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Research Article

The Initial Value Problem for the Quadratic Nonlinear Klein-Gordon Equation

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We study the initial value problem for the quadratic nonlinear Klein-Gordon equation $\mathcal{L}u = \langle i\partial_x\rangle^{-1}\overline{u}^2$, $(t,x) \in \mathbf{R}\times\mathbf{R}$, $u(0,x) = u_0(x)$, $x \in \mathbf{R}$, where $\mathcal{L} = \partial_t + i\langle i\partial_x\rangle$ and $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. Using the Shatah normal forms method, we obtain a sharp asymptotic behavior of small solutions without the condition of a compact support on the initial data which was assumed in the previous works.

1. Introduction

Let us consider the Cauchy problem for the nonlinear Klein-Gordon equation with a quadratic nonlinearity in one dimensional case

$$\begin{gathered} \mathcal{L}u = \lambda\langle i\partial_x\rangle^{-1}\overline{u}^2, \quad (t,x) \in \mathbf{R}\times\mathbf{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{gathered} \tag{1.1}$$

where $\lambda \in \mathbf{C}$, $\mathcal{L} = \partial_t + i\langle i\partial_x\rangle$, and $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$.

Our purpose is to obtain the large time asymptotic profile of small solutions to the Cauchy problem (1.1) without the restriction of a compact support on the initial data which was assumed in the previous work [1]. One of the important tools of paper [1] was based on the transformation of the equation by virtue of the hyperbolic polar coordinates following to paper [2]. The application of the hyperbolic polar coordinates implies the restriction to the interior of the light cone, and therefore, requires the compactness of the initial data. Problem

(1.1) is related to the Cauchy problem

$$v_{tt} + v - v_{xx} = \mu v^2, \quad (t, x) \in \mathbf{R} \times \mathbf{R},$$
$$v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), \quad x \in \mathbf{R}, \tag{1.2}$$

where v_0 and v_1 are the real-valued functions, and $\mu \in \mathbf{R}$. Indeed we can put $u = (1/2)(v + i\langle i\partial_x \rangle^{-1} v_t)$, then u satisfies

$$\mathcal{L}u = \frac{i}{2}\mu \langle i\partial_x \rangle^{-1} (u + \overline{u})^2, \quad (t, x) \in \mathbf{R} \times \mathbf{R},$$
$$u(0, x) = u_0(x), \quad x \in \mathbf{R}, \tag{1.3}$$

where $u_0 = (1/2)(v_0 + i\langle i\partial_x \rangle^{-1} v_1)$.

There are a lot of works devoted to the study of the cubic nonlinear Klein-Gordon equation

$$v_{tt} + v - v_{xx} = \mu v^3, \quad (t, x) \in \mathbf{R} \times \mathbf{R},$$
$$v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), \quad x \in \mathbf{R} \tag{1.4}$$

with $\mu \in \mathbf{R}$. When $\mu < 0$, the global existence of solutions to (1.4) can be easily obtained in the energy space, which is, however, insufficient for determining the large-time asymptotic behavior of solutions. The sharp $\mathbf{L}^\infty$- time decay estimates of solutions and nonexistence of the usual scattering states for (1.4) were shown in [3] by using hyperbolic polar coordinates under the conditions that the initial data are sufficiently regular and have a compact support.

The initial value problem for the nonlinear Klein-Gordon equation with various cubic nonlinearities depending on $v, v_t, v_x, v_{xx}, v_{tx}$ and having a suitable nonresonance structure was studied in [4–6], where small solutions were found in the neighborhood of the free solutions when the initial data are small and regular and decay rapidly at infinity. Hence the cubic nonlinearities are not necessarily critical; however the resonant nonlinear term v^3 was excluded in these works. In paper [4], the nonresonant nonlinearities were classified into two types, one of them can be treated by the nonlinear transformation which is different from the method of normal forms [7] and the other reveals an additional time decay rate via the operator $x\partial_t + t\partial_x$ which was used in [2]. This nonlinear transformation was refined in [8] and applied to a system of nonlinear Klein-Gordon equations in one or two space dimensions with nonresonant nonlinearities. It seems that the method of normal forms is very useful in the case of a single equation; however it does not work well in the case of a system of nonlinear Klein-Gordon equations. Some sufficient conditions on quadratic or cubic nonlinearities were given in [1], which allow us to prove global existence and to find sharp asymptotics of small solutions to the Cauchy problem including (1.2) with small and regular initial data having a compact support. Moreover it was proved that the asymptotic profile differs from that of the linear Klein-Gordon equation. See also [9, 10] in which asymptotic behavior of solutions to (1.4) was studied as in [1] by using hyperbolic polar coordinates. Compactness condition on the data was removed in [11] in the case of the cubic nonlinearity v^3 and a real-valued solution. Final value problem with the cubic nonlinearity was studied in [12] for a real-valued solution. As far as we know the problem of finding the large-time asymptotics is still open for the case of the cubic nonlinearity v^3 and the complex valued initial data. When the initial data are complex-valued, global existence and $\mathbf{L}^\infty$-time decay estimates of small solutions to the Klein-Gordon equation with cubic nonlinearity $|v|^2 v$ were obtained in paper [13] under the conditions that the initial data are smooth and have a compact support.

The scattering problem and the time decay rates of small solutions to (1.4) with super-critical nonlinearities $|v|^{p-1}v$ and $|v|^p$ with $p > 3$ were studied in papers of [14, 15]. Finally, we note that the Klein-Gordon equation (1.4) with quadratic nonlinearities in two space dimensions was studied in [16], where combining the method of the normal forms of [7] and the time decay estimate through the operator $x\partial_t + t\partial_x$ of [17], it was shown that every quadratic nonlinearity is nonresonant.

We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathcal{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, with the norm $\|\phi\|_{\mathbf{L}^p} = (\int_{\mathbf{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{\phi \in \mathbf{L}^p; \left\| \langle x \rangle^s \langle i\partial_x \rangle^m \phi \right\|_{\mathbf{L}^p} < \infty \right\}, \tag{1.5}$$

for $m, s \in \mathbf{R}$, $1 \le p \le \infty$, where $\langle x \rangle = \sqrt{1+x^2}$. For simplicity we write $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$. The index 0 we usually omit if it does not cause a confusion. We denote by $\mathcal{F}\phi = \mathcal{F}_{x\to\xi}\phi \equiv \widehat{\phi} = 1/\sqrt{2\pi}\int_{\mathbf{R}} e^{-ix\xi}\phi(x)dx$ the Fourier transform of the function ϕ. Then the inverse Fourier transformation is $\mathcal{F}^{-1}\phi = \mathcal{F}^{-1}_{\xi\to x}\phi = (1/\sqrt{2\pi})\int_{\mathbf{R}} e^{ix\xi}\phi(\xi)d\xi$.

Our main result of this paper is the following.

Theorem 1.1. *Let $u_0 \in \mathbf{H}^{3,1}$ and the norm $\|u_0\|_{\mathbf{H}^{3,1}} = \varepsilon$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the Cauchy problem* (1.1) *has a unique global solution*

$$u(t) \in \mathbf{C}\Big([0,\infty); \mathbf{H}^{3,1}\Big) \tag{1.6}$$

satisfying the time decay estimate

$$\|u(t)\|_{\mathbf{H}^{1,0}_\infty} \le C\varepsilon(1+t)^{-1/2}. \tag{1.7}$$

Furthermore there exists a unique final state $\widehat{W}_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\begin{aligned} \left\| u(t) - e^{-i\langle i\partial_x\rangle t}\mathcal{F}^{-1}\widehat{W}_+ e^{-2i|\lambda|^2\Omega|\widehat{W}_+|^2 \log t} \right\|_{\mathbf{H}^{1,0}} &\le C\varepsilon^{3/2} t^{\gamma-1/4}, \\ \left\| \mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \widehat{W}_+ e^{-2i|\lambda|^2\Omega|\widehat{W}_+|^2 \log t} \right\|_{\mathbf{H}^{0,1}_\infty} &\le C\varepsilon^{3/2} t^{\gamma-1/4}, \end{aligned} \tag{1.8}$$

where $\gamma \in (0, 1/4)$, $\Omega(\xi) \equiv \langle \xi \rangle^2 / \langle 2\xi \rangle (2\langle \xi \rangle + \langle 2\xi \rangle)$.

An important tool for obtaining the time decay estimates of solutions to the nonlinear Klein-Gordon equation is implementation of the operator

$$\mathcal{J} = \langle i\partial_x \rangle e^{-i\langle i\partial_x\rangle t} x e^{i\langle i\partial_x\rangle t} = \mathcal{F}^{-1}\langle \xi \rangle e^{-i\langle \xi\rangle t} i\partial_\xi e^{i\langle \xi\rangle t}\mathcal{F} = \langle i\partial_x \rangle x + it\partial_x, \tag{1.9}$$

which is analogous to the operator $x + it\partial_x = e^{(-it/2)\partial_x^2} x e^{(it/2)\partial_x^2}$ in the case of the nonlinear Schrödinger equations used in [18]. The operator $\mathcal{J}$ was used previously in paper of [15] for constructing the scattering operator for nonlinear Klein-Gordon equations with a supercritical nonlinearity. We have $[x, \langle i\partial_x \rangle^{\alpha}] = \alpha \langle i\partial_x \rangle^{\alpha-2} \partial_x$; therefore the commutator $[\mathcal{L}, \mathcal{J}] = \mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L} = 0$, where $\mathcal{L} = \partial_t + i\langle i\partial_x \rangle$ is a linear part of (1.1). Since $\mathcal{J}$ is not a purely differential operator, it is apparently difficult to calculate the action of $\mathcal{J}$ on the nonlinearity in (1.1). So, instead we use the first-order differential operator

$$\mathcal{P} = t\partial_x + x\partial_t \tag{1.10}$$

which is closely related to $\mathcal{J}$ by the identity $\mathcal{P} = \mathcal{L}x - i\mathcal{J}$ and acts easily on the nonlinearity. Moreover, it almost commutes with $\mathcal{L}$, since $[\mathcal{L}, \mathcal{P}] = -i\langle i\partial_x \rangle^{-1} \partial_x \mathcal{L}$.

