alpha} x$ if and only if $\|x_n - p_n^\alpha x\|_{E_n^\alpha} \to 0$ as $n \to \infty$. One can see that $\|x_n - p_n^\alpha x\|_{E_n^\alpha} = \|(-A_n)^\alpha x_n - p_n(-A)^\alpha x\|$ and $\|p_n^\alpha x\|_{E_n^\alpha} = \|p_n(-A)^\alpha x\|_{E_n} \to \|(-A)^\alpha x\|_E = \|x\|_{E^\alpha}$ for any $x \in D((-A)^\alpha)$ and $n \to \infty$.

Consider in Banach spaces E_n^α the family of parabolic problems

$$\begin{aligned} u_n'(t) &= A_n u_n(t) + f_n(u_n(t)), \quad t \geq 0, \\ u_n(0) &= u_n^0 \in E_n^\alpha, \end{aligned} \tag{3.7}$$

where $u_n^0 \xrightarrow{p^\alpha} u^0$, operators (A_n, A) are compatible, $f_n(\cdot) : E_n^\alpha \to E_n$ are globally bounded and globally Lipschitz continuous both uniformly in $n \in \mathbb{N}$, and continuously Fréchet differentiable.

Under the above assumptions, the mild solution $u_n(\cdot)$ of (3.7) is defined for all $t \geq 0$ (see [1, 17]), and we define it by $u_n(\cdot) = T_n(\cdot)u_n^0 : \mathbb{R}^+ \to E_n$. The nonlinear semigroup $T_n(\cdot)$ satisfies the variation of constants formula

$$T_n(t)u_n^0 = e^{tA_n}u_n^0 + \int_0^t e^{(t-s)A_n} f_n\Big(T_n(s)u_n^0\Big)ds, \quad t \geq 0. \tag{3.8}$$

Recall that a hyperbolic equilibrium point u^* is a solution of equation $Au + f(u) = 0$, or equivalently $u^* = -A^{-1}f(u^*)$. Since the operator A has a compact resolvent, the operator $A^{-1}f(\cdot)$ is compact. In case of $\Delta_{cc} \neq \emptyset$, the operators $A_n^{-1}f_n(\cdot) \xrightarrow{pp} A^{-1}f(\cdot)$ compactly. From [15], it follows that equations $u_n = -A_n^{-1}f_n(u_n)$ have solutions $\{u_n^*\}, A_n u_n^* + f_n(u_n^*) = 0$, such that $u_n^* \xrightarrow{p} u^*$.

Now, consider the problems (3.7) near hyperbolic equilibrium points u_n^*. In this case one gets

$$v_n'(t) = A_{u_n^*,n}v_n(t) + F_{u_n^*,n}(v_n(t)), \quad v_n(0) = v_n^0, \ t \geq 0, \tag{3.9}$$

where $A_{u_n^*,n} = A_n + f_n'(u_n^*)$, $F_{u_n^*,n}(v_n(t)) = f_n(v_n(t) + u_n^*) - f_n(u_n^*) - f_n'(u_n^*)v_n(t)$. From now on, we consider hyperbolic point u^* and hyperbolic points $u_n^* \xrightarrow{p^\alpha} u^*$.

We decompose E_n^α using the projection

$$P_n(\sigma_n^+) := P_n\big(\sigma_n^+, A_{u_n^*,n}\big) := \frac{1}{2\pi i}\int_{\partial U(\sigma_n^+)} \big(\zeta I_n - A_{u_n^*,n}\big)^{-1} d\zeta \tag{3.10}$$

defined by the set σ_n^+, which is enclosed in a contour consisting of a part of $i\mathbb{R}$ and the contour from condition (B_1) for operators $A_{u_n^*,n}$.

There are some positive $M_2, \gamma > 0$, because of analyticity of C_0-semigroup $e^{tA_{u_n^*,n}}$ and condition $\Delta_{cc} \neq \emptyset$, which is applied to operator $A_{u_n^*,n}$, such that [7, 18]

$$\begin{aligned} \left\| e^{tA_{u_n^*,n}} z_n \right\|_{E_n^\alpha} &\leq M_2 e^{-\gamma t}\|z_n\|_{E_n^\alpha}, \quad t \geq 0, \\ \left\| e^{tA_{u_n^*,n}} v_n \right\|_{E_n^\alpha} &\leq M_2 e^{\gamma t}\|v_n\|_{E_n^\alpha}, \quad t \leq 0, \end{aligned} \tag{3.11}$$

for all $v_n \in P_n(\sigma_n^+)E_n^\alpha$ and $z_n \in (I_n - P_n(\sigma_n^+))E_n^\alpha$. One has to note that the neighborhood of any part of $i\mathbb{R}$ does not intersect $\sigma(A_{u_n^*,n})$. Moreover, the condition $\Delta_{cc} \neq \emptyset$ implies that $P_n(\sigma_n^+) \xrightarrow{pp} P(\sigma^+)$ compactly, and therefore, as was shown in [14], $\dim P_n(\sigma_n^+) = \dim P(\sigma^+)$ for $n \geq n_0$. One can consider for the $T = T(v_n^0) = \sup\{t \geq 0 : v_n(t, v_n^0) \in \mathcal{U}_{E_n^\alpha}(0;\rho)\}$ with $v_n^0 \in \mathcal{U}_{E_n^\alpha}(0;\rho)$ the problem

