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Research Article

Generalized Binomial Probability Distributions Attached to Landau Levels on the Riemann Sphere

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A family of generalized binomial probability distributions attached to Landau levels on the Riemann sphere is introduced by constructing a kind of generalized coherent states. Their main statistical parameters are obtained explicitly. As an application, photon number statistics related to coherent states under consideration are discussed.

1. Introduction

The *binomial states* (BS) are the field states that are defined as a finite linear superposition of field number states weighted by a binomial counting probability distribution [1, 2]. Precisely, these states are labeled by points z of the Riemann sphere $\mathbb{S}^2 \equiv \mathbb{C} \cup \{\infty\}$, and are of the from

$$|z, B\rangle = \left(1 + |z|^2\right)^{-B} \sum_{j=0}^{2B} \left(\frac{(2B)!}{j!(2B-j)!}\right)^{1/2} z^j |j\rangle, \tag{1.1}$$

where $B \in \mathbb{Z}_+$ is a fixed integer parameter and $|j\rangle$ are number states of the field mode. Define $\mu_z := |z|^2(1+|z|^2)^{-1}$, then the probability for the production of j photons is given by the squared modulus of the projection of the BS $|z, B\rangle$ onto the number state $|j\rangle$ as

$$\left|\langle j \mid z, B\rangle\right|^2 = \frac{(2B)!}{j!(2B-j)!} \mu_z^j (1-\mu_z)^{2B-j}. \tag{1.2}$$

The latter is recognized as the binomial probability density $\mathcal{B}(2B,\mu_z)$ where $\{\mu_z,(1-\mu_z)\}$ are the probabilities of the two possible outcomes of a Bernoulli trial [3]. Also, observe that the coefficients in the finite superposition of number states in (1.1)

$$h_j^B(z) := \left(1+|z|^2\right)^{-B}\left(\frac{(2B)!}{j!(2B-j)!}\right)^{1/2} z^j, \quad j=0,1,2,\ldots,2B, \tag{1.3}$$

constitute an orthonormal basis of the null space

$$\mathcal{A}_B\left(\mathbb{S}^2\right) := \left\{\varphi\in L^2\left(\mathbb{S}^2\right),\ H_B[\varphi]=0\right\} \tag{1.4}$$

of the second-order differential operator

$$H_B := -\left(1+|z|^2\right)^2\frac{\partial^2}{\partial z\partial\overline{z}} - B\left(1+|z|^2\right)\left(z\frac{\partial}{\partial z}-\overline{z}\frac{\partial}{\partial\overline{z}}\right)+B^2|z|^2-B, \tag{1.5}$$

which constitutes (in suitable units and up to additive constant) a realization in $L^2(\mathbb{S}^2)$ of the Schrödinger operator with uniform magnetic field on $\mathbb{S}^2$, with a field strength proportional to B (see [4]). The given orthonormal basis $h_j^B(z)$ together with the reproducing kernel

$$K_B(z,w) = (2B+1)(1+z\overline{w})^{2B}\left(1+|z|^2\right)^{-B}\left(1+|w|^2\right)^{-B} \tag{1.6}$$

of the Hilbert space $\mathcal{A}_B(\mathbb{S}^2)$ in (1.4) can be used to interpret the projection of the BS $|z,B\rangle$ onto the number state $|j\rangle$ mentioned in (1.2) by writing

$$\langle j \mid z,B\rangle = (K_B(z,z))^{-(1/2)}h_j^B(z). \tag{1.7}$$

Note also that the space $\mathcal{A}_B(\mathbb{S}^2)$ is nothing else than the eigenspace associated with the first eigenvalue of the spectrum of H_B acting on $L^2(\mathbb{S}^2)$, which consists of an infinite set of eigenvalues *(spherical Landau levels)* of the form $\epsilon_m^B:=(2m+1)B+m(m+1),\quad m=0,1,2,\ldots,$ with finite multiplicity; that is, the associated L^2-eigenspace

$$\mathcal{A}_{B,m}\left(\mathbb{S}^2\right) := \left\{\varphi\in L^2\left(\mathbb{S}^2\right),\ H_B[\varphi]=\epsilon_m^B\varphi\right\} \tag{1.8}$$

is of finite dimension equals to $d_{B,m} := 2B+2m+1$.

Here, we take the advantage of the fact that each of the eigenspaces in (1.8) admits an orthogonal basis, denoted $h_j^{B,m}(z)$, $j=0,1,2,\ldots,2B+2m$, whose elements are expressed in terms of Jacobi polynomials $P_n^{(\tau,\varsigma)}(\cdot)$, as well as a reproducing kernel $K_{B,m}(z,w)$ in an explicit form (see [5]) to consider a set of coherent states by adopting a generalized coherent states form *"à la Iwata"* [6] as

$$|z,B,m\rangle = (K_{B,m}(z,z))^{-(1/2)} \sum_{j=0}^{2B+2m} \frac{h_j^{B,m}(z)}{\sqrt{\rho_{B,m}(j)}} |j\rangle, \tag{1.9}$$

where $\rho_{B,m}(j)$ denotes the norm square of $h_j^{B,m}(z)$ in $L^2(\mathbb{S}^2)$. The coherent states in (1.9) possess a form similar to (1.1) and will enable us, starting from the observation made in (1.7), to attach to each eigenspace $\mathscr{A}_{B,m}(\mathbb{S}^2)$ a photon counting probability distribution in the same way as for the space $\mathscr{A}_B(\mathbb{S}^2) \equiv \mathscr{A}_{B,0}(\mathbb{S}^2)$ through the quantities

$$p_j(2B,\mu_z,m) = \frac{m!(2B+m)!}{j!(2B+2m-j)!} \mu_z^{j-m} (1-\mu_z)^{2B+m-j} \left(P_m^{(j-m,2B+m-j)}(1-2\mu_z)\right)^2, \quad j = 0,1,\ldots,2B+2m. \tag{1.10}$$

The latter can be considered as a kind of generalized binomial probability distribution $X \sim \mathcal{B}(2B,\mu_z,m)$ depending on an additional parameter $m = 0,1,2,\ldots$. Thus, we study the main properties of the family of probability distributions in (1.10) and we examine the quantum photon counting statistics with respect to the location in the Riemann sphere of the point z labeling the generalized coherent states introduced formally in (1.9).