Also we use the method of normal forms of [7] by which we transform the quadratic nonlinearity into a cubic one with a nonlocal operator. We multiply both sides of equation (1.1) by the free Klein-Gordon evolution group $\mathcal{F}\mathcal{U}(-t) = \mathcal{F}e^{it\langle i\partial_x \rangle} = e^{it\langle \xi \rangle}\mathcal{F}$ and put $v(t,\xi) = e^{it\langle \xi \rangle}\widehat{u}$ to get

$$v_t(t,\xi) = \lambda e^{it\langle \xi \rangle} \mathcal{F}\left(\langle i\partial_x \rangle^{-1} \overline{u}^2\right) = \frac{\lambda}{\sqrt{2\pi}\langle \xi \rangle} \int_{\mathbf{R}} e^{itA} \overline{v(t,\eta-\xi) v(t,-\eta)} d\eta, \tag{1.11}$$

where $A(\xi,\eta) = \langle \xi \rangle + \langle \xi - \eta \rangle + \langle \eta \rangle$. Integrating (1.11) with respect to time, we find

$$v(t,\xi) = v(0,\xi) + \frac{\lambda}{\sqrt{2\pi}\langle \xi \rangle} \int_0^t d\tau \int_{\mathbf{R}} e^{i\tau A} \overline{v(\tau,\eta-\xi) v(\tau,-\eta)} d\eta. \tag{1.12}$$

Then we integrate by parts with respect to τ, taking into account (1.11),

$$\begin{aligned}
&v(t,\xi) + i\lambda \int_{\mathbf{R}} e^{itA} \overline{v(t,\eta-\xi) v(t,-\eta)} \frac{d\eta}{\sqrt{2\pi}\langle \xi \rangle A(\xi,\eta)} \\
&= v(0,\xi) + i\lambda \int_{\mathbf{R}} \overline{v(0,\eta-\xi) v(0,-\eta)} \frac{d\eta}{\sqrt{2\pi}\langle \xi \rangle A(\xi,\eta)} \\
&\quad - 2i|\lambda|^2 \int_0^t d\tau \int_{\mathbf{R}} \frac{d\eta}{\sqrt{2\pi}\langle \xi \rangle A(\xi,\eta)} e^{i\tau(A - \langle \eta \rangle)} \overline{v(\tau,\eta-\xi)} \mathcal{F}_{x\to\eta}\left(\langle i\partial_x \rangle^{-1} u^2\right).
\end{aligned} \tag{1.13}$$

Returning to the function $u(t,x) = \mathcal{U}(t)\mathcal{F}^{-1}_{\xi\to x} v = \mathcal{F}^{-1}_{\xi\to x}(e^{-it\langle \xi \rangle} v(t,\xi))$, we obtain the following equation:

$$\mathcal{L}(u + i\lambda \mathcal{G}(\overline{u},\overline{u})) = -2i|\lambda|^2 \mathcal{G}\left(\overline{u}, \langle i\partial_x \rangle^{-1} u^2\right), \tag{1.14}$$

with the symmetric bilinear operator

$$\mathcal{G}(\phi,\psi) = \mathcal{F}^{-1}_{\xi\to x} \int_{\mathbf{R}} \widehat{\widehat{g}}(\xi-\eta,\eta) \widehat{\phi}(\xi-\eta) \widehat{\psi}(\eta) d\eta, \tag{1.15}$$

where

$$\widehat{\widehat{g}}(\zeta,\eta)=\frac{1}{\sqrt{2\pi}\langle\zeta+\eta\rangle(\langle\zeta+\eta\rangle+\langle\zeta\rangle+\langle\eta\rangle)},\tag{1.16}$$

and $\zeta=\xi-\eta$. Our main point in this paper is to show that the right-hand side of (1.14) can be decomposed into two terms; one of them is a cubic nonlinearity

$$-2i|\lambda|^2\frac{1}{t}\mathcal{U}(t)\mathcal{F}^{-1}\Omega(\xi)|\mathcal{F}\mathcal{U}(-t)u(t,\xi)|^2\mathcal{F}\mathcal{U}(-t)u(t,\xi),\tag{1.17}$$

and the other one is a remainder term with an estimate like $O(t^{-5/4}\|\mathcal{F}\mathcal{U}(-t)u(t)\|^3_{\mathbf{H}^{1,3}})$.

Remark 1.2. We believe that all quadratic nonlinear terms $u^2,\overline{u}^2,|u|^2$ of problem (1.3) also could be considered by this approach. In the same way as in the derivation of (1.14) we get from (1.3)

$$\begin{aligned}\mathcal{L}\Big(u+\frac{i}{2}\mu(\mathcal{G}(\overline{u},\overline{u})+\mathcal{G}_1(u,u)+2\mathcal{G}_2(u,\overline{u}))\Big)\\ &=i\mu^2\mathcal{G}\Big(\overline{u},\langle i\partial_x\rangle^{-1}(u+\overline{u})^2\Big)\\ &\quad+i\mu^2\mathcal{G}_1\Big(u,\langle i\partial_x\rangle^{-1}(u+\overline{u})^2\Big)\\ &\quad+i\mu^2\mathcal{G}_2\Big(u+\overline{u},\langle i\partial_x\rangle^{-1}(u+\overline{u})^2\Big),\end{aligned}\tag{1.18}$$

where

$$\begin{aligned}\mathcal{G}_j(\phi,\varphi)&=\mathcal{F}^{-1}_{\xi\to x}\int_{\mathbf{R}}\widehat{\widehat{g}}_j(\xi-\eta,\eta)\widehat{\phi}(\xi-\eta)\widehat{\varphi}(\eta)d\eta,\\ \widehat{\widehat{g}}_j(\zeta,\eta)&=\frac{1}{\sqrt{2\pi}A_j(\zeta+\eta,\eta)\langle\zeta+\eta\rangle}\end{aligned}\tag{1.19}$$

with $A_1(\xi,\eta)=\langle\xi\rangle-\langle\xi-\eta\rangle-\langle\eta\rangle$, $A_2(\xi,\eta)=\langle\xi\rangle-\langle\xi-\eta\rangle+\langle\eta\rangle$. Some more regularity conditions are necessary to treat the bilinear operators $\mathcal{G}_j$. Also we have to show that

$$\begin{aligned}&\mathcal{G}\Big(\overline{u},\langle i\partial_x\rangle^{-1}\Big(\overline{u}^2+2|u|^2\Big)\Big),\qquad \mathcal{G}_1\Big(u,\langle i\partial_x\rangle^{-1}\Big(u^2+\overline{u}^2\Big)\Big),\\ &\mathcal{G}_2\Big(u,\langle i\partial_x\rangle^{-1}\Big(u^2+\overline{u}^2\Big)\Big),\qquad \mathcal{G}_2\Big(\overline{u},\langle i\partial_x\rangle^{-1}\Big(\overline{u}^2+2|u|^2\Big)\Big)\end{aligned}\tag{1.20}$$

are the nonresonant terms (i.e., remainders) and to remove the resonant terms

$$\mathcal{G}\Big(\overline{u},\langle i\partial_x\rangle^{-1}u^2\Big),\quad \mathcal{G}_1\Big(u,\langle i\partial_x\rangle^{-1}|u|^2\Big),\quad \mathcal{G}_2\Big(u,\langle i\partial_x\rangle^{-1}|u|^2\Big),\quad \mathcal{G}_2\Big(\overline{u},\langle i\partial_x\rangle^{-1}u^2\Big)\tag{1.21}$$

by an appropriate phase function. We will dedicate a separate paper to this problem.

We prove our main result in Section 3. In the next section we prove several lemmas used in the proof of the main result.

2. Preliminaries

First we give some estimates for the symmetric bilinear operator

$$\mathcal{G}(\phi,\varphi) = \mathcal{F}^{-1}_{\xi\to x}\int_{\mathbf{R}} \widehat{\widehat{g}}(\xi-\eta,\eta)\widehat{\phi}(\xi-\eta)\widehat{\varphi}(\eta)d\eta, \tag{2.1}$$

where

$$\widehat{\widehat{g}}(\zeta,\eta) = \frac{1}{\sqrt{2\pi}\langle\zeta+\eta\rangle(\langle\zeta+\eta\rangle+\langle\zeta\rangle+\langle\eta\rangle)}. \tag{2.2}$$

Denote the kernel as follows:

$$g(y,z) = \frac{1}{4\pi^2}\iint_{\mathbf{R}^2}\frac{e^{iy\zeta+iz\eta}}{(\langle\zeta+\eta\rangle+\langle\zeta\rangle+\langle\eta\rangle)\langle\zeta+\eta\rangle}d\eta d\zeta. \tag{2.3}$$

Lemma 2.1. *The representation is true*

$$\mathcal{G}(\phi,\varphi) = \iint_{\mathbf{R}^2} g(y,z)\phi(x-y)\varphi(x-z)dydz, \tag{2.4}$$

where the kernel $g(y,z)$ obeys the following estimate:

$$|g(y,z)| \le C\langle y\rangle^{-3}\langle z\rangle^{-3}\ln^2\left(1+\frac{1}{|z-y|}\right) \tag{2.5}$$

for all $y,z\in\mathbf{R}, y\ne z$. Moreover the following estimates are valid:

$$\begin{aligned}\|\mathcal{G}(\phi,\varphi)\|_{\mathbf{L}^p} &\le C\|\phi\|_{\mathbf{L}^q}\|\varphi\|_{\mathbf{L}^r},\\ \|\mathcal{P}\mathcal{G}(\phi,\varphi)\|_{\mathbf{L}^p} &\le C\|\mathcal{P}\phi\|_{\mathbf{L}^q}\|\varphi\|_{\mathbf{L}^r}+C\|\mathcal{P}\varphi\|_{\mathbf{L}^q}\|\phi\|_{\mathbf{L}^r}\\ &\quad +C\|\partial_t\phi\|_{\mathbf{L}^q}\|\varphi\|_{\mathbf{L}^r}+C\|\partial_t\varphi\|_{\mathbf{L}^q}\|\phi\|_{\mathbf{L}^r}\end{aligned} \tag{2.6}$$

for $1\le p\le\infty, 1<q\le p/\alpha, 1<r\le p/(1-\alpha), \alpha\in[0,1]$, provided that the right-hand sides are bounded.