$$\begin{gathered} v_n'(t) = A_{u_n^*,n}v_n(t) + F_{u_n^*,n}(v_n(t)), \quad 0 \leq t \leq T, \\ (I_n - P_n(\sigma_n^+))v_n(0) = (I_n - P_n(\sigma_n^+))v_n^0, \qquad P_n(\sigma_n^+)v_n(T) = P_n(\sigma_n^+)v_n^T. \end{gathered} \tag{3.12}$$

The mild solution of problem (3.12) is given by the formula (for $0 \leq t \leq T$)

$$v_n(t) = e^{(t-T)A_{u_n^*,n}} P_n(\sigma_n^+) v_n^T + e^{tA_{u_n^*,n}} (I_n - P_n(\sigma_n^+)) v_n^0 + \int_0^t e^{(t-s)A_{u_n^*,n}} (I_n - P_n(\sigma_n^+)) F_{u_n^*,n}(v_n(s)) ds + \int_t^T e^{(t-s)A_{u_n^*,n}} P_n(\sigma_n^+) F_{u_n^*,n}(v_n(s)) ds. \tag{3.13}$$

Now we are in a position to state our main result on uniform in index n estimates for the terms of discrete solutions (3.13). □

Theorem 3.16. *Let operators A_n, A be generators of analytic C_0-semigroups and let condition (B_1) be satisfied. Assume also that the semigroup $e^{tA_{u^*}}$ is hyperbolic, $\sigma(A_{u^*}) \cap \{\lambda : \operatorname{Re}\lambda \geq 0\} = P\sigma(A_{u^*})$, $\dim P(\sigma+) < \infty$, and for $\rho > 0$ such that $\{\lambda : -\rho \leq \operatorname{Re}\lambda \leq \rho\} \subset \rho(A_{u^*})$, operators $\lambda I_n - A_{u_n^*,n}$ are Fredholm operators of ind 0, and operators $\lambda I_n - A_{u_n^*,n}, \lambda I - A_{u^*}$ are regularly consistent for any $\operatorname{Re}\lambda \geq -\rho$. Then, $P_n(\sigma_n^+) \xrightarrow{\mathcal{PP}} P(\sigma^+)$ compactly and*

$$\begin{aligned} \left\| e^{tA_{u_n^*,n}} (I_n - P_n(\sigma_n^+)) \right\|_{E_n} &\leq M_2 e^{-\gamma t}, \quad t \geq 0, \\ \left\| e^{tA_{u_n^*,n}} P_n(\sigma_n^+) \right\|_{E_n} &\leq M_2 e^{\gamma t}, \quad t \leq 0, \end{aligned} \tag{3.14}$$

where $\gamma > 0$.

Proof. The condition (B_1) implies that $(\lambda I_n - A_{u_n^*,n})^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A_{u^*})^{-1}$ as $-\rho \leq \operatorname{Re}\lambda \leq \rho$ for $|\lambda|$ big enough. For the other $-\rho \leq \operatorname{Re}\lambda \leq \rho$, the convergence $(\lambda I_n - A_{u_n^*,n})^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A_{u^*})^{-1}$ follows from analogy of Theorem 3.9 for closed operators. Now the compact convergence $P_n(\sigma_n^+) \xrightarrow{\mathcal{PP}} P(\sigma^+)$ can be obtained in the same way as in [3] and the estimates (3.14) follow say as in [7, 18]. The theorem is proved. □

Remark 3.17. Of course, Theorem 3.16 holds for the case of any operator A which generates analytic hyperbolic C_0-semigroup with condition $\sigma(A) \cap \{\lambda : \operatorname{Re}\lambda \geq 0\} = P\sigma(A)$, $\dim P(\sigma+) < \infty$ and corresponding conditions on approximation of operators. The structure of operator like $A + f'(u^*)$ is not necessary.