The paper is organized as follows. In Section 2, we recall briefly the principal statistical properties of the binomial states. Section 3 deals with some needed facts on the Schrödinger operator with uniform magnetic field on the Riemann sphere with an explicit description of the corresponding eigenspaces. Section 4 is devoted to a coherent states formalism. This formalism is applied so as to construct a set of generalized coherent states attached to each spherical Landau level. In Section 5, we introduce the announced generalized binomial probability distribution and we give its main parameters. In Section 6, we discuss the classicality/nonclassicality of the generalized coherent states with respect to the location of their labeling points belonging to the Riemann sphere.

2. The Binomial States

The binomial states in their first form were introduced by Stoler et al. [1] to define a pure state of a single mode of the electromagnetic field for which the photon number density is binomial. Like the generalized coherent states (whose coefficients of its j states expansion are allowed to have additional arbitrary phases), generalized binomial states (GBS) can be defined by

$$|n,\mu,\theta\rangle = \sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!} \mu^j (1-\mu)^{n-j} \right)^{1/2} e^{ij\theta} |j\rangle. \tag{2.1}$$

The corresponding photon counting probability is given by

$$p_j(n,\mu) = \frac{n!}{j!(n-j)!}\mu^j(1-\mu)^{n-j} \tag{2.2}$$

which follows the binomial law $Y \sim \mathcal{B}(n,\mu)$ with parameters n and μ; $n \in \mathbb{Z}_+$, $0 < \mu < 1$. The connection with our notations in (1.1) and (1.2) can be made by setting $n = 2B$, $z = |z|e^{i\theta}$ and $|z|^2 = \mu(1-\mu)^{-1}$. Note that in limits $\mu \to 0$ and $\mu \to 1$ the binomial state reduces to number states $|0\rangle$ and $|n\rangle$, respectively. In a different limit of $n \to +\infty$ and $\mu \to 0$ with $n\mu \to \lambda$, the probability distribution (2.2) goes to the Poisson distribution $\mathcal{P}(\lambda)$

$$p_j(\lambda) = \frac{\lambda^j}{j!}e^{-\lambda}, \quad j = 0,1,2,\ldots, \tag{2.3}$$

which characterizes the coherent states of the harmonic oscillator. In fact, and as pointed out in [1], the binomial states interpolate between *number states* (nonclassical states) and *coherent states* (classical states). It partakes of the properties of both and reduces to each in different limits.

The characteristic function of the random variable $Y \sim \mathcal{B}(n,\mu)$ is given by

$$\mathcal{C}_Y(t) = \left((1-\mu) + \mu e^{it}\right)^n \tag{2.4}$$

from which one obtains the mean value and the variance as

$$E(Y) = n\mu, \qquad \mathrm{Var}(Y) = n\mu(1-\mu). \tag{2.5}$$

Therefore, the Mandel parameter, which measures deviation from the Poissonian distribution,

$$Q = \frac{\mathrm{Var}(Y)}{E(Y)} - 1 = -\mu, \tag{2.6}$$

is always negative. Thus photon statistics in the binomial states is always *sub-Poissonian*.

Remark 2.1. One of the peculiarities of the GBS in (2.1) is that they can be exploited as reference states within schemes devoted at measuring the canonical phase of quantum electromagnetic fields. Moreover, they are the electromagnetic correspondent of the well-known coherent atomic states [7]. For more information and applications involving these binomial states in the context of cavity quantum electrodynamic, see, for example, [8, 9]. We should note that they also admit also a *ladder operator definition* which means that they are eigenstate of a proper combination of the number operator and the annihilation operator via the Holstein-Primakoff realization of the Lie algebra $su(2)$ [10].

3. An Orthonormal Basis of $\mathcal{A}_{B,m}(\mathbb{S}^2)$

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere with the standard metric of constant Gaussian curvature $\kappa = 1$. We identify the sphere $\mathbb{S}^2$ with the extended complex plane $\mathbb{C}\cup\{\infty\}$, called the Riemann sphere, via the stereographic coordinate $z = x + iy$; $x, y \in \mathbb{R}$. We shall work within a fixed coordinate neighborhood with coordinate z obtained by deleting the "point at infinity" $\{\infty\}$. Near this point we use instead of z the coordinate z^{-1}.

In the stereographic coordinate z, the Hamiltonian operator of the Dirac monopole with charge $q = 2B$ reads [4, page 598]

$$H_B := -\left(1+|z|^2\right)^2 \frac{\partial^2}{\partial z \partial \overline{z}} - Bz\left(1+|z|^2\right)\frac{\partial}{\partial z} + B\overline{z}\left(1+|z|^2\right)\frac{\partial}{\partial \overline{z}} + B^2\left(1+|z|^2\right) - B^2. \tag{3.1}$$