Proof. To prove representation (2.4), we substitute the direct Fourier transforms

$$\widehat{\phi}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-i(x-y)\zeta}\phi(x-y)dy,$$
$$\widehat{\varphi}(\eta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-i(x-z)\eta}\varphi(x-z)dz \tag{2.7}$$

into the definition of the operator $\mathcal{G}$. Then changing $\zeta = \xi - \eta$, we find

$$\mathcal{G}(\phi,\varphi) = (2\pi)^{-3/2}\iint_{\mathbf{R}^2}\phi(x-y)\varphi(x-z)\left(\iint_{\mathbf{R}^2}\widehat{\widehat{g}}(\zeta,\eta)e^{iy\zeta+iz\eta}d\eta d\zeta\right)dydz$$
$$= \iint_{\mathbf{R}^2} g(y,z)\phi(x-y)\varphi(x-z)dydz, \tag{2.8}$$

where the kernel $g(y,z)$ is

$$g(y,z) = (2\pi)^{-3/2}\iint_{\mathbf{R}^2}\widehat{\widehat{g}}(\zeta,\eta)e^{iy\zeta+iz\eta}d\eta d\zeta$$
$$= \frac{1}{4\pi^2}\iint_{\mathbf{R}^2}\frac{e^{iy\zeta+iz\eta}}{(\langle\zeta+\eta\rangle+\langle\zeta\rangle+\langle\eta\rangle)\langle\zeta+\eta\rangle}d\eta d\zeta. \tag{2.9}$$

Changing the variables of integration $\zeta = \xi/2 - \eta'$ and $\eta = \xi/2 + \eta'$ (the prime we will omit), we get

$$g(y,z) = \frac{1}{4\pi^2}\iint_{\mathbf{R}^2}\frac{e^{iy\zeta+iz\eta}}{(\langle\zeta+\eta\rangle+\langle\zeta\rangle+\langle\eta\rangle)\langle\zeta+\eta\rangle}d\eta d\zeta$$
$$= \frac{1}{4\pi^2}\int_{\mathbf{R}}\frac{d\xi}{\langle\xi\rangle}e^{i((y+z)/2)\xi}\int_{\mathbf{R}}B(\xi,\eta)e^{i(z-y)\eta}d\eta, \tag{2.10}$$

where $B(\xi,\eta) = 1/(\langle\xi\rangle + \langle\xi/2-\eta\rangle + \langle\xi/2+\eta\rangle)$. We change $s = y + z/2$ and $\rho = (z-y)$ and denote

$$\widetilde{g}(s,\rho) = \frac{1}{4\pi^2}\int_{\mathbf{R}}\frac{d\xi}{\langle\xi\rangle}e^{is\xi}\int_{\mathbf{R}}B(\xi,\eta)e^{i\rho\eta}d\eta. \tag{2.11}$$

For the case of $|\rho| \le 1, |s| \le 1$ we integrate by parts using the identity $e^{i\rho\eta} = A(\eta)\partial_\eta(\eta e^{i\rho\eta})$, where $A(\eta) = 1/(1+i\rho\eta)$. Then we get

$$\widetilde{g}(s,\rho) = -\frac{1}{4\pi^2}\int_{\mathbf{R}}\frac{d\xi}{\langle\xi\rangle}e^{is\xi}\int_{\mathbf{R}}e^{i\rho\eta}\eta\partial_\eta(A(\eta)B(\xi,\eta))d\eta. \tag{2.12}$$

Note that

$$\eta\partial_\eta B(\xi,\eta) = \frac{\eta}{\left(\langle\xi\rangle + \langle\xi/2-\eta\rangle + \langle\xi/2+\eta\rangle\right)^2}\left(\frac{\eta-\xi/2}{\langle\eta-\xi/2\rangle} + \frac{\eta+\xi/2}{\langle\xi/2+\eta\rangle}\right). \tag{2.13}$$

Then

$$\left|\langle\xi\rangle^{-1}\eta\partial_\eta(A(\eta)B(\xi,\eta))\right| \le \frac{C}{\langle\xi\rangle(1+|\rho||\eta|)(\langle\xi\rangle+\langle\eta\rangle)}. \tag{2.14}$$

Hence we can estimate the kernel $\tilde{g}(s,\rho)$ as follows:

$$\begin{aligned} |\tilde{g}(s,\rho)| &\le C\int_{\mathbf{R}}\frac{d\xi}{\langle\xi\rangle}\int_{\mathbf{R}}\frac{d\eta}{(1+|\rho||\eta|)(\langle\xi\rangle+\langle\eta\rangle)} \\ &\le C\int_1^\infty \frac{d\eta}{1+|\rho|\eta}\int_1^\infty\frac{d\xi}{\xi(\eta+\xi)} = C\int_1^\infty\frac{d\eta}{1+|\rho|\eta}\frac{1}{\eta}\left(\ln\frac{\xi}{\eta+\xi}\right)\Big|_1^\infty d\xi \\ &\le C\int_1^\infty\frac{\ln\eta}{(1+|\rho|\eta)\eta}d\eta \le C\int_1^{1/|\rho|}\ln\eta\frac{d\eta}{\eta} + \frac{C}{|\rho|}\int_{1/|\rho|}^\infty \ln\eta\frac{d\eta}{\eta^2} \\ &\le C\ln^2\left(1+\frac{1}{|\rho|}\right) \end{aligned} \tag{2.15}$$

for the case of $|\rho| \le 1, |s| \le 1$. For the case of $|\rho| \le 1, |s| \ge 1$ we integrate three times by parts with respect to ξ

$$\tilde{g}(s,\rho) = C|s|^{-3}\int_{\mathbf{R}} d\xi e^{is\xi}\int_{\mathbf{R}} e^{i\rho\eta}\partial_\xi^3\left(\langle\xi\rangle^{-1}\eta\partial_\eta(A(\eta)B(\xi,\eta))\right)d\eta. \tag{2.16}$$

Note that

$$\left|\partial_\xi^3\left(\langle\xi\rangle^{-1}\eta\partial_\eta(A(\eta)B(\xi,\eta))\right)\right| \le \frac{C}{\langle\xi\rangle(1+|\rho||\eta|)(\langle\xi\rangle+\langle\eta\rangle)}. \tag{2.17}$$

Hence we can estimate the kernel $g(y,z)$ as follows:

$$|\tilde{g}(s,\rho)| \le C|s|^{-3}\int_{\mathbf{R}}\frac{d\xi}{\langle\xi\rangle}\int_{\mathbf{R}}\frac{d\eta}{(1+|\rho||\eta|)(\langle\xi\rangle+\langle\eta\rangle)} \le C|s|^{-3}\ln^2\left(1+\frac{1}{|\rho|}\right) \tag{2.18}$$

for all $|\rho| \le 1, |s| \ge 1$. For the case $|\rho| \ge 1, |s| \le 1$ we integrate by parts three times with respect to η

$$\tilde{g}(s,\rho) = C|\rho|^{-3}\int_{\mathbf{R}}\frac{d\xi}{\langle\xi\rangle}e^{is\xi}\int_{\mathbf{R}} e^{i\rho\eta}\partial_\eta^3 B(\xi,\eta)d\eta. \tag{2.19}$$

Note that $|\partial_\eta^3 B(\xi,\eta)| \le C(\langle\xi\rangle + \langle\eta\rangle)^{-3}$. Hence

$$|\tilde{g}(s,\rho)| \le C|\rho|^{-3}\int_{\mathbf{R}} \frac{d\xi}{\langle\xi\rangle^2}\int_{\mathbf{R}} \frac{d\eta}{\langle\eta\rangle^2} \le C|\rho|^{-3}. \tag{2.20}$$

Finally for the case of $|\rho| \ge 1, |s| \ge 1$ we integrate by parts three times with respect to ξ and η

$$\tilde{g}(s,\rho) = C|s\rho|^{-3}\int_{\mathbf{R}} d\xi e^{is\xi}\int_{\mathbf{R}} e^{i\rho\eta}\partial_\xi^3\partial_\eta^3\left(\langle\xi\rangle^{-1}B(\xi,\eta)\right)d\eta. \tag{2.21}$$

Since

$$\left|\partial_\xi^3\partial_\eta^3\left(\langle\xi\rangle^{-1}B(\xi,\eta)\right)\right| \le \frac{C}{\langle\xi\rangle^2(\langle\xi\rangle + \langle\eta\rangle)^2}, \tag{2.22}$$

then we can estimate the kernel $\tilde{g}(s,\rho)$ as follows:

$$|\tilde{g}(s,\rho)| \le C|s\rho|^{-3}\int_{\mathbf{R}} \frac{d\xi}{\langle\xi\rangle^2}\int_{\mathbf{R}} \frac{d\eta}{\langle\eta\rangle^2} \le C|s\rho|^{-3} \tag{2.23}$$

for all $|\rho| \ge 1, |s| \ge 1$. Hence estimate (2.5) is true.