Theorem 3.18. *Let the operators A_n, A be generators of analytic C_0-semigroups and let the condition (B_1) be satisfied. Assume also that the C_0-semigroup $e^{tA_{u^*}}$ is hyperbolic, $\sigma(A_{u^*}) \cap \{\lambda : \operatorname{Re}\lambda \geq 0\} = P\sigma(A_{u^*})$, $\dim P(\sigma+) < \infty$. Assume also that $\Delta_{cc}(A_n, A) \neq \emptyset$ and resolvents of A_n, A are compact operators. Then,* (3.14) *holds.*

Proof. We set $B_n = e^{1A_{u_n^*,n}}$ and $B = e^{1A_{u^*}}$ and apply Proposition 3.15. It is known [3] that $\Delta_{cc} \neq \emptyset$ is equivalent to compact convergence $B_n \xrightarrow{\mathcal{PP}} B$. Then, the condition (i) of Theorem 3.12 is satisfied and one gets discrete dichotomy for B_n. From the other side, since we have exactly the form $B_n = e^{1A_{u_n^*,n}}$, this means by Theorem 2.7 and Proposition 2.4 that operators $\mathcal{L}_n = -d/dt + A_{u_n^*,n}$ are invertible, and thus by Theorem 2.3 the estimates (3.14) are followed. The theorem is proved. □

3.3. Dichotomy for Condensing Operators in Semidiscretization

Let $\Lambda \subseteq \mathbf{C}$ be some open connected set, and let $B \in B(E)$. For an isolated point $\lambda \in \sigma(B)$, the corresponding maximal invariant space (or generalized eigenspace) will be denoted by $\mathcal{W}(\lambda; B) = Q(\lambda)E$, where $Q(\lambda) = (1/2\pi i)\int_{|\zeta-\lambda|=\delta}(\zeta I - B)^{-1}d\zeta$ and δ is small enough so that there are no points of $\sigma(B)$ in the disc $\{\zeta : |\zeta - \lambda| \leq \delta\}$ different from λ. The isolated point $\lambda \in \sigma(B)$ is a *Riesz point* of B if $\lambda I - B$ is a Fredholm operator of index zero and $Q(\lambda)$ is of finite rank. Denote

$$\mathcal{W}(\lambda, \delta; B_n) = \bigcup_{\substack{\lambda_n \in \sigma(B_n), \\ |\lambda_n - \lambda| < \delta}} \mathcal{W}(\lambda_n, B_n). \tag{3.15}$$

It is clear that $\mathcal{W}(\lambda, \delta; B_n) = Q_n(\lambda)E_n$, where

$$Q_n(\lambda) = \frac{1}{2\pi i}\int_{|\zeta-\lambda|=\delta}(\zeta I_n - B_n)^{-1}d\zeta. \tag{3.16}$$

Definition 3.19. The function $\mu(\cdot)$ is said to be the measure of noncompactness if for any bounded sequence $\{x_n\}, x_n \in E_n$, one has

$$\mu(\{x_n\}) = \inf\left\{\epsilon > 0 : \forall \mathbb{N}' \subseteq \mathbb{N},\ \exists \mathbb{N}'' \subseteq \mathbb{N}',\ x' \in E \text{ such that } \left\|x_n - p_n x'\right\| \leq \epsilon, n \in \mathbb{N}''\right\}. \tag{3.17}$$

Definition 3.20. We say that the operators $B_n \in B(E_n)$ are jointly condensing with constant $q > 0$ with respect to measure $\mu(\cdot)$ if for any bounded sequence $\{x_n\}, x_n \in E_n$, one has

$$\mu(\{B_n x_n\}) \leq q\mu(\{x_n\}). \tag{3.18}$$

It is known [19, page 82], that outside a closed disc of radius q centered at zero each operator B_n has only isolated points of spectrum, each of which can only be an eigenvalue of finite multiplicity.

Proposition 3.21. *Let $B_n \xrightarrow{\mathcal{PP}} B$ for $B_n \in B(E_n)$, $B \in B(E)$ and $\mu(\{B_n x_n\}) \leq q\mu(\{x_n\})$, for any bounded sequence $\{x_n\}, x_n \in E_n$. Assume that $\sigma(B) \cap \Psi = \emptyset$, where bounded closed set $\Psi \subset \mathbf{C} \setminus \{\lambda : |\lambda| \leq q\}$ and $\sigma(B) \setminus \{\lambda : |\lambda| \leq \mathbf{q}\}$ consists only of discrete spectrum. Then, there is a constant $C > 0$ such that $\|(\lambda I_n - B_n)^{-1}\| \leq C, \lambda \in \Psi, n \in \mathbb{N}$.*

Proof. Any point $\lambda \in \Psi$ belongs to $P\sigma(B_n) \cup \rho(B_n)$. This means that one gets for a sequence $\|x_n\| = 1$, $x_n \in E_n$, two cases: $(\lambda I_n - B_n)x_n = 0$ or $\|(\lambda I_n - B_n)x_n\| \geq \gamma_{\lambda,n}\|x_n\|$ with some $\gamma_{\lambda,n} > 0, \lambda \in \Psi$. We are going to show that in reality we have $\|(\lambda I_n - B_n)x_n\| \geq \gamma_\Psi\|x_n\|$, $\lambda \in \Psi$.