This operator acts on the sections of the $U(1)$-bundle with the first Chern class q. We have denoted by $B \in \mathbb{Z}_+$ the strength of the quantized magnetic field. We shall consider the Hamiltonian H_B in (3.1) acting in the Hilbert space $L^2(\mathbb{S}^2) := L^2(\mathbb{S}^2, (1+|z|^2)^{-2} d\nu(z))$, $d\nu(z) = \pi^{-1} dx\, dy$ being the Lebesgue measure on $\mathbb{C} \equiv \mathbb{R}^2$. Its spectrum consists of an infinite number of eigenvalues (*spherical Landau levels*) of the form

$$\epsilon_m^B := (2m+1)B + m(m+1), \quad m = 0, 1, 2, \ldots, \tag{3.2}$$

with finite degeneracy $2B + 2m + 1$ (see [4, page 598]). In order to present expressions of the corresponding eigensections in the coordinate z, we first mention that the shifted operator $H_B - B$ on $L^2(\mathbb{S}^2)$ is intertwined with the invariant Laplacian

$$\Delta_{2B} := -\left(1+|z|^2\right)^2 \frac{\partial^2}{\partial z \partial \overline{z}} + 2B\overline{z}\left(1+|z|^2\right)\frac{\partial}{\partial \overline{z}} \tag{3.3}$$

acting in the Hilbert space $L^{2,B}(\mathbb{S}^2) := L^2(\mathbb{S}^2, (1+|z|^2)^{-2-2B} d\nu(z))$. Namely, we have

$$H_B - B = \left(1+|z|^2\right)^{-B} \Delta_{2B} \left(1+|z|^2\right)^{B}, \tag{3.4}$$

and therefore any ket $|\phi\rangle$ of $L^{2,B}(\mathbb{S}^2)$ is represented by

$$\left(1+|z|^2\right)^{-B} \langle z \mid \phi\rangle \quad \text{in } L^2\left(\mathbb{S}^2\right). \tag{3.5}$$

As mentioned in the introduction, we denote by $\mathcal{A}_{B,m}(\mathbb{S}^2)$ the eigenspace of H_B in $L^2(\mathbb{S}^2)$, corresponding to the eigenvalue ϵ_m^B given in (3.2). Then, by [5] together with (3.5) and the intertwining relation (3.4), we obtain the following orthogonal basis of this eigenspace:

$$h_j^{B,m}(z) := \left(1+|z|^2\right)^{-B} z^j Q_{B,m,j}\left(\frac{|z|^2}{1+|z|^2}\right), \quad 0 \le j \le 2B + 2m, \tag{3.6}$$

where $Q_{B,m,j}(\cdot)$ is the polynomial function given by

$$Q_{B,m,j}(t) = t^{-j}(1-t)^{j-2B}\left(\frac{d}{dt}\right)^{m}\left[t^{j+m}(1-t)^{2B-j+m}\right]. \tag{3.7}$$

According to the Jacobi's formula [11]

$$\left(\frac{d}{dx}\right)^{m}\left(x^{c+m-1}(1-x)^{b-c}\right) = \frac{\Gamma(c+m)}{\Gamma(c)}x^{c-1}(1-x)^{b-c-m}{}_2F_1(-m,b;c;x), \tag{3.8}$$

${}_2F_1(a,b,c;x)$ being the Gauss hypergeometric function, it follows that

$$Q_{B,m,j}(t) = \frac{(m+j)!}{j!}\,{}_2F_1(-m,2B+m+1,j+1;t). \tag{3.9}$$

The latter can also be expressed in terms of Jacobi polynomials via the transformation [11, page 283]

$${}_2F_1\left(k+\nu+\varrho+1,-k,1+\nu;\frac{1-t}{2}\right) = \frac{k!\Gamma(1+\nu)}{\Gamma(k+1+\nu)}P_k^{(\nu,\varrho)}(t). \tag{3.10}$$

So that the orthogonal basis in (3.6) reads simply in terms of Jacobi polynomial as

$$h_j^{B,m}(z) = m!\left(1+|z|^2\right)^{-B}z^{j-m}P_m^{(j-m,2B+m-j)}\left(\frac{1-|z|^2}{1+|z|^2}\right). \tag{3.11}$$

Also, one obtains the norm square of the eigenfunction $h_j^{B,m}$ given in (3.6) as

$$\rho_{B,m}(j) := \left\|h_j^{B,m}\right\|^2_{L^2(\mathbb{S}^2)} = \frac{m!(m+j)!(2B+m-j)!}{(2B+2m+1)(2B+m)!}. \tag{3.12}$$

Finally, by Theorem 1 of [5, page 231] and thanks to (3.5), we obtain the following expression for the reproducing kernel of the eigenspace $\mathcal{A}_{B,m}(\mathbb{S}^2)$:

$$\begin{aligned} K_{B,m}(z,w) = (2B+2m+1) &= \frac{(1+z\overline{w})^{2B}}{\left(1+|z|^2\right)^{B}\left(1+|w|^2\right)^{B}} \\ &\quad\times {}_2F_1\left(-m,m+2B+1,1;\frac{|z-w|^2}{\left(1+|z|^2\right)\left(1+|w|^2\right)}\right). \end{aligned} \tag{3.13}$$

Remark 3.1. In the case $m = 0$, the elements of the orthogonal basis reduce further to $h_j^{B,0}(z) = (1+|z|^2)^{-B} z^j$ and the reproducing kernel reads

$$K_{B,0}(z,w) = (2B+1)(1+z\overline{w})^{2B}\left(1+|z|^2\right)^{-B}\left(1+|w|^2\right)^{-B}. \tag{3.14}$$

Remark 3.2. In the case of the n-dimensional projective space $\mathbb{P}(\mathbb{C}^n)(= S^1 \setminus S^{2n+1})$ equipped with the Fubini-Study metric, explicit formulae for the reproducing kernels of the eigenspaces associated with the Schrödinger operator with constant magnetic field written in the local coordinates (of the chart $\mathbb{C}^n$) as

$$H_B := \left(1+|z|^2\right)\left\{\sum_{i,j=1}^{n}\left(\delta_{ij}+z_i\overline{z}_j\right)\frac{\partial^2}{\partial z_i \partial \overline{z}_j} - B\sum_{j=1}^{n}\left(z_j\frac{\partial}{\partial z_j} - \overline{z}_j\frac{\partial}{\partial \overline{z}_j}\right) - B^2\right\} + B^2 \tag{3.15}$$

have been obtained in [12].