By virtue of estimate (2.5) applying the Hölder inequality with $1/p = 1/p_1 + 1/p_2$ and the Young inequality with $1/p_1 + 1 = 1/q + 1/q_1$ and $1/p_2 + 1 = 1/r + 1/r_1$, we find

$$\begin{aligned} \|\mathcal{G}(\phi,\psi)\|_{\mathbf{L}^p} &\le C\left\|\int_{\mathbf{R}} |\phi(x-y)|\frac{dy}{\langle y\rangle^3}\right\|_{\mathbf{L}_x^{p_1}}\left\|\int_{\mathbf{R}} \ln^2\left(1+\frac{1}{|z-y|}\right)|\psi(x-z)|\frac{dz}{\langle z\rangle^3}\right\|_{\mathbf{L}_x^{p_2}\mathbf{L}_y^\infty} \\ &\le C\|\phi\|_{\mathbf{L}^q}\|\psi\|_{\mathbf{L}^r}, \end{aligned} \tag{2.24}$$

where $1 \le p \le \infty, 1 < q \le p/\alpha, 1 < r \le p/(1-\alpha), \alpha \in [0,1]$.

We now estimate the operator $\mathcal{P} = t\partial_x + x\partial_t$ as follows:

$$\begin{aligned} \mathcal{P}\mathcal{G}(\phi,\psi) = \mathcal{G}(\mathcal{P}\phi,\psi) &+ \iint_{\mathbf{R}^2} yg(y,z)\psi(x-z)\partial_t\phi(x-y)dydz \\ &+ \mathcal{G}(\phi,\mathcal{P}\psi) + \iint_{\mathbf{R}^2} zg(y,z)\phi(x-y)\partial_t\psi(x-z)dydz. \end{aligned} \tag{2.25}$$

Then by virtue of estimate (2.4) applying the Hölder and Young inequalities, we get

$$\begin{aligned}\|\mathcal{P}\mathcal{G}(\phi,\psi)\|_{\mathbf{L}^p} &\le \|\mathcal{G}(\mathcal{P}\phi,\psi)\|_{\mathbf{L}^p} + \|\mathcal{G}(\phi,\mathcal{P}\psi)\|_{\mathbf{L}^p} \\ &\quad + \left\| \iint_{\mathbf{R}^2} yg(y,z)\psi(x-z)\partial_t\phi(x-y)\,dy\,dz \right\|_{\mathbf{L}^p} \\ &\quad + \left\| \iint_{\mathbf{R}^2} zg(y,z)\phi(x-y)\partial_t\psi(x-z)\,dy\,dz \right\|_{\mathbf{L}^p} \\ &\le C\|\mathcal{P}\phi\|_{\mathbf{L}^q}\|\psi\|_{\mathbf{L}^r} + C\|\mathcal{P}\psi\|_{\mathbf{L}^q}\|\phi\|_{\mathbf{L}^r} + C\|\partial_t\phi\|_{\mathbf{L}^q}\|\psi\|_{\mathbf{L}^r} \\ &\quad + C\|\partial_t\psi\|_{\mathbf{L}^q}\|\phi\|_{\mathbf{L}^r}.\end{aligned} \tag{2.26}$$

Lemma 2.1 is proved. □

We now decompose the free Klein-Gordon evolution group $\mathcal{U}(t) = e^{-i\langle i\partial_x\rangle t} = \mathcal{F}^{-1}E(t)\mathcal{F}$, where $E(t) = e^{-it\langle\xi\rangle}$ similarly to the factorization of the free Schrödinger evolution group. We denote the dilation operator by

$$\mathfrak{D}_\omega\phi = \frac{1}{\sqrt{i\omega}}\phi\left(\frac{x}{\omega}\right), \qquad (\mathfrak{D}_\omega)^{-1} = i\mathfrak{D}_{1/\omega}. \tag{2.27}$$

Define the multiplication factor $M(t) = e^{-it\langle ix\rangle\theta(x)}$, where $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| \ge 1$. We introduce the operator

$$\mathcal{B}\phi = \frac{\theta(x)}{\langle ix\rangle^{3/2}}\phi\left(\frac{x}{\langle ix\rangle}\right). \tag{2.28}$$

The inverse operator $\mathcal{B}^{-1}$ acts on the functions $\phi(x)$ defined on $(-1,1)$ as follows:

$$\mathcal{B}^{-1}\phi = \frac{1}{\langle\xi\rangle^{3/2}}\phi\left(\frac{\xi}{\langle\xi\rangle}\right) \tag{2.29}$$

for all $\xi \in \mathbf{R}$, since $\xi = x/\langle ix\rangle \in \mathbf{R}$ and $x = \xi/\langle\xi\rangle \in (-1,1)$. We now introduce the operators

$$\begin{aligned}\mathcal{V}(t) &= \mathcal{B}^{-1}\overline{M}(t)\mathfrak{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}, \\ \mathcal{W}(t) &= \overline{M}(t)(1-\theta)\mathfrak{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle},\end{aligned} \tag{2.30}$$

so that we have the representation for the free Klein-Gordon evolution group

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1} &= e^{-it\langle i\partial_x\rangle}\mathcal{F}^{-1} = \mathcal{F}^{-1}e^{-it\langle\xi\rangle} = \mathfrak{D}_tM(t)(\mathcal{B}\mathcal{V}(t) + \mathcal{W}(t)) \\ &= \mathfrak{D}_tM(t)\mathcal{B} + \mathfrak{D}_tM(t)\mathcal{B}(\mathcal{V}(t)-1) + \mathfrak{D}_tM(t)\mathcal{W}(t).\end{aligned} \tag{2.31}$$

The first term $\mathfrak{D}_t M(t)\mathcal{B}\phi$ of the right-hand side of (2.31) describes inside the light cone the well-known leading term of the large-time asymptotics of solutions of the linear Klein-Gordon equation $\mathcal{L}u = 0$ with initial data ϕ. The second term of the right-hand side of (2.31) is a remainder inside of the light cone, whereas the last term represents the large time asymptotics outside of the light cone which decays more rapidly in time. We also have

$$\begin{aligned} \mathcal{F}\mathcal{U}(-t) &= \mathcal{F}e^{it\langle i\partial_x\rangle} = e^{it\langle\xi\rangle}\mathcal{F} = \mathcal{V}^{-1}(t)\mathcal{B}^{-1}\overline{M}(t)\mathfrak{D}_t^{-1} + \mathcal{W}^{-1}(t)\mathfrak{D}_t^{-1} \\ &= \mathcal{B}^{-1}\overline{M}(t)\mathfrak{D}_t^{-1} + \left(\mathcal{V}^{-1}(t) - 1\right)\mathcal{B}^{-1}\overline{M}(t)\mathfrak{D}_t^{-1} + \mathcal{W}^{-1}(t)\mathfrak{D}_t^{-1}, \end{aligned} \tag{2.32}$$

where the right-inverse operators are

$$\begin{aligned} \mathcal{V}^{-1}(t) &= \overline{E}(t)\mathcal{F}\mathfrak{D}_t M(t)\mathcal{B}, \\ \mathcal{W}^{-1}(t) &= \overline{E}(t)\mathcal{F}\mathfrak{D}_t(1-\theta), \end{aligned} \tag{2.33}$$

where $E(t) = e^{-it\langle\xi\rangle}$.

In the next lemma we state the estimates of the operators $\mathcal{V}(t) = \mathcal{B}^{-1}\overline{M}(t)\mathfrak{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$.

Lemma 2.2. *The estimates hold as follows:*

$$\begin{aligned} &\|\mathcal{V}(t)\phi\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^2}, \\ &\|\langle\xi\rangle^{\rho}\partial_\xi\mathcal{V}(t)\phi\|_{\mathbf{L}^2} \le C\|\langle\xi\rangle^{\alpha}\partial_\xi\phi\|_{\mathbf{L}^2} + C\left\|\langle\xi\rangle^{\alpha-1}\phi\right\|_{\mathbf{L}^2}, \end{aligned} \tag{2.34}$$

where $\rho < \alpha, \alpha \in [1,2]$, *and*

$$\left\|\langle\xi\rangle^{\rho}\partial_\xi\mathcal{V}^{-1}(t)\phi\right\|_{\mathbf{L}^2} \le C\left\|\langle\xi\rangle^{1/4+\alpha}\partial_\xi\phi\right\|_{\mathbf{L}^2} + C\left\|\langle\xi\rangle^{\alpha-3/4}\phi\right\|_{\mathbf{L}^2}, \tag{2.35}$$

where $\rho < \alpha, \alpha \in [0,1]$, *provided the right-hand sides are finite.*

Proof. Changing the variable of integration $x = \xi\langle\xi\rangle^{-1}$, we see that

$$\left\|\mathcal{B}^{-1}\phi\right\|_{\mathbf{L}^2}^2 = \int_{\mathbf{R}}\left|\phi\left(\frac{\xi}{\langle\xi\rangle}\right)\right|^2\frac{d\xi}{\langle\xi\rangle^3} = \int_{-1}^{1}|\phi(x)|^2dx = \|\phi\|_{\mathbf{L}^2(-1,1)}^2. \tag{2.36}$$

Hence $\|\mathcal{V}(t)\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$.