Assume in contradiction that there are sequences $\{\lambda_n\}, \lambda_n \in \Psi, \{x_n\}, \|x_n\| = 1$, such that

$$(\lambda_n I_n - B_n)x_n \xrightarrow{\mathcal{P}} 0 \text{ as } n \in \mathbb{N}. \tag{3.19}$$

Then, $\lambda_n \to \lambda_0 \in \Psi,\ n \in \mathbb{N}' \subseteq \mathbb{N}$. One has for $\tilde{r} = \inf\{|\xi| : \xi \in \Psi\}$

$$\mu(\{x_n\}) \le \frac{|\lambda_n|}{\tilde{r}}\mu(\{x_n\}) \le \frac{\mu(\{B_n x_n\})}{\tilde{r}} \le \frac{q}{\tilde{r}}\mu(\{x_n\}), \tag{3.20}$$

which means because of $q/\tilde{r} < 1$, that $\mu(\{x_n\}) = 0$, that is, $\{x_n\}$ is $\mathcal{P}$-compact. Now $x_n \xrightarrow{\mathcal{P}} x_0,\ n \in \mathbb{N}'' \subseteq \mathbb{N}'$ and $B_n x_n \xrightarrow{\mathcal{P}} Bx_0,\ \lambda_n x_n \xrightarrow{\mathcal{P}} \lambda_0 x_0,\ n \in \mathbb{N}''$, that is, $\lambda_0 x_0 = Bx_0$ with $\|x_0\| = 1$, which contradicts our assumption $\sigma(B) \cap \Psi = \emptyset$. The proposition is proved. □

Proposition 3.22. *Let $B_n \xrightarrow{\mathcal{PP}} B$ and $\mu(\{B_n x_n\}) \le q\mu(\{x_n\})$ for any bounded sequence $\{x_n\}, x_n \in E_n$. Assume that any $\lambda_0 \in \sigma(B)$, $|\lambda_0| > q$, is an isolated eigenvalue with the finite dimensional projector $Q(\lambda_0)$. Then, there are sequence $\{\lambda_n\}, \lambda_n \in \sigma(B_n)$, and sequence of projectors $Q_n(\lambda_0) \in B(E_n)$ such that $\lambda_n \to \lambda_0$ and $Q_n(\lambda_0) \xrightarrow{\mathcal{PP}} Q(\lambda_0)$ converge compactly.*

Proof. Note first that for $\Gamma_r = \{\lambda : |\lambda - \lambda_0| = r\} \subset \mathbf{C} \setminus \{\lambda : |\lambda| \le \mathbf{q}\}$, where r can be taken small enough, we have by Proposition 3.21

$$(\lambda I_n - B_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - B)^{-1} \quad \text{as } \lambda \in \Gamma_r,\ n \in \mathbb{N}. \tag{3.21}$$

Therefore, $Q_n(\lambda_0) \xrightarrow{\mathcal{PP}} Q(\lambda_0)$. To show compact convergence of these projectors one can note that

$$\mu(\{(\lambda_0 I_n - B_n)x_n\}) \ge |\lambda_0|\mu(\{x_n\}) - \mu(\{B_n x_n\}) \ge |\lambda_0|\mu(\{x_n\}) - q\mu(\{x_n\}) \ge \gamma\mu(\{x_n\}), \tag{3.22}$$

where $\gamma = |\lambda_0| - q > 0$. This means that

$$\mu\Big(\Big\{(\lambda_0 I_n - B_n)^k x_n\Big\}\Big) \ge \gamma^k \mu(\{x_n\}) \quad \text{for any } k \in \mathbb{N}. \tag{3.23}$$

By functional calculus

$$(\lambda_0 I_n - B_n)^k Q_n(\lambda_0) x_n = \frac{1}{2\pi i}\int_{\Gamma_r} (\lambda_0 - \lambda)^k (\lambda I_n - B_n)^{-1} x_n d\lambda. \tag{3.24}$$

From this representation using (3.23), one has

$$\gamma^k \mu(\{Q_n(\lambda_0)x_n\}) \le \Big\|(\lambda_0 I_n - B_n)^k Q_n(\lambda_0) x_n\Big\| \le \frac{C}{2\pi} r^k\ \|x_n\|. \tag{3.25}$$

It is clear that from $r/\gamma < 1$ it follows that $(r/\gamma)^k \to 0$ as $k \to \infty$. This means that $Q_n(\lambda_0) \xrightarrow{\mathcal{PP}} Q(\lambda_0)$ compactly. The proposition is proved. □

Theorem 3.23. *Let the conditions (A) and (B_1) be satisfied, and let the analytic C_0-semigroup $e^{tA}, t \in \mathbb{R}_+$, be hyperbolic such that the set $\sigma(A) \cap \{\lambda : \operatorname{Re}\lambda \ge 0\}$ consists of a finite number of points*

$P\sigma(A)$ and $\dim P(\sigma+) < \infty$. Assume also that $\mu(\{B_n x_n\}) \le q\mu(\{x_n\})$, for any bounded sequence $\{x_n\}, x_n \in E_n$, with $q < 1$, where $B_n = e^{1A_n}$. Then, conclusions of Theorem 3.16, that is, (3.14), are hold and $P_n(\sigma_n+) \xrightarrow{\mathcal{PP}} P(\sigma+)$ converge compactly.