4. Generalized Coherent States

We follow the formalism presented in [13] for constructing generalized coherent states, which can be considered as a generalization of the canonical ones when written as series expansion in the Fock basis according to [6]. Let $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ be a functional Hilbert space with an orthonormal basis $\{\phi_n\}_{n=1}^{d}$ and $\mathscr{A}^2$ a finite d-dimensional subspace of the Hilbert space $L^2(\Omega, ds)$, of square integrable functions on a given measured space (Ω, ds), with an orthogonal basis $\{\Phi_n\}_{n=1}^{d}$. Then, $\mathscr{A}^2$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$K(x,y) := \sum_{n=1}^{d}\frac{\Phi_n(x)\overline{\Phi_n(y)}}{\rho_n}; \quad x,y \in \Omega, \tag{4.1}$$

where we have set $\rho_n := \|\Phi_n\|^2_{L^2(\Omega,ds)}$. Associated to the data of $(\mathscr{A}^2, \Phi_n)$ and $(\mathscr{H}, \phi_n)$, the generalized coherent states are the elements of $\mathscr{H}$ defined by

$$\Phi_x := (\omega_d(x))^{-(1/2)}\sum_{n=1}^{d}\frac{\Phi_n(x)}{\sqrt{\rho_n}}\phi_n; \quad x \in \Omega, \tag{4.2}$$

where $\omega_d(x)$ stands for $\omega_d(x) := K(x,x)$. Note that the choice of the Hilbert space $\mathscr{H}$ defines a quantization of Ω into $\mathscr{H}$ by considering the inclusion map $x \mapsto \Phi_x$. Furthermore, it is straightforward to check that $\langle\Phi_x, \Phi_x\rangle_{\mathscr{H}} = 1$ and to show that the corresponding coherent state transform (CST) $\mathcal{W}$ on $\mathscr{H}$,

$$\mathcal{W}[f](x) := (\omega_d(x))^{1/2}\langle\Phi_x, f\rangle_{\mathscr{H}}; \quad f \in \mathscr{H}, \tag{4.3}$$

defines an isometry from $\mathscr{H}$ into $\mathscr{A}^2$. Thereby we have a resolution of the identity of $\mathscr{H}$, that is, we have the following integral representation:

$$f(\cdot) = \int_\Omega \langle \Phi_x, f \rangle_{\mathscr{H}} \Phi_x(\cdot)\omega_d(x)ds(x); \quad f \in \mathscr{H}. \tag{4.4}$$

Remark 4.1. Note that formula (4.2) can be considered as a generalization (in the finite dimensional case) of the series expansion of the well-known canonical coherent states

$$|\zeta\rangle = \left(e^{|\zeta|^2}\right)^{-(1/2)} \sum_{k\geq 0} \frac{\zeta^k}{\sqrt{k!}} |k\rangle \tag{4.5}$$

with $\phi_k := |k\rangle$ being the number states of the harmonic oscillator, which also known as Fock states.

We can now construct for each spherical Landau level ϵ_m^B given in (3.2) a set of generalized coherent states (GCS) according to formula (4.2) as

$$\vartheta_{z,B,m} \equiv |z, B, m\rangle = (K_{B,m}(z,z))^{-(1/2)} \sum_{j=0}^{2B+2m} \frac{h_j^{B,m}(z)}{\sqrt{\rho_{B,m}(j)}} |\phi_j\rangle \tag{4.6}$$

with the following precisions:

(i) $(\Omega, ds) := (\mathbb{S}^2, (1+|z|^2)^{-2} d\nu(z))$, $\mathbb{S}^2$ being identified with $\mathbb{C} \cup \{\infty\}$.

(ii) $\mathscr{A}^2 := \mathscr{A}_{B,m}(\mathbb{S}^2)$ is the eigenspace of H_B in $L^2(\mathbb{S}^2)$ with dimension $d_{B,m} = 2B+2m+1$.

(iii) $\omega(z) = K_{B,m}(z,z) = 2B + 2m + 1$ (in view of (3.13)).

(iv) $h_j^{B,m}(z)$ are the eigenfunctions given by (3.11) in terms of the Jacobi polynomials.

(v) $\rho_{B,m}(j)$ being the norm square of $h_j^{B,m}$, given in (3.12).

(vi) $\mathscr{H} := \mathcal{P}_{B+m}$ the space of polynomials of degree less than $d_{B,m}$, which carries a unitary irreducible representation of the compact Lie group SU(2) (see [14]). The scalar product in $\mathcal{P}_{B+m}$ is written as

$$\langle \psi, \phi \rangle_{\mathcal{P}_{B+m}} = d_{B,m} \int_{\mathbb{C}} ds(\zeta) \left(1 + |\zeta|^2\right)^{-2(B+m)-2} \psi(\zeta)\overline{\phi(\zeta)}. \tag{4.7}$$

(vii) $\{\phi_j;\ 0 \leq j \leq 2B + 2m\}$ is an orthonormal basis of $\mathcal{P}_{B+m}$, whose elements are given explicitly by

$$\phi_j(\xi) := \sqrt{\frac{(2(B+m))!}{(2B+m-j)!(j+m)!}} \xi^{j+m} \quad \zeta \in \mathbb{C}. \tag{4.8}$$

Definition 4.2. For given positive integers $2B$ and m, and fixed $z \in \mathbb{S}^2$, the wave function of the GCS in (4.6) are expressed as

$$\vartheta_{z,B,m}(\xi) = \frac{1}{\left(1+|z|^2\right)^{-B}} \sum_{j=0}^{2B+2m} \frac{\sqrt{m!(2B+m)!(2B+2m)!}}{j!(2B+2m-j)!} z^{j-m} P_m^{(j-m,2B+m-j)}\left(\frac{1-|z|^2}{1+|z|^2}\right)\xi^j, \tag{4.9}$$

for every $\zeta \in \mathbb{C}$.