Consider the estimate for the derivative $\partial_\xi\mathcal{V}(t)\phi$. Define $S(\xi,\eta) = \langle\eta\rangle - (\xi\eta+1)/\langle\xi\rangle$. Note that $\partial S/\partial\eta = \eta/\langle\eta\rangle - \xi/\langle\xi\rangle = ((1+\langle\eta\rangle\langle\xi\rangle - \eta\xi)/\langle\xi\rangle\langle\eta\rangle(\langle\eta\rangle+\langle\xi\rangle))(\eta-\xi)$ and $\partial S/\partial\xi = (\xi-\eta)/\langle\xi\rangle^3$. Hence integrating by parts one time yields

$$\begin{aligned} \partial_\xi\int_{\mathbf{R}}e^{-itS}\phi(\eta)d\eta &= -it\langle\xi\rangle^{-3}\int_{\mathbf{R}}e^{-itS}\phi(\eta)(\xi-\eta)d\eta \\ &= \langle\xi\rangle^{-3}\int_{\mathbf{R}}e^{-itS}(\phi'(\eta)g(\xi,\eta) + \phi(\eta)g_\eta(\xi,\eta))d\eta, \end{aligned} \tag{2.37}$$

where $g(\xi,\eta)=(\xi-\eta)/(\xi/\langle\xi\rangle-\eta/\langle\eta\rangle)=\langle\xi\rangle\langle\eta\rangle(\langle\eta\rangle+\langle\xi\rangle)/(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)$. Also we have

$$(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)^{-1}\le C\frac{\langle\xi\rangle\langle\eta\rangle}{(\langle\eta\rangle+\langle\xi\rangle)^{2}}. \tag{2.38}$$

Hence the estimate is true

$$g(\xi,\eta)=\frac{\langle\xi\rangle\langle\eta\rangle(\langle\eta\rangle+\langle\xi\rangle)}{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}\le C\frac{\langle\xi\rangle^{2}\langle\eta\rangle^{2}}{\langle\eta\rangle+\langle\xi\rangle}. \tag{2.39}$$

Since $\partial_\xi\langle\xi\rangle^{a}=a\xi\langle\xi\rangle^{a-2}$, we get

$$\begin{aligned}\langle\xi\rangle^{\rho}\partial_\xi\mathcal{V}(t)\phi&=\langle\xi\rangle^{\rho}\sqrt{\frac{it}{2\pi}}\partial_\xi\langle\xi\rangle^{-3/2}\int_{\mathbf{R}}e^{-itS}\phi(\eta)d\eta\\&=-\frac{3}{2}\xi\langle\xi\rangle^{\rho-2}\mathcal{V}(t)\phi+\langle\xi\rangle^{\rho-9/2}\sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}}e^{-itS}\phi'(\eta)g(\xi,\eta)d\eta\\&\quad+\langle\xi\rangle^{\rho-9/2}\sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}}e^{-itS}\phi(\eta)g_\eta(\xi,\eta)d\eta.\end{aligned} \tag{2.40}$$

Consider the $\mathbf{L}^2$-estimate of the integral

$$\sqrt{t}\langle\xi\rangle^{\rho-9/2}\int_{\mathbf{R}}e^{it(\xi\eta/\langle\xi\rangle)}\psi(\eta)g(\xi,\eta)d\eta. \tag{2.41}$$

We have

$$\begin{aligned}&\left\|\sqrt{t}\int_{\mathbf{R}}e^{it(\xi\eta/\langle\xi\rangle)}\psi(\eta)g(\xi,\eta)\langle\xi\rangle^{\rho-9/2}d\eta\right\|_{\mathbf{L}^2}^{2}\\&\quad=t\int_{\mathbf{R}}d\eta\psi(\eta)\int_{\mathbf{R}}d\zeta\overline{\psi(\zeta)}\int_{\mathbf{R}}e^{it(\xi/\langle\xi\rangle)(\eta-\zeta)}g(\xi,\eta)g(\xi,\zeta)\langle\xi\rangle^{2\rho-9}d\xi\\&\quad=\int_{\mathbf{R}}d\eta\psi(\eta)\int_{\mathbf{R}}d\zeta\overline{\psi(\zeta)}G(\eta,\zeta),\end{aligned} \tag{2.42}$$

where the kernel

$$\begin{aligned}G(\eta,\zeta)&=t\int_{\mathbf{R}}e^{it(\xi/\langle\xi\rangle)(\eta-\zeta)}g(\xi,\eta)g(\xi,\zeta)\langle\xi\rangle^{2\rho-9}d\xi\\&=t\langle\eta\rangle\langle\zeta\rangle\int_{\mathbf{R}}e^{it(\xi/\langle\xi\rangle)(\eta-\zeta)}\frac{\langle\eta\rangle+\langle\xi\rangle}{(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)}\frac{\langle\zeta\rangle+\langle\xi\rangle}{(1+\langle\zeta\rangle\langle\xi\rangle-\zeta\xi)}\langle\xi\rangle^{2\rho-7}d\xi.\end{aligned} \tag{2.43}$$

After a change $\xi = x/\langle ix\rangle$, we find

$$G(\eta,\zeta) = t\langle\eta\rangle\langle\zeta\rangle\int_{-1}^{1} e^{itx(\eta-\zeta)}\frac{\langle\eta\rangle + 1/\langle ix\rangle}{(1+\langle\eta\rangle/\langle ix\rangle - \eta x/\langle ix\rangle)}\frac{\langle\zeta\rangle + 1/\langle ix\rangle}{(1+\langle\zeta\rangle/\langle ix\rangle - \zeta x/\langle ix\rangle)}\langle ix\rangle^{4-2\rho}dx. \tag{2.44}$$

We now change the contour $z = x + iy \in \Gamma$ of integration in G as $y = \operatorname{sign}(\eta-\zeta)\langle ix\rangle^2$; then

$$G(\eta,\zeta) = t\langle\eta\rangle\langle\zeta\rangle\int_{\Gamma} e^{itz(\eta-\zeta)}\frac{\langle\eta\rangle + 1/\langle iz\rangle}{(1+\langle\eta\rangle/\langle iz\rangle - \eta z/\langle iz\rangle)}\frac{\langle\zeta\rangle + 1/\langle iz\rangle}{(1+\langle\zeta\rangle/\langle iz\rangle - \zeta z/\langle iz\rangle)}\langle iz\rangle^{4-2\rho}dz. \tag{2.45}$$

Since $|\langle iz\rangle| = ((\langle ix\rangle^2 + y^2)^2 + 4x^2y^2)^{1/4} \sim \langle ix\rangle$, using (2.38) and the inequality $(\langle\eta\rangle + \langle\xi\rangle) \geq \langle\eta\rangle^{\alpha-1}\langle\xi\rangle^{2-\alpha}$ for $\alpha \in [1,2]$, we get $|(\langle\eta\rangle + 1/\langle iz\rangle)/(1+\langle\eta\rangle/\langle iz\rangle - \eta z/\langle iz\rangle)| \leq C\langle\eta\rangle^{\alpha-1}\langle ix\rangle^{\alpha-2}$; also we have

$$\left|e^{itz(\eta-\zeta)}\right| = e^{-\langle ix\rangle^2 t|\eta-\zeta|} \leq \frac{C\langle ix\rangle^{-2\delta}}{(1+t|\eta-\zeta|)^{\delta}} \tag{2.46}$$

for $\delta > 1$. Therefore we obtain the estimate

$$|G(\eta,\zeta)| \leq \frac{Ct\langle\eta\rangle^{\alpha}\langle\zeta\rangle^{\alpha}}{(1+t|\eta-\zeta|)^{\delta}}\int_{-1}^{1}\langle ix\rangle^{2(\alpha-\rho-\delta)}dx \leq \frac{Ct\langle\eta\rangle^{\alpha}\langle\zeta\rangle^{\alpha}}{(1+t|\eta-\zeta|)^{\delta}}, \tag{2.47}$$

if we choose $\rho < \alpha, \alpha \in [1,2]$ and $1 < \delta < 1 + \alpha - \rho$. Thus we get

$$\begin{aligned}
\left\|\sqrt{t}\int_{\mathbf{R}} e^{it(\xi\eta/\langle\xi\rangle)}\psi(\eta)g(\xi,\eta)\langle\xi\rangle^{\rho-9/2}d\eta\right\|_{\mathbf{L}^2}^2 &= \int_{\mathbf{R}} d\eta\psi(\eta)\int_{\mathbf{R}} d\zeta\overline{\psi(\zeta)}G(\eta,\zeta) \\
&\leq C\|\langle\eta\rangle^{\alpha}\psi(\eta)\|_{\mathbf{L}^2}\left\|\int_{\mathbf{R}} d\zeta\overline{\psi(\zeta)}\langle\zeta\rangle^{\alpha}\frac{t}{(1+t|\eta-\zeta|)^{\delta}}\right\|_{\mathbf{L}^2} \\
&\leq Ct\|\langle\cdot\rangle^{\alpha}\psi\|_{\mathbf{L}^2}^2\int_{\mathbf{R}}\frac{d\eta}{(1+t|\eta|)^{\delta}} \leq C\|\langle\cdot\rangle^{\alpha}\psi\|_{\mathbf{L}^2}^2
\end{aligned} \tag{2.48}$$

since $\delta > 1$.

In the same manner we consider the estimate of the integral

$$\begin{aligned}
&\left\|\sqrt{t}\int_{\mathbf{R}} e^{it(\xi\eta/\langle\xi\rangle)}\,\varphi(\eta)g_{\eta}(\xi,\eta)\langle\xi\rangle^{\rho-9/2}d\eta\right\|_{\mathbf{L}^2}^2 \\
&\quad = t\int_{\mathbf{R}} d\eta\varphi(\eta)\int_{\mathbf{R}} d\zeta\overline{\varphi(\zeta)}\int_{\mathbf{R}} e^{it(\xi/\langle\xi\rangle)(\eta-\zeta)}g_{\eta}(\xi,\eta)g_{\zeta}(\xi,\zeta)\langle\xi\rangle^{2\rho-9}d\xi \\
&\quad = \int_{\mathbf{R}} d\eta\varphi(\eta)\int_{\mathbf{R}} d\zeta\overline{\varphi(\zeta)}\widetilde{G}(\eta,\zeta),
\end{aligned} \tag{2.49}$$

where the kernel

$$\begin{aligned}
\widetilde{G}(\eta,\zeta) &= t\int_{\mathbf{R}} e^{it(\xi/\langle\xi\rangle)(\eta-\zeta)}g_{\eta}(\xi,\eta)g_{\zeta}(\xi,\zeta)\langle\xi\rangle^{2\rho-9}d\xi \\
&= t\int_{\mathbf{R}} e^{it(\xi/\langle\xi\rangle)(\eta-\zeta)}\frac{(\eta+\xi+(\eta/\langle\eta\rangle)\langle\xi\rangle)}{(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)}\frac{(\zeta+\xi+(\zeta/\langle\zeta\rangle)\langle\xi\rangle)}{(1+\langle\zeta\rangle\langle\xi\rangle-\zeta\xi)}\langle\xi\rangle^{2\rho-7}d\xi,
\end{aligned} \tag{2.50}$$

since by a direct calculation

$$\partial_{\eta}\frac{\langle\eta\rangle(\langle\eta\rangle+\langle\xi\rangle)}{(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)} = \frac{\eta+\xi+(\eta/\langle\eta\rangle)\langle\xi\rangle}{(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)}. \tag{2.51}$$