Proof. Because of the spectral mapping theorem, the spectrum of operator $B = e^{1A}$ which is located outside of unit disc $\mathbb{T}$ consists of finite number of points of the set $P\sigma(e^{1A}) = \{\zeta : \zeta = e^{\lambda}, \lambda \in P\sigma(A) \cap \{\xi : \operatorname{Re}\xi \ge 0\}\}$. Moreover, since $q < 1$, for any Ψ which contains $\mathbb{T}$ and $\Psi \subset \rho(B), B = e^{1A}$, one has $\|(\lambda I_n - B_n)^{-1}\| \le constant$ as $\lambda \in \Psi$ by Proposition 3.22. Now from Theorem 3.12 it follows that B_n have discrete dichotomy. By Theorem 2.5, the operator $\mathfrak{D}_n$ is invertible, and by Proposition 2.4 and by Theorem 2.3 we get that the semigroups $e^{tA_n}, t \in \mathbb{R}_+$, have an exponential dichotomy uniformly in index $n \in \mathbb{N}$ and (2.7), that is, (3.14) holds. By Theorem 3.9 $Q_n(\lambda_0) \xrightarrow{\mathcal{PP}} Q(\lambda_0)$ compactly, which implies that conditions of Theorem 3.16 are satisfied. The theorem is proved. □

3.4. Discretisation in Time Variable and $\Delta_{cc} \neq \emptyset$

Consider now the discretization of the problem (3.9) in time by the following scheme:

$$\frac{V_n(t+\tau_n) - V_n(t)}{\tau_n} = A_{u_n^*,n} V_n(t+\tau_n) + F_{u_n^*,n}(V_n(t)), \quad t = k\tau_n, \tag{3.26}$$

with initial data $V_n(0) = v_n^0$. The solution of such problem is given by formula

$$\begin{aligned} V_n(t+\tau_n) &= \left(I_n - \tau_n A_{u_n^*,n}\right)^{-1} V_n(t) + \tau_n \left(I_n - \tau_n A_{u_n^*,n}\right)^{-1} F_{u_n^*,n}(V_n(t)) \\ &= \left(I_n - \tau_n A_{u_n^*,n}\right)^{-k} V_n(0) + \tau_n \sum_{j=0}^{k} \left(I_n - \tau_n A_{u_n^*,n}\right)^{-(k-j+1)} F_{u_n^*,n}\left(V_n(j\tau_n)\right), \quad t = k\tau_n, \end{aligned} \tag{3.27}$$

where $V_n(0) = v_n^0$.

The problem (3.12) also can be discretized following (3.26) approach, so we have

$$\frac{V_n(t+\tau_n) - V_n(t)}{\tau_n} = A_{u_n^*,n} V_n(t+\tau_n) + F_{u_n^*,n}(V_n(t)), \quad t = k\tau_n, \tag{3.28}$$

$$(I_n - P_n)V_n(0) = (I_n - P_n)v_n^0, \qquad P_n V_n(T) = P_n v_n^T. \tag{3.29}$$

The solution of problem (3.28) could be obtained by using the formulas

$$\begin{aligned} (I_n - P_n)V_n(t+\tau_n) &= \left(I_n - \tau_n A_{u_n^*,n}\right)^{-1} (I_n - P_n) V_n(t) + \tau_n \left(I_n - \tau_n A_{u_n^*,n}\right)^{-1} (I_n - P_n) F_{u_n^*,n}\left(V_n(j\tau_n)\right), \\ \left(I_n - \tau_n A_{u_n^*,n}\right) P_n V_n(t+\tau_n) &= P_n V_n(t) + \tau_n P_n F_{u_n^*,n}(V_n(k\tau_n)), \quad t = k\tau_n. \end{aligned} \tag{3.30}$$

So one has a representation of solution of problem (3.28) as

$$
\begin{aligned}
V_n\left(t; v_n^0, v_n^T\right) &= (I_n - P_n)V_n(t) + P_n V_n(t) \\
&= \left(I_n - \tau_n A_{u_n^*,n}\right)^{-k+1}(I_n - P_n)v_n^0 \\
&\quad + \tau_n \sum_{j=0}^{k-1}\left(I_n - \tau_n A_{u_n^*,n}\right)^{-(k-j)}(I_n - P_n)F_{u_n^*,n}\left(V_n\left(j\tau_n\right)\right) \\
&\quad + \left(I_n - \tau_n A_{u_n^*,n}\right)^{K-k}P_n v_n^T - \tau_n \sum_{j=k}^{K-1}\left(I_n - \tau_n A_{u_n^*,n}\right)^{K-1-j}P_n F_{u_n^*,n}\left(V_n\left(j\tau_n\right)\right), \quad t = k\tau_n.
\end{aligned}
\tag{3.31}
$$

From (3.31), it is clear that corresponding estimates on powers of operators $(I_n - \tau_n A_{u_n^*,n})^{-k+1}(I_n - P_n), (I_n - \tau_n A_{u_n^*,n})^{K-k}P_n$ play the main role in approximation of solutions of (1.4) in the vicinity of u^*.