According to (4.4), the system of GCS $|\vartheta_{z,B,m}\rangle$ solves then the unity of the Hilbert space $\mathcal{P}_{B+m}$ as

$$\mathbf{1}_{\mathcal{P}_{B+m}} = d_{B,m}\int_{\mathbb{C}} d\nu(z)\left(1+|z|^2\right)^{-2}|\vartheta_{z,B,m}\rangle\langle\vartheta_{z,B,m}|. \tag{4.10}$$

They also admit a closed form [15], as

$$\vartheta_{z,B,m}(\xi) = \sqrt{\frac{(2B+2m)!}{(2B+m)!m!}}\left(\frac{(1+\xi z)^2}{1+|z|^2}\right)^B\left(\frac{(\xi-\overline{z})(1+\xi z)}{1+|z|^2}\right)^m. \tag{4.11}$$

Remark 4.3. Note that for $m=0$, the expression above reduces to

$$\langle \xi \mid z, B, 0\rangle = \left(1+|z|^2\right)^{-B}(1+\xi z)^{2B}, \tag{4.12}$$

which are wave functions of Perelomov's coherent states based on SU(2) (see [16, page 62]).

5. Generalized Binomial Probability Distributions

According to (4.6), we see that the squared modulus of the projection of coherent state $\vartheta_{z,B,m}$ onto the state ϕ_j, is given by

$$\left|\langle\vartheta_{z,B,m},\phi_j\rangle_{\mathscr{H}}\right|^2 = \frac{1}{\rho_j^{B,m} d_{B,m}}\left|h_j^{B,m}(z)\right|^2. \tag{5.1}$$

This is in fact the probability of finding j photons in the coherent state $\vartheta_{z,B,m}$. More explicitly, in view of (3.11), the quantity in (5.1) reads

$$\left|\langle\vartheta_{z,B,m},\phi_j\rangle_{\mathscr{H}}\right|^2 = \frac{m!(2B+m)!}{j!(2B+2m-j)!}\left(1+|z|^2\right)^{-2B}|z|^{2(j-m)}\left(P_m^{(j-m,2B+m-j)}\left(\frac{1-|z|^2}{1+|z|^2}\right)\right)^2. \tag{5.2}$$

As annonced in the introduction, we denote the expression in (5.2) by $p_j(2B,\mu_z,m)$ for $j=0,1,2,\ldots$, with $\mu_z=|z|^2(1+|z|^2)^{-1}$ or equivalently $|z|^2=\mu_z(1-\mu_z)^{-1}$. Motivated by this quantum probability, we then state the following.

Definition 5.1. For fixed integers $B,m\in\mathbb{Z}_+$, the discrete random variable X having the probability distribution

$$p_j(2B,\mu_z,m)=\frac{m!(2B+m)!}{j!(2B+2m-j)!}\mu_z^{(j-m)}(1-\mu_z)^{2B+m-j}\left(P_m^{(j-m,2B+m-j)}(1-2\mu_z)\right)^2, \tag{5.3}$$

with $j=0,1,2,\ldots,2B+2m$, and denoted by $X\sim\mathcal{B}(2B,\mu_z,m)$, $0<\mu_z<1$, will be called the generalized binomial probability distribution associated to the weighted Hilbert space $\mathscr{A}_{B,m}(S^2)$.

Remark 5.2. Note that for $m=0$, the above expression in (5.3) reduces to

$$p_j(2B,\mu_z,0)=\frac{(2B)!}{j!(2B-j)!}\mu_z^j(1-\mu_z)^{2B-j},\quad j=0,1,2,\ldots,2B, \tag{5.4}$$

which is the standard binomial distribution with parameters $2B$ and $0<\mu_z<1$.

A convenient way to summarize all the properties of a probability distribution X is to explicit its characteristic function:

$$\mathcal{C}_X(t)=:E\left(e^{itX}\right), \tag{5.5}$$

where t is a real number, $i:=\sqrt{-1}$ is the imaginary unit, and E denotes the expected value.

Proposition 5.3. *For fixed $m=0,1,2,\ldots$, the characteristic function of $X\sim\mathcal{B}(2B,\mu_z,m)$ is given by*

$$\mathcal{C}_m(t)=e^{imt}\left((1-\mu_z)+\mu_z e^{it}\right)^{2B}P_m^{(0,2B)}(1-4\mu_z(1-\mu_z)(1-\cos(t))) \tag{5.6}$$

for every $t\in\mathbb{R}$.