In the same way as in (2.47) we have

$$\left|\widetilde{G}(\eta,\zeta)\right| \le \frac{Ct\langle\eta\rangle^{\alpha-1}\langle\zeta\rangle^{\alpha-1}}{(1+t|\eta-\zeta|)^{\delta}} \tag{2.52}$$

if we choose $\rho<\alpha, \alpha\in[1,2]$, and $1<\delta<1+\alpha-\rho$. Therefore we get

$$\left\|\sqrt{t}\int_{\mathbf{R}} e^{it(\xi\eta/\langle\xi\rangle)}\varphi(\eta)g_{\eta}(\xi,\eta)\langle\xi\rangle^{\rho-9/2}d\eta\right\|_{\mathbf{L}^2}^2 \le C\left\|\langle\eta\rangle^{\alpha-1}\varphi(\eta)\right\|_{\mathbf{L}^2}^2\int_{\mathbf{R}}\frac{td\eta}{(1+t|\eta|)^{\delta}} \le C\left\|\langle\cdot\rangle^{\alpha-1}\varphi\right\|_{\mathbf{L}^2}^2. \tag{2.53}$$

Hence $\|\langle\xi\rangle^{\rho}\partial_{\xi}\mathcal{U}(t)\phi\|_{\mathbf{L}^2} \le C\|\langle\xi\rangle^{\alpha}\partial_{\xi}\phi\|_{\mathbf{L}^2} + C\|\langle\xi\rangle^{\alpha-1}\phi\|_{\mathbf{L}^2}$. Thus we have the second estimate of the lemma.

Consider the estimate for the derivative $\partial_{\xi}\mathcal{U}^{-1}(t)\phi$. Note that $\partial S(\eta,\xi)/\partial\xi = \xi/\langle\xi\rangle - \eta/\langle\eta\rangle = ((1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)/\langle\xi\rangle\langle\eta\rangle(\langle\eta\rangle+\langle\xi\rangle))(\xi-\eta)$ and $\partial S(\eta,\xi)/\partial\eta = (\eta-\xi)/\langle\eta\rangle^3$. Hence integrating by parts one time yields

$$\begin{aligned}
\langle\xi\rangle^{\rho}\partial_{\xi}\mathcal{U}^{-1}(t)\phi &= -\frac{\sqrt{t}}{\sqrt{2i\pi}}\langle\xi\rangle^{\rho-1}\int_{\mathbf{R}}\frac{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}{(\langle\eta\rangle+\langle\xi\rangle)}\langle\eta\rangle^{1/2}\phi(\eta)de^{itS(\eta,\xi)} \\
&= \frac{\sqrt{t}}{\sqrt{2i\pi}}\langle\xi\rangle^{\rho-1}\int_{\mathbf{R}} e^{itS(\eta,\xi)}(\phi'(\eta)g_1(\xi,\eta)+\phi(\eta)g_{1\eta}(\xi,\eta))d\eta,
\end{aligned} \tag{2.54}$$

where $S(\eta,\xi)=\langle\xi\rangle-(\xi\eta+1)/\langle\eta\rangle$ and $g_1(\xi,\eta)=(((1+\langle\eta\rangle\langle\xi\rangle-\eta\xi))/(\langle\eta\rangle+\langle\xi\rangle))\langle\eta\rangle^{1/2}$. The estimate is true

$$g_1(\xi,\eta)=\frac{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}{\langle\eta\rangle+\langle\xi\rangle}\langle\eta\rangle^{1/2}\leq C\frac{\langle\eta\rangle^{3/2}\langle\xi\rangle}{\langle\eta\rangle+\langle\xi\rangle}. \tag{2.55}$$

Consider the $\mathbf{L}^2$-estimate of the integral

$$\langle\xi\rangle^{\rho-1}\sqrt{t}\int_{\mathbf{R}}e^{-it(\xi\eta/\langle\eta\rangle)}\varphi(\eta)g_1(\xi,\eta)d\eta. \tag{2.56}$$

We have, changing $\eta/\langle\eta\rangle=y$ and $\zeta/\langle\zeta\rangle=z$,

$$\begin{aligned}
&\left\|\langle\xi\rangle^{\rho-1}\sqrt{t}\int_{\mathbf{R}}e^{-it(\xi\eta/\langle\eta\rangle)}\varphi(\eta)g_1(\xi,\eta)d\eta\right\|_{\mathbf{L}^2}^2\\
&\quad=t\int_{\mathbf{R}}d\eta\varphi(\eta)\int_{\mathbf{R}}d\zeta\overline{\varphi(\zeta)}\int_{\mathbf{R}}e^{-it\xi(\eta/\langle\eta\rangle-\zeta/\langle\zeta\rangle)}g_1(\xi,\eta)g_1(\xi,\zeta)\langle\xi\rangle^{2\rho-2}d\xi\\
&\quad=t\int_{-1}^{1}dy\langle iy\rangle^3\varphi\left(\frac{y}{\langle iy\rangle}\right)\int_{-1}^{1}dz\langle iz\rangle^3\overline{\varphi\left(\frac{z}{\langle iz\rangle}\right)}\\
&\qquad\times\int_{\mathbf{R}}e^{-it\xi(y-z)}g_1\left(\xi,\frac{y}{\langle iy\rangle}\right)g_1\left(\xi,\frac{z}{\langle iz\rangle}\right)\langle\xi\rangle^{2\rho-2}d\xi\\
&\quad=\int_{-1}^{1}dy\langle iy\rangle^3\varphi\left(\frac{y}{\langle iy\rangle}\right)\int_{-1}^{1}dz\langle iz\rangle^3\overline{\varphi\left(\frac{z}{\langle iz\rangle}\right)}G_1(y,z),
\end{aligned}\tag{2.57}$$

where the kernel

$$\begin{aligned}
&G_1(y,z)\\
&\quad=t\int_{\mathbf{R}}e^{-it\xi(y-z)}g_1\left(\xi,\frac{y}{\langle iy\rangle}\right)g_1\left(\xi,\frac{z}{\langle iz\rangle}\right)\langle\xi\rangle^{2\rho-2}d\xi\\
&\quad=t\langle iy\rangle^{-1/2}\langle iz\rangle^{-1/2}\int_{\mathbf{R}}e^{-it\xi(y-z)}\frac{1+\langle\xi\rangle/\langle iy\rangle-y\xi/\langle iy\rangle}{1/\langle iy\rangle+\langle\xi\rangle}\frac{1+\langle\xi\rangle/\langle iz\rangle-z\xi/\langle iz\rangle}{1/\langle iz\rangle+\langle\xi\rangle}\langle\xi\rangle^{2\rho-2}d\xi.
\end{aligned}\tag{2.58}$$

We now change the contour $\xi=\tilde{x}+i\tilde{y}\in\Gamma$ of integration in G as $\tilde{y}=\operatorname{sign}(y-z)\langle\tilde{x}\rangle^2$; then

$$\begin{aligned}
&G_1(y,z)\\
&\quad=t\langle iy\rangle^{-1/2}\langle iz\rangle^{-1/2}\int_{\Gamma}e^{-it\xi(y-z)}\frac{1+\langle\xi\rangle/\langle iy\rangle-y\xi/\langle iy\rangle}{1/\langle iy\rangle+\langle\xi\rangle}\frac{1+\langle\xi\rangle/\langle iz\rangle-z\xi/\langle iz\rangle}{1/\langle iz\rangle+\langle\xi\rangle}\langle\xi\rangle^{2\rho-2}d\xi.
\end{aligned}\tag{2.59}$$

Since $|\langle\xi\rangle| = ((\langle\tilde{x}\rangle^2 - \tilde{y}^2)^2 + 4\tilde{x}^2\tilde{y}^2)^{1/2} \sim \langle\tilde{x}\rangle$ and by (2.55) $|(1+\langle\xi\rangle/\langle iy\rangle - y\xi/\langle iy\rangle)/(1/\langle iy\rangle + \langle\xi\rangle)| \le C\langle\xi\rangle^{1-\alpha}\langle iy\rangle^{-\alpha}$ for $\alpha \in [0,1]$, and also

$$\left|e^{-it\xi(y-z)}\right| \le \frac{C}{(1+t\langle\tilde{x}\rangle|y-z|)^2} \le \frac{C}{t\langle\tilde{x}\rangle^{\delta}|y-z|^{\delta}\langle y-z\rangle} \tag{2.60}$$

for $\delta < 1$, we obtain the estimate

$$\begin{aligned}|G_1(y,z)| &\le \frac{C\langle iy\rangle^{-1/2-\alpha}\langle iz\rangle^{-1/2-\alpha}}{|y-z|^{\delta}\langle y-z\rangle}\int_{\mathbf{R}}\langle\tilde{x}\rangle^{-2\alpha+2\rho-\delta}d\tilde{x} \\ &\le \frac{C\langle iy\rangle^{-1/2-\alpha}\langle iz\rangle^{-1/2-\alpha}}{|y-z|^{\delta}\langle y-z\rangle},\end{aligned} \tag{2.61}$$

if we choose $\rho < \alpha$, and $1 > \delta > 1 - 2\alpha + 2\rho$. Therefore we get

$$\begin{aligned}&\left\|\sqrt{t}\int_{\mathbf{R}} e^{it(\xi\eta/\langle\xi\rangle)}\psi(\eta)g(\xi,\eta)\langle\xi\rangle^{\rho-9/2}d\eta\right\|_{\mathbf{L}^2}^2 \\ &\quad= \int_{-1}^{1} dy\langle iy\rangle^{5/2-\alpha}\psi\left(\frac{y}{\langle iy\rangle}\right)\int_{-1}^{1} dz\langle iz\rangle^{5/2-\alpha}\overline{\psi\left(\frac{z}{\langle iz\rangle}\right)}\frac{C}{|y-z|^{\delta}\langle y-z\rangle} \\ &\quad\le C\int_{-1}^{1} dy\langle iy\rangle^{5/2-2\alpha}\left|\psi\left(\frac{y}{\langle iy\rangle}\right)\right|^2 \\ &\quad\le C\int_{-1}^{1} d\xi\langle\xi\rangle^{1/2+2\alpha}|\psi(\xi)|^2 \le C\left\|\langle\cdot\rangle^{1/4+\alpha}\psi\right\|_{\mathbf{L}^2}^2.\end{aligned} \tag{2.62}$$