Theorem 3.24. *Let operators A_n, A be generators of analytic C_0-semigroups and let condition (B_1) be satisfied. Assume also that the analytic C_0-semigroup $e^{tA_{u^*}}, t \in \mathbb{R}_+$, is hyperbolic and for $\rho > 0$ such that $\{\lambda : -\rho \le \operatorname{Re}\lambda \le \rho\} \subset \rho(A)$, operators $\lambda I_n - A_{u_n^*,n}$ are Fredholm operators of ind 0, and operators $\lambda I_n - A_{u_n^*,n}, \lambda I - A_{u^*}$ are regularly consistent for any $\operatorname{Re}\lambda \ge -\rho$. Then, $P_n(\sigma+) \xrightarrow{\mathcal{PP}} P(\sigma+)$ compactly and*

$$
\begin{aligned}
\left\|\left(I_n - \tau_n A_{u_n^*,n}\right)^{-k_n}P_n\right\|_{E_n} &\le M_2 e^{-\gamma t}, \quad t \ge 0, \\
\left\|\left(I_n - \tau_n A_{u_n^*,n}\right)^{k_n}(I_n - P_n)\right\|_{E_n} &\le M_2 e^{\gamma t}, \quad t \le 0,
\end{aligned}
\tag{3.32}
$$

for some $\gamma > 0$.

Proof. From Theorem 3.16 it follows that the analytic C_0-semigroups $e^{tA_{u_n^*,n}}$ have dichotomy uniformly in $n \in \mathbb{N}$. Now one can see as in [20] that $\|(I_n - \tau_n A_{u_n^*,n})^{-k_n} - e^{tA_{u_n^*,n}}\| \le M\tau_n e^{\omega t}/t$, where $t = k_n\tau_n = 1$. Using perturbation dichotomy theorem from [1, p. 254], one gets that (3.32) holds. The Theorem is proved. □

Theorem 3.25. *Let $\Delta_{cc}(A_n, A) \ne \emptyset$ and resolvents of A_n, A be compact operators. Assume also that the analytic C_0-semigroup $e^{tA_{u^*}}, t \in \mathbb{R}_+$, is hyperbolic and condition (B_1) is satisfied. Then,*

$$
\begin{aligned}
\left\|\left(I_n - \tau_n A_{u_n^*,n}\right)^{-k_n}P_n\right\|_{E_n} &\le M_2 r^{[t]}, \quad t = k_n\tau_n \ge 0, \\
\left\|\left(I_n - \tau_n A_{u_n^*,n}\right)^{k_n}(I_n - P_n)\right\|_{E_n} &\le M_2 r^{-[t]}, \quad -t = -k_n\tau_n \le 0,
\end{aligned}
\tag{3.33}
$$

where $r < 1$.

Proof. Compact convergence of resolvents $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ implies [18] that

$$
\left(\lambda I_n - A_{u_n^*,n}\right)^{-1} \xrightarrow{\mathcal{PP}} \left(\lambda I - A_{u^*}\right)^{-1} \text{ compactly.} \tag{3.34}
$$

We set $B_n = (I_n - \tau_n A_{u_n^*,n})^{-k_n}, \tau_n k_n = 1$, and $B = e^{1A_{u^*}}$. Then, $B_n \xrightarrow{\mathcal{PP}} B$, since the operators $A_{u_n^*,n}, A_{u^*}$ are consistent. Note that by condition (B_1) one has [21, 22] $\|\tau_n k_n A_{u_n^*,n}(I_n - \tau_n A_{u_n^*,n})^{-k_n}\|_{B(E_n)} \le constant$, which implies $B_n = A_{u_n^*,n}^{-1}\tau_n k_n A_{u_n^*,n}(I_n - \tau_n A_{u_n^*,n})^{-k_n} \xrightarrow{\mathcal{PP}} B$ compactly, since $A_{u_n^*,n}^{-1} \xrightarrow{\mathcal{PP}} A_{u^*}^{-1}$ compactly. Now by applying Proposition 3.15 one gets discrete dichotomy for B_n by Theorem 3.12. The theorem is proved. □

3.5. Discretisation in Time Variable and Condensing Property

Theorem 3.26. *Let condition* (A) *and condition* (B_1) *be satisfied. Assume that* $\mu(\{B_n x_n\}) \le q\,\mu(\{x_n\})$, *for any bounded sequence* $\{x_n\}, x_n \in E_n$, *with* $q < 1$ *and* $B_n = e^{A_{u_n^*,n}}$. *Assume also that the analytic* C_0*-semigroup* $e^{tA_{u^*}}, t \in \mathbb{R}_+$, *is hyperbolic.Then,*

$$\begin{aligned} \left\| (I_n - \tau_n A_{u_n^*,n})^{-k_n} P_n \right\|_{E_n} &\le M_2 r^{[t]}, \quad t = k_n\tau_n \ge 0, \\ \left\| (I_n - \tau_n A_{u_n^*,n})^{k_n} (I_n - P_n) \right\|_{E_n} &\le M_2 r^{-[t]}, \quad -t = -k_n\tau_n \le 0, \end{aligned} \tag{3.35}$$

where $r < 1$.