Proof. Recall first that for every given fixed nonnegative integer m, the characteristic function $\mathcal{C}_m(t)$ in (5.5) can be written as

$$\begin{aligned}\mathcal{C}_m(t)&=\sum_{j=0}^{2B+2m}e^{ijt}p_j(2B,\mu_z,m)\\&=\sum_{j=0}^{2B+2m}e^{ijt}\frac{m!(2B+m)!}{j!(2B+2m-j)!}\mu_z^{(j-m)}(1-\mu_z)^{2B+m-j}\left(P_m^{(j-m,2B+m-j)}(1-2\mu_z)\right)^2.\end{aligned} \tag{5.7}$$

The last equality follows using (5.3). Next, by making the change $k = B + m - j$ in (5.7), it follows that

$$\mathcal{C}_m(t) = \sum_{k=-(B+m)}^{B+m} e^{i(B+m-k)t} \frac{m!(2B+m)!}{(B+m+k)!(B+m-k)!} \mu_z^{B-k} (1-\mu_z)^{B+k} \left(P_m^{(B-k,B+k)}(1-2\mu_z)\right)^2. \tag{5.8}$$

Instead of the Jacobi polynomials, it is convenient to consider the closely related function $\mathcal{P}_{r,s}^l(x)$ introduced in [14, page 270]. They can be defined through the formula [14, equation (1), page 288],

$$P_{n-r}^{(r-s,r+s)}(x) = 2^r \left(\frac{(n-s)!(n+s)!}{(n-r)!(n+r)!}\right)^{1/2} (1-x)^{(s-r)/2}(1+x)^{-(s+r)/2} \mathcal{P}_{r,s}^n(x), \tag{5.9}$$

with $m = n-B$ (i.e., $n = B+m$) and $x = 1-2\mu_z$. We can then express the square of $P_m^{(B-k,B+k)}(x)$ as follows:

$$\left(P_m^{(B-k,B+k)}(1-2\mu_z)\right)^2 = \frac{(B+m-k)!(B+m+k)!}{m!(2B+m)!} \mu_z^{-B+k}(1-\mu_z)^{-B-k}\left(\mathcal{P}_{B,k}^{B+m}(1-2\mu_z)\right)^2. \tag{5.10}$$

Therefore, (5.8) reduces further to

$$\mathcal{C}_m(t) = e^{i(B+m)t} \sum_{k=-(B+m)}^{B+m} e^{-ikt} \left(\mathcal{P}_{B,k}^{B+m}(1-2\mu_z)\right)^2 \tag{5.11}$$

$$\begin{aligned} &\overset{(\star)}{=} (-1)^B e^{i(B+m)t} \sum_{k=-(B+m)}^{B+m} e^{-ik(t-\pi)} \mathcal{P}_{B,k}^{B+m}(1-2\mu_z) \mathcal{P}_{k,B}^{B+m}(1-2\mu_z) \\ &= (-1)^B e^{i(B+m)t} e^{-iB(\varphi+\psi)} \mathcal{P}_{B,B}^{B+m}(\cos(\theta)). \end{aligned} \tag{5.12}$$

The transition $(\star)$ above holds using the fact that [14, page 288]

$$\mathcal{P}_{j,k}^l(x) = (-1)^{j+k} \mathcal{P}_{k,j}^l(x). \tag{5.13}$$

While the last equality can be checked easily using the addition formula [14, equation (3), page 326]

$$\sum_{k=-s}^{s} e^{-ik\tau} \mathcal{P}_{j,k}^s(\cos(\theta_1)) \mathcal{P}_{k,l}^s(\cos(\theta_2)) = e^{-i(j\varphi+l\psi)} \mathcal{P}_{j,l}^s(\cos(\theta)). \tag{5.14}$$

Here the involved complex angles φ, ψ, and θ are given through equations (8), (8′), and (8″) in [14, page 270]. In our case, they yield the followings:

$$\cos(\theta) = \cos^2(2\alpha) + \sin^2(2\alpha)\cos(t),$$
$$e^{i((\varphi+\psi)/2)} = \frac{-i\left(\cos^2(\alpha) + \sin^2(\alpha)e^{-it}\right)e^{it/2}}{\cos(\theta/2)} \tag{5.15}$$

for $\theta_1 = \theta_2 = 2\alpha$, so that

$$e^{-iB(\varphi+\psi)} = (-1)^B\left(\cos\left(\frac{\theta}{2}\right)\right)^{-2B}\left(\cos^2(\alpha) \ + \sin^2(\alpha)e^{it}\right)^{2B}e^{-iBt}. \tag{5.16}$$

Next, using the fact that

$$2^{-s}(1+x)^s P_{n-s}^{(0,2s)}(x) = \mathcal{P}_{s,s}^{n}(x), \tag{5.17}$$

which is a special case of (5.9), with $s = B$, $n - s = m$, and $x = \cos(\theta)$, we obtain

$$\mathcal{P}_{B,B}^{B+m}(\cos(\theta)) = \left(\cos\left(\frac{\theta}{2}\right)\right)^{2B} P_m^{(0,2B)}(\cos(\theta)). \tag{5.18}$$

Finally, by substituting (5.16) and (5.18) in (5.12), taking into account that $\sin^2(\alpha) = \mu_z$ and $\cos^2(\alpha) = 1 - \mu_z$, we see that the characteristic function $\mathcal{C}_m(t)$ reads simply as

$$\mathcal{C}_m(t) = e^{imt}\left(\cos^2(\alpha) + \sin^2(\alpha)e^{it}\right)^{2B} P_m^{(0,2B)}(\cos(\theta)), \tag{5.19}$$

where $\cos(\theta) = 1 - 4\mu_z(1-\mu_z)(1-\cos(t))$. □

Remark 5.4. Note that by taking $m = 0$ in (5.19), the characteristic function reduces to

$$C_Y(t) = \left(\cos^2(\alpha) + \sin^2(\alpha)e^{it}\right)^{2B} = \left((1-\mu_z) + \mu_z e^{it}\right)^{2B}, \tag{5.20}$$

which is the well-known characteristic function of the binomial random variable $Y \sim \mathcal{B}(2B, \mu_z)$ with parameters $n = 2B \in \mathbb{Z}_+$ and $0 < \mu_z < 1$ as in (2.4).