In the same manner we consider the estimate of the integral with $\phi(\eta)g_{1\eta}(\xi,\eta)$. Hence

$$\left\|\langle\xi\rangle^{\rho}\partial_{\xi}\mathcal{V}^{-1}(t)\phi\right\|_{\mathbf{L}^2} \le C\left\|\langle\xi\rangle^{1/4+\alpha}\partial_{\xi}\phi\right\|_{\mathbf{L}^2} + C\left\|\langle\xi\rangle^{\alpha-3/4}\phi\right\|_{\mathbf{L}^2}, \tag{2.63}$$

where $\alpha > \rho$. Thus we have the second estimate of the lemma. Lemma 2.2 is proved. □

In the next lemma we prove an auxiliary asymptotics for the integral $\int_0^{\infty} e^{-itz^2}\Phi(\xi,z)dz$.

Lemma 2.3. *The estimate holds as follows:*

$$\left|\langle\xi\rangle^{\alpha}\left(\int_0^{\infty} e^{-itz^2}\Phi(\xi,z)dz - \Phi(\xi,0)\sqrt{\frac{\pi}{4it}}\right)\right| \le Ct^{-3/4}\|\langle\xi\rangle^{\alpha}\Phi_z(\xi,z)\|_{\mathbf{L}^{\infty}_{\xi}\mathbf{L}^2_z} \tag{2.64}$$

provided the right-hand side is finite, where $\alpha \in \mathbf{R}$.

Proof. We represent the integral

$$\int_0^\infty e^{-itz^2}\Phi(\xi,z)dz = \Phi(\xi,0)\int_0^\infty e^{-itz^2}dz + R = \Phi(\xi,0)\sqrt{\frac{\pi}{4it}} + R, \tag{2.65}$$

where the remainder term is

$$R = \int_0^\infty e^{-itz^2}(\Phi(\xi,z) - \Phi(\xi,0))dz. \tag{2.66}$$

In the remainder term we integrate by parts via identity $e^{-itz^2} = (1/(1-2itz^2))\partial_z(ze^{-itz^2})$

$$\begin{aligned}\left\|\langle\xi\rangle^\alpha R\right\|_{\mathbf{L}^\infty_\xi} &\le C\left\|\langle\xi\rangle^\alpha\int_0^\infty \frac{1}{1+tz^2}(\Phi(\xi,z)-\Phi(\xi,0))dz\right\|_{\mathbf{L}^\infty_\xi}\\ &\quad + C\left\|\langle\xi\rangle^\alpha\int_0^\infty \frac{z}{1+tz^2}\Phi_z(\xi,z)dz\right\|_{\mathbf{L}^\infty_\xi}\\ &\le C\left\|\langle\xi\rangle^\alpha\Phi_z(\xi,z)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z}\left\|\sqrt{z}\left(1+tz^2\right)^{-1}\right\|_{\mathbf{L}^1_z}\\ &\quad + C\left\|\langle\xi\rangle^\alpha\Phi_z(\xi,z)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z}\left\|z\left(1+tz^2\right)^{-1}\right\|_{\mathbf{L}^2_z}\\ &\le Ct^{-3/4}\left\|\langle\xi\rangle^\alpha\Phi_z(\xi,z)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z},\end{aligned} \tag{2.67}$$

since

$$|\Phi(\xi,z)-\Phi(\xi,0)| \le \int_0^z |\Phi_z(\xi,z)|dz \le C\sqrt{|z|}\|\Phi_z(\xi,z)\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z}. \tag{2.68}$$

Thus we have the estimate for the remainder in the asymptotic formula. Lemma 2.3 is proved. □

In the next lemma we obtain the asymptotics for the operator $\mathcal{V}(t) = \mathcal{B}^{-1}\overline{M}(t)\mathfrak{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$ and the right-inverse operator $\mathcal{V}^{-1}(t) = \overline{E}(t)\mathcal{F}\mathfrak{D}_t M(t)\mathcal{B}$.

Lemma 2.4. *The estimates hold as follows:*

$$\begin{aligned}\left\|\langle\xi\rangle^\beta(\mathcal{V}(t)-1)\phi\right\|_{\mathbf{L}^\infty} &\le Ct^{-1/4}\left(\left\|\langle\xi\rangle^{\beta+3/4}\partial_\xi\phi\right\|_{\mathbf{L}^2} + \left\|\langle\xi\rangle^{\beta-1/4}\phi\right\|_{\mathbf{L}^2}\right),\\ \left\|\langle\xi\rangle^{\beta-3/4}\left(\mathcal{V}^{-1}(t)-1\right)\phi\right\|_{\mathbf{L}^\infty} &\le Ct^{-1/4}\left(\left\|\langle\xi\rangle^{3/2+\beta}\partial_\xi\phi\right\|_{\mathbf{L}^2} + \left\|\langle\xi\rangle^{1/2+\beta}\phi\right\|_{\mathbf{L}^2}\right),\end{aligned} \tag{2.69}$$

for all $t \ge 1$, where $0 \le \beta \le 3/2$, provided the right-hand sides are finite.

Proof. We have the identities

$$S(\xi,\eta) = \langle\eta\rangle - \langle\xi\rangle - \frac{\xi}{\langle\xi\rangle}(\eta-\xi) = \frac{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}{(\langle\eta\rangle+\langle\xi\rangle)^2\langle\xi\rangle}(\eta-\xi)^2,$$
$$S_\eta(\xi,\eta) = \frac{\eta}{\langle\eta\rangle} - \frac{\xi}{\langle\xi\rangle} = \frac{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}{\langle\xi\rangle\langle\eta\rangle(\langle\eta\rangle+\langle\xi\rangle)}(\eta-\xi), \tag{2.70}$$

and $S_{\eta\eta}(\xi,\eta) = \langle\eta\rangle^{-3}$. We now change $z = \sqrt{S(\xi,\eta)}$. We denote The inverse functions by $\eta_j(z)$, so that $\eta_1(z) : (0,\infty) \to (\xi,\infty)$ and $\eta_2(z) : (0,\infty) \to (-\infty,\xi)$.

Thus the stationary point $z = 0$ transforms into $\eta = \xi$. Hence we can write the representation

$$\mathcal{V}(t)\phi = \langle\xi\rangle^{-3/2}\sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}} e^{-itS(\xi,\eta)}\phi(\eta)d\eta$$
$$= 2\langle\xi\rangle^{-3/2}\sqrt{\frac{it}{2\pi}}\sum_{j=1}^{2}\int_0^\infty e^{-itz^2}\phi(\eta_j(z))\frac{\sqrt{S(\xi,\eta_j(z))}}{|S_\eta(\xi,\eta_j(z))|}dz. \tag{2.71}$$

By a direct calculation we find

$$\frac{dz}{d\eta} = \frac{|S_\eta(\xi,\eta)|}{\sqrt{S(\xi,\eta)}} = \frac{\sqrt{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}}{\langle\eta\rangle\langle\xi\rangle^{1/2}}. \tag{2.72}$$

Therefore we obtain

$$\mathcal{V}(t)\phi = \int_0^\infty e^{-itz^2}\Phi(\xi,z)dz, \tag{2.73}$$

where

$$\Phi(\xi,z) = 2\langle\xi\rangle^{-1}\sqrt{\frac{it}{2\pi}}\sum_{j=1}^{2}\left.\frac{\langle\eta\rangle\phi(\eta)}{\sqrt{1+\langle\xi\rangle\langle\eta\rangle-\xi\eta}}\right|_{\eta=\eta_j(z)}. \tag{2.74}$$

Application of Lemma 2.3 yields

$$\mathcal{V}(t)\phi = \phi(\xi) + O\left(t^{-3/4}\|\Phi_z(\xi,z)\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z}\right). \tag{2.75}$$

By a direct calculation we have

$$\Phi_z(\xi,z)\sqrt{\frac{dz}{d\eta}} = \sqrt{\frac{it}{2\pi}}\sum_{j=1}^{2}\frac{2\langle\eta\rangle^{3/2}\phi'(\eta)}{\langle\xi\rangle^{3/4}(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)^{3/4}} + \sqrt{\frac{it}{2\pi}}\sum_{j=1}^{2}\frac{\eta+(\eta\langle\xi\rangle+\xi\langle\eta\rangle)/(\langle\eta\rangle+\langle\xi\rangle)}{\langle\xi\rangle^{3/4}\langle\eta\rangle^{1/2}(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)^{3/4}}\phi(\eta). \tag{2.76}$$

By (2.38) we get the estimate

$$\begin{aligned}\left\|\langle\xi\rangle^{\beta}\Phi_z(\xi,z)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z} &= \left\|\langle\xi\rangle^{\beta}\Phi_z(\xi,z)\sqrt{\frac{dz}{d\eta}}\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_\eta} \\ &\le C\sqrt{t}\left\|\frac{\langle\xi\rangle^{\beta}\langle\eta\rangle^{9/4}}{(\langle\eta\rangle+\langle\xi\rangle)^{3/2}}\phi'(\eta)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_\eta} + C\sqrt{t}\left\|\frac{\langle\xi\rangle^{\beta}\langle\eta\rangle^{5/4}}{(\langle\eta\rangle+\langle\xi\rangle)^{3/2}}\phi(\eta)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_\eta} \\ &\le C\sqrt{t}\left\|\langle\eta\rangle^{3/4+\beta}\phi'\right\|_{\mathbf{L}^2} + C\sqrt{t}\left\|\langle\eta\rangle^{\beta-1/4}\phi\right\|_{\mathbf{L}^2}\end{aligned} \tag{2.77}$$

for $0\le\beta\le 3/2$ since $(\langle\eta\rangle+\langle\xi\rangle)\ge C\langle\xi\rangle^{(2/3)\beta}\langle\eta\rangle^{1-(2/3)\beta}$. This yields the first estimate of the lemma.