Proof. By Theorem 3.23 it follows that (3.11) holds. By perturbation theorem from [1], one gets (3.35) which follows in the same way as in the proof of Theorem 3.24. The theorem is proved. □

4. Example

The condition $\mu(B_n x_n) \le q\mu(x_n)$ with $q < 1$ in Theorems 3.23 and 3.26 can be checked for instance in case of compact convergence of operators $A_n^{-1} f_n'(u_n^*) \xrightarrow{\mathcal{PP}} A^{-1} f'(u^*)$. We present here an example where the analogy of such condition is naturally satisfied.

Example 4.1. Consider in $L^2(\mathbb{R})$ an operator

$$(Av)(x) = v''(x) + av'(x) + bv(x), \quad x \in (-\infty, \infty). \tag{4.1}$$

Since we took $E = L^2(R)$, we can take $(p_n v)(x) = (1/h)\int_{-h/2}^{h/2} v(x+y)dy$, and the main condition $\|p_n v\|_{L_h^2(Z)} \to \|v\|_{L^2(R)}$ is satisfied [15].

As in Section 5.4 of [1], one can see that $\sigma_{\mathrm{ess}}(-A) \subset \{\lambda : \operatorname{Re}\lambda - ((Im\lambda)^2/a^2 \ge -b\}$. For the case of $a = 0$, we have $\sigma(A) \in (-\infty, b)$. So for $b < 0$ the operator A is a negative self-adjoint operator. Then, we have the same for some difference scheme, say for central difference scheme,

$$A_n v_n(x) = \frac{v_n(x+h) - 2v_n(x) + v_n(x-h)}{h^2} + bv_n(x), \tag{4.2}$$

that is, $\omega_{\text{ess}}(A_n) \le \omega_1 < 0$ uniformly in $h > 0$. Moreover, it is easy to see that

$$\left\| e^{tA_n} \right\| \le M e^{\omega_2 t}, \quad t \ge 0, \text{ with } \omega_2 < 0, \tag{4.3}$$

that is,

$$\mu\left(\left\{ e^{tA_n} x_n \right\}\right) \le \gamma \mu(\{x_n\}) \quad \text{with } \gamma < 1 \text{ for some } t = t_0 > 0. \tag{4.4}$$

Now to get to the range of Theorem 3.23, let us denote $B = e^{t_0 A}$, $B_n = e^{t_0 A_n}$. For analytic C_0-semigroups, the spectrums of operators A and B are strictly related, which is also concerned to point spectrum $P\sigma(B) = e^{t_0 P\sigma(A)}$. This means that operators A_n have for almost all n the spectrums $\sigma(A_n) \cap \{\lambda : \operatorname{Re}\lambda > 0\}$ which approximate the spectrum $\sigma(A) \cap \{\lambda : \operatorname{Re}\lambda > 0\}$.

Let us now consider in $L^2(\mathbb{R})$ the case of perturbed operator with smooth function $b(x)$

$$\widetilde{A} v(x) = v''(x) + b(x) v(x) \tag{4.5}$$

and its approximation, say like

$$\left(\widetilde{A}_n v_n\right)(x) = \frac{v_n(x+h) - 2v_n(x) + v_n(x-h)}{h^2} + b(x) v_n(x), \tag{4.6}$$

with condition $b(x) \to b$ as $x \to \pm\infty$ for simplicity. The operator $((\widetilde{A} - A)v)(x) = (b(x) - b)v(x)$ is an additive perturbation. We assume that C_0-semigroup $e^{t\widetilde{A}}, t \in \mathbb{R}_+$, is hyperbolic. The perturbation $\widetilde{A} - A$ is a relatively compact perturbation like in [23]. The same must happen to A_n because of $\widetilde{A}_n = A_n + (\widetilde{A}_n - A_n)$ and

$$e^{t_0 \widetilde{A}_n} = e^{t_0 A_n} + \int_0^{t_0} A_n^{\alpha} e^{(t_0 - s) A_n} A_n^{-\alpha} \left(\widetilde{A}_n - A_n\right) e^{s \widetilde{A}_n} ds. \tag{4.7}$$

The crucial point is that such perturbation gives us from (4.4) the estimate

$$\mu\left(\left\{ e^{t\widetilde{A}_n} x_n \right\}\right) \le \gamma \mu(\{x_n\}) \quad \text{with } \gamma < 1 \text{ for some t} = t_0 > 0, \tag{4.8}$$

since the integral part in (4.7) could be estimated by any small $\epsilon > 0$ as $\mu(\int_0^{t_0 - \epsilon} + \int_{t_0 - \epsilon}^{t_0}) \le c\epsilon^{1-\alpha}$.