Now, the characteristic function contains important information about the random variable X. For example, various moments may be obtained by repeated differentiation of $\mathcal{C}_m(t)$ in (5.6) with respect to the variable t and evaluation at the origin as

$$E\left(X^k\right) = \frac{1}{i^k}\frac{\partial^k}{\partial t^k}(C_m(t))\bigg|_{t=0}. \tag{5.21}$$

Corollary 5.5. *Let $m, 2B \in \mathbb{Z}_+$. The mean value and the variance of $X \sim \mathcal{B}(2B, \mu_z, m)$ are given respectively by*

$$E(X) = m + 2B\mu_z,$$
$$\operatorname{Var}(X) = 2B\mu_z(1-\mu_z) + 2\mu_z(1-\mu_z)m(2B+m+1). \tag{5.22}$$

Proof. Let recall first that for every fixed integer $m = 0, 1, 2, \ldots$, we have

$$E(X) = \left.\frac{\partial \mathcal{C}_m}{i\partial t}\right|_{t=0},$$
$$\operatorname{Var}(X) = E\left(X^2\right) - [E(X)]^2 = \left.\frac{\partial^2 \mathcal{C}_m}{i^2\partial t^2}\right|_{t=0} - \left[\left.\frac{\partial \mathcal{C}_m}{i\partial t}\right|_{t=0}\right]^2. \tag{5.23}$$

Thus direct computation gives rise to

$$\frac{\partial \mathcal{C}_m}{i\partial t}(t) = \left[m + \frac{2B\mu_z e^{it}}{([1-\mu_z] + \mu_z e^{it})} - 4\mu_z(1-\mu_z)\sin(t)\left(\frac{\left(\partial P_m^{(0,2B)}(x)/i\partial x\right)\Big|_{x=\cos(\theta)}}{P_m^{(0,2B)}(\cos(\theta))}\right)\right]\mathcal{C}_m(t),$$

$$\begin{aligned}
&\frac{\partial^2 \mathcal{C}_m}{i^2\partial t^2}(t)\\
&= \left[\frac{\partial}{i\partial t}\left(m + \frac{2B\mu_z e^{it}}{([1-\mu_z] + \mu_z e^{it})} - 4\mu_z(1-\mu_z)\sin(t)\left(\frac{\left(\partial P_m^{(0,2B)}(x)/i\partial x\right)\Big|_{x=\cos(\theta)}}{P_m^{(0,2B)}(\cos(\theta))}\right)\right)\right]\\
&\quad\times \mathcal{C}_m(t) + \left[m + \frac{2B\mu_z e^{it}}{([1-\mu_z] + \mu_z e^{it})} - 4\mu_z(1-\mu_z)\sin(t)\left(\frac{\left(\partial P_m^{(0,2B)}(x)/i\partial x\right)\Big|_{x=\cos(\theta)}}{P_m^{(0,2B)}(\cos(\theta))}\right)\right]\frac{\partial \mathcal{C}_m}{i\partial t}(t)\\
&= \left[\frac{2B\mu_z e^{it}([1-\mu_z] + \mu_z e^{it}) - 2B\mu_z^2 e^{it}}{([1-\mu_z] + \mu_z e^{it})^2} + 4\mu_z(1-\mu_z)\cos(t)\left(\frac{\left(\partial P_m^{(0,2B)}(x)/\partial x\right)\Big|_{x=\cos(\theta)}}{P_m^{(0,2B)}(\cos(\theta))}\right)\right]\\
&\quad\times \mathcal{C}_m(t) - 4\mu_z(1-\mu_z)\sin(t)\frac{\partial}{i\partial t}\left(\frac{\left(\partial P_m^{(0,2B)}(x)/i\partial x\right)\Big|_{x=\cos(\theta)}}{P_m^{(0,2B)}(\cos(\theta))}\right)\mathcal{C}_m(t)\\
&\quad+ \left[m + \frac{2B\mu_z e^{it}}{([1-\mu_z] + \mu_z e^{it})} - 4\mu_z(1-\mu_z)\sin(t)\left(\frac{\left(\partial P_m^{(0,2B)}(x)/i\partial x\right)\Big|_{x=\cos(\theta)}}{P_m^{(0,2B)}(\cos(\theta))}\right)\right]\frac{\partial \mathcal{C}_m}{i\partial t}(t).
\end{aligned} \tag{5.24}$$

To conclude, we have to use successively the facts that for $t = 0$, we have $\cos(\theta) = 1$ and $\mathcal{C}_m(0) = 1$, together with

$$\frac{\partial P_m^{(a,b)}}{i\partial x}(x) = \frac{a+b+m+1}{2} P_{m-1}^{(a+1,b+1)}(x), \qquad P_m^{(a,b)}(1) = \frac{\Gamma(a+m+1)}{m!\Gamma(a+1)}. \tag{5.25}$$

Thus, we have

$$E(X) = \left.\frac{\partial \mathcal{C}_m}{i\partial t}\right|_{t=0} = (m + 2B\mu_z)\mathcal{C}_m(0) = m + 2B\mu_z. \tag{5.26}$$

We have also

$$\begin{aligned} \frac{\partial^2 \mathcal{C}_m}{i^2\partial t^2}|_{t=0} = {} & \left[2B\mu_z(1-\mu_z) + 2\mu_z(1-\mu_z)(2B+m+1)\left(\frac{P_{m-1}^{(1,2B+1)}(1)}{P_m^{(0,2B)}(1)}\right)\right] \\ & \times \mathcal{C}_m(0) + [m + 2B\mu_z]\frac{\partial \mathcal{C}_m}{i\partial t}|_{t=0}, \end{aligned} \tag{5.27}$$

and therefore

$$\begin{aligned} \operatorname{Var}(X) &= \left.\frac{\partial^2 \mathcal{C}_m}{i^2\partial t^2}\right|_{t=0} - \left(\left.\frac{\partial \mathcal{C}_m}{i\partial t}\right|_{t=0}\right)^2 \\ &= 2B\mu_z(1-\mu_z) + 2\mu_z(1-\mu_z)m(2B+m+1). \end{aligned} \tag{5.28}$$

□

Remark 5.6. Note that by taking $m = 0$ in (5.22), we recover the standard values

$$E(Y) = 2B\mu_z, \qquad \operatorname{Var}(Y) = 2B\mu_z(1-\mu_z) \tag{5.29}$$

of the binomial probability distribution as given in (2.5).