Then any point of spectrum of $\widetilde{A}$ which is located to the right of b belongs to $P\sigma(\widetilde{A})$ and it is of finite dimensional generalized eigenspace. The same is true for $\widetilde{A}_n$ with $B_n = e^{t\widetilde{A}_n}$, since we have (4.8). Using property (4.8), we get from Theorems 3.8 and 3.9 the regular consistence of operators $\lambda I_n - \widetilde{A}_n, \lambda I - \widetilde{A}$ for any $\lambda \in i\mathbb{R}$ and any $\operatorname{Re}\lambda > b$.

If as before P is a dichotomy projector, then one has $\dim P < \infty$ and $P\widetilde{A} = \widetilde{A}P$. We can also state that $P_n \to P$ compactly by Proposition 3.22. This means that say from dichotomy of $\widetilde{A}$ we get dichotomy for $\widetilde{A}_n$ uniformly in n by Theorem 3.16.

The similar situation for concrete differential operator was considered in [24].

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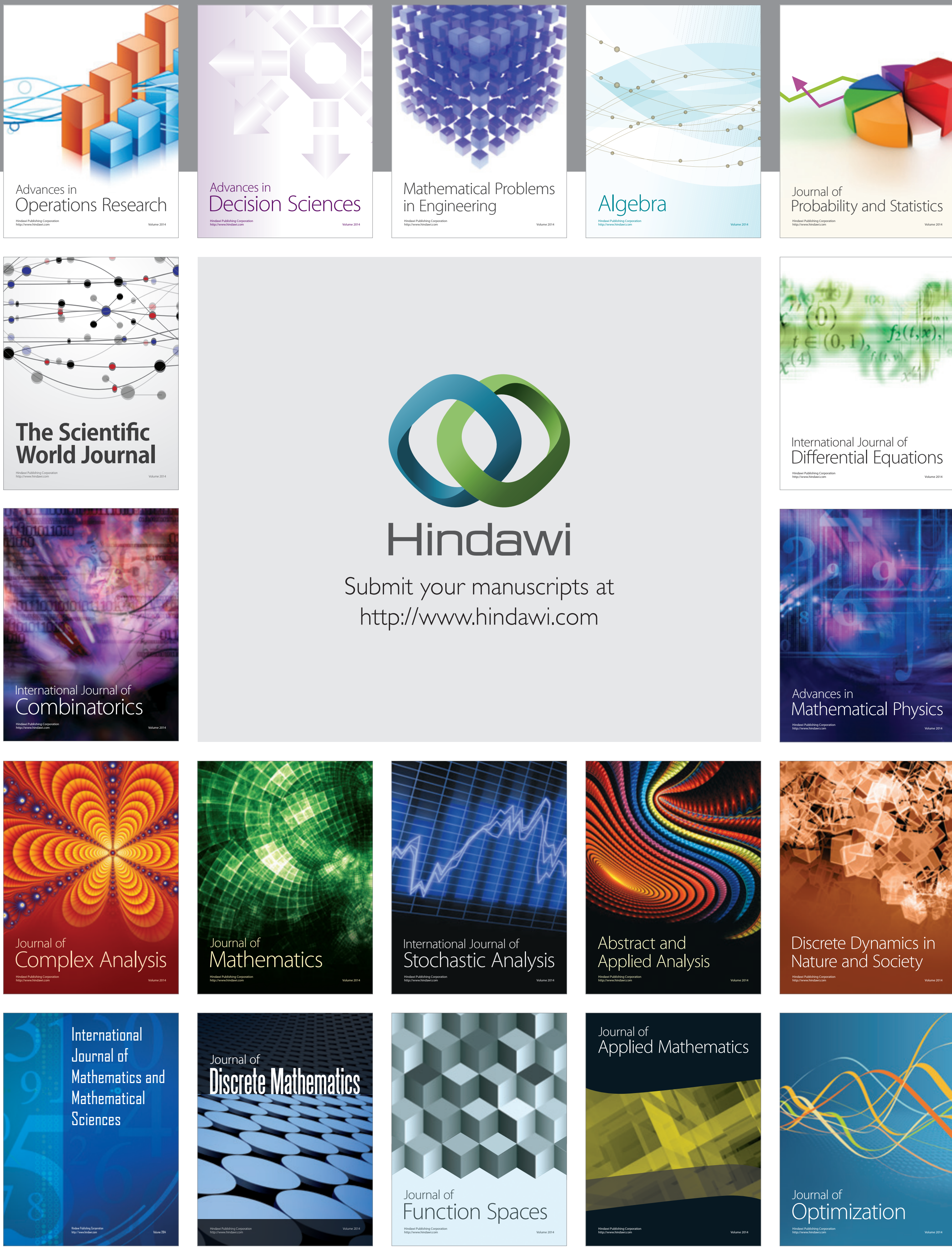
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