6. Photon Counting Statistics

For an arbitrary quantum state one may ask to what extent is "nonclassical" in a sense that its properties differ from those of coherent states? In other words, is there any parameter that may reflect the degree of nonclassicality of a given quantum state? In general, to define a measure of nonclassicality of a quantum states one can follow several different approach. An earlier attempt to shed some light on the nonclassicality of a quantum state was pioneered by Mandel [17], who investigated radiation fields and introduced the parameter

$$Q = \frac{\operatorname{Var}(X)}{E(X)} - 1, \tag{6.1}$$

to measure deviation of the photon number statistics from the Poisson distribution, characteristic of coherent states. Indeed, $Q = 0$ characterizes Poissonian statistics. If $Q < 0$,

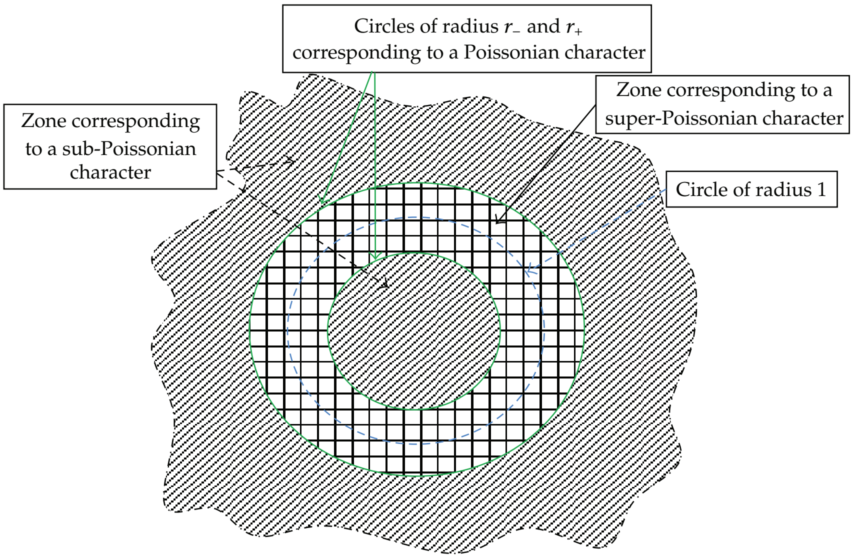

Figure 1: Quantum photon counting statistics.

we have *sub-Poissonian* statistics, otherwise, statistics are *super-Poissonian*. In our context and for $m = 0$, as mentioned in Section 2, the fact that the binomial probability distribution has a negative Mandel parameter, according to (5.29), and thereby the binomial states obey sub-Poissonian statistics. For $m \neq 0$, we make use of the obtained statistical parameters $E(X)$ and $\mathrm{Var}(X)$ to calculate Mandel parameter corresponding to the random variable $X \sim \mathcal{B}(2B, \mu_z, m)$. The discussion with respect to the sign of this parameter gives rise to the following statement.

Proposition 6.1. *Let m and B be nonnegative integers and set*

$$r_{\pm}(B,m) := \left(1 \pm \left(1 - \frac{1}{m(2B+m)}\right)^{1/2}\right)^{1/2}. \tag{6.2}$$

Then, $r_-(B,m) \leq 1 \leq r_+(B,m)$ and the photon counting statistics are

(i) *sub-Poissonian for points z such that $|z| < r_-(B,m)$ and $|z| > r_+(B,m)$,*

(ii) *Poissonian for points z such that $|z| = r_-(B,m)$ or $|z| = r_+(B,m)$,*

(iii) *super-Poissonian for z such that $r_-(B,m) < |z| < r_+(B,m)$,*

Proof. Assume that $m \neq 0$. Making use of (5.22), we see that the Mandel parameter can be written as follows $Q(X) = Q_m(\mu_z) = -T_m(\mu_z)/(2B\mu_z + m)$, where we have set

$$\begin{aligned} T_m(\mu_z) &= 2(B + m[2B+m+1])\mu_z^2 - 2m[2B+m+1]\mu_z + m \\ &= \left(\mu_z - \frac{md_{B,m}}{2(B+md_{B,m})}\right)^2 - \frac{m(d_{B,m}-1)(2Bm+m^2-1)}{4(B+md_{B,m})^2} \end{aligned} \tag{6.3}$$

with $d_{B,m} := 2B + 2m + 1$. Hence, it is clear that $T_m(\mu_z) = 0$, viewed as second-degree polynomials in μ_z, admits exactly two real solutions given by

$$\mu_z^{\pm}(B,m) := \frac{md_{B,m}}{2(B+md_{B,m})}\left(1 \pm \left(1 - \frac{2(B+md_{B,m})}{md_{B,m}^2}\right)^{1/2}\right). \tag{6.4}$$

Now, assertions (i), (ii), and (iii) follow by discussing the sign of the parameter $Q_m(\mu_z)$ (i.e., the sign of $-T_m(\mu_z)$) with respect to the modulus of $z \in \mathbb{C} \cup \{\infty\}$, keeping in mind that $|z|^2 = \mu_z/(1-\mu_z)$. □

Figure 1 illustrates the quantum photon counting statistics with respect to the location in the extended complex plane of the point z as discussed in Proposition 6.1. Here $r_{\pm} := r_{\pm}(B,m)$ are as in (6.2).

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