We now prove the second estimate. We have

$$\mathcal{V}^{-1}(t)\phi = \frac{\sqrt{t}}{\sqrt{2i\pi}}\int_{-1}^{1}e^{it(\langle\xi\rangle-\xi y-\langle iy\rangle)}\langle iy\rangle^{-3/2}\phi\left(\frac{y}{\langle iy\rangle}\right)dy, \tag{2.78}$$

then changing $y=\eta\langle\eta\rangle^{-1}$, we find

$$\mathcal{V}^{-1}(t)\phi(x) = \frac{\sqrt{t}}{\sqrt{2i\pi}}\int_{\mathrm{R}}e^{itS(\eta,\xi)}\phi(\eta)\langle\eta\rangle^{-3/2}d\eta, \tag{2.79}$$

where

$$S(\eta,\xi) = \langle\xi\rangle - \frac{\xi\eta+1}{\langle\eta\rangle} = \frac{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}{(\langle\eta\rangle+\langle\xi\rangle)^2\langle\eta\rangle}(\eta-\xi)^2. \tag{2.80}$$

As above we change $z=\sqrt{S(\eta,\xi)}$ and represent

$$\begin{aligned}\mathcal{V}^{-1}(t)\phi(x)&=\frac{\sqrt{t}}{\sqrt{2i\pi}}\int_{\mathbf{R}}e^{itS(\eta,\xi)}\phi(\eta)\langle\eta\rangle^{-3/2}d\eta\\&=\frac{\sqrt{2t}}{\sqrt{i\pi}}\sum_{j=1}^{2}\int_0^{\infty}e^{itz^2}\phi(\eta_j(z))\langle\eta_j(z)\rangle^{-3/2}\frac{\sqrt{S(\eta_j(z),\xi)}}{|S_\eta(\eta_j(z),\xi)|}dz.\end{aligned}\tag{2.81}$$

By a direct calculation we find $S_\eta(\eta,\xi)=(\eta-\xi)/\langle\eta\rangle^3$ and

$$\frac{dz}{d\eta}=\frac{|S_\eta(\eta,\xi)|}{2\sqrt{S(\eta,\xi)}}=\frac{\langle\eta\rangle+\langle\xi\rangle}{2\langle\eta\rangle^{5/2}\sqrt{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}}.\tag{2.82}$$

Therefore we obtain

$$\mathcal{V}^{-1}(t)\phi(x)=\int_0^{\infty}e^{itz^2}\Phi(\xi,z)dz,\tag{2.83}$$

where

$$\Phi(\xi,z)=\frac{2\sqrt{2t}}{\sqrt{i\pi}}\sum_{j=1}^{2}\frac{\langle\eta\rangle\sqrt{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}}{\langle\eta\rangle+\langle\xi\rangle}\phi(\eta)\Bigg|_{\eta=\eta_j(z)}.\tag{2.84}$$

Application of Lemma 2.3 yields

$$\mathcal{V}^{-1}(t)\phi(x)=\phi(\xi)+O\left(t^{-3/4}\|\Phi_z(\xi,z)\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z}\right).\tag{2.85}$$

By a direct calculation we have

$$\begin{aligned}\Phi_z(\xi,z)\sqrt{\frac{dz}{d\eta}}&=\frac{2\sqrt{2t}}{\sqrt{i\pi}}\sum_{j=1}^{2}\frac{(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)^{3/4}}{(\langle\eta\rangle+\langle\xi\rangle)^{3/2}}\langle\eta\rangle^{9/4}\phi'(\eta)\\&\quad+\frac{2\sqrt{2t}}{\sqrt{i\pi}}\sum_{j=1}^{2}\frac{\langle\eta\rangle^{5/4}(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi)^{1/4}}{\sqrt{\langle\eta\rangle+\langle\xi\rangle}}\left(\frac{\langle\eta\rangle\sqrt{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}}{\langle\eta\rangle+\langle\xi\rangle}\right)_\eta\phi(\eta),\end{aligned}\tag{2.86}$$

Since

$$\begin{aligned}
&\left|\left(\frac{\langle\eta\rangle\sqrt{1+\langle\eta\rangle\langle\xi\rangle-\eta\xi}}{\langle\eta\rangle+\langle\xi\rangle}\right)_\eta\right|\frac{\langle\eta\rangle^{5/4}\left(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi\right)^{1/4}}{\sqrt{\langle\eta\rangle+\langle\xi\rangle}}\\
&\quad=\left|\frac{\langle\xi\rangle\eta\langle\eta\rangle^{1/4}\left(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi\right)^{3/4}}{\left(\langle\eta\rangle+\langle\xi\rangle\right)^{5/2}}+\frac{\langle\eta\rangle^{5/4}\left(\eta\langle\xi\rangle-\xi\langle\eta\rangle\right)}{2\left(\langle\eta\rangle+\langle\xi\rangle\right)^{3/2}\left(1+\langle\eta\rangle\langle\xi\rangle-\eta\xi\right)^{1/4}}\right| \qquad (2.87)\\
&\quad\le C\frac{\langle\xi\rangle^{3/4}\langle\eta\rangle^2}{\left(\langle\xi\rangle+\langle\eta\rangle\right)^{3/2}},
\end{aligned}$$

therefore we get the estimate

$$\begin{aligned}
\left\|\langle\xi\rangle^{\beta-3/4}\Phi_z(\xi,z)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_z}&=\left\|\langle\xi\rangle^{\beta-3/4}\Phi_z(\xi,z)\sqrt{\frac{dz}{d\eta}}\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_\eta}\\
&\le C\sqrt{t}\left(\left\|\frac{\langle\xi\rangle^\beta\langle\eta\rangle^3}{\left(\langle\eta\rangle+\langle\xi\rangle\right)^{3/2}}\phi'(\eta)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_\eta}+\left\|\frac{\langle\xi\rangle^\beta\langle\eta\rangle^2}{\left(\langle\xi\rangle+\langle\eta\rangle\right)^{3/2}}\phi(\eta)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_\eta}\right)\\
&\le C\sqrt{t}\left(\left\|\langle\eta\rangle^{3/2+\beta}\phi'\right\|_{\mathbf{L}^2}+\left\|\langle\eta\rangle^{1/2+\beta}\phi\right\|_{\mathbf{L}^2}\right)
\end{aligned} \qquad (2.88)$$

for $0\le\beta\le 2/3$, since $(\langle\eta\rangle+\langle\xi\rangle)\ge C\langle\xi\rangle^{2/3\beta}\langle\eta\rangle^{1-2/3\beta}$. This yields the second estimate of the lemma. Lemma 2.4 is proved. □

In the next lemma we find the estimates for the operator $\mathcal{W}(t)=(1-\theta)\overline{M}(t)\mathfrak{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$.

Lemma 2.5. *The estimates hold as follows:*

$$\left\|\mathcal{W}(t)\phi\right\|_{\mathbf{L}^r}\le Ct^{-1/2+(1-\delta)(1-1/r)}\left(\left\|\langle\xi\rangle^{(1+2\delta)(1-1/r)}\partial_\xi\phi\right\|_{\mathbf{L}^1}+\left\|\langle\xi\rangle^{(1+2\delta)(1-1/r)-1}\phi\right\|_{\mathbf{L}^1}\right) \qquad (2.89)$$

for all $t\ge 1$, where $1<r\le\infty$, $\delta\in[0,1]$, provided the right-hand side is finite.

Proof. Note that $\mathcal{W}(t)\phi = 0$ for $|x| \le 1$. To prove the estimate, we integrate by parts via the identity $e^{it(x\xi-\langle\xi\rangle)} = (1/it(x-\xi/\langle\xi\rangle))\ \partial_\xi(e^{it(x\xi-\langle\xi\rangle)})$

$$\begin{aligned}\mathcal{W}(t)\phi &= \frac{\sqrt{it}}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\langle\xi\rangle)}\phi(\xi)d\xi \\ &= -\frac{1}{\sqrt{2\pi it}}\int_{\mathbf{R}} e^{it(x\xi-\langle\xi\rangle)}\frac{1}{x-\xi/\langle\xi\rangle}\phi'(\xi)d\xi \\ &\quad -\frac{1}{\sqrt{2\pi it}}\int_{\mathbf{R}} e^{it(x\xi-\langle\xi\rangle)}\frac{\langle\xi\rangle^{-3}}{(x-\xi/\langle\xi\rangle)^2}\phi(\xi)d\xi\end{aligned} \tag{2.90}$$

for all $|x| \ge 2$. Since $1/|x-\xi/\langle\xi\rangle| \le C\langle x\rangle^{-1}$ for all $|x| \ge 2$, then we get

$$\begin{aligned}|\mathcal{W}(t)\phi| &\le Ct^{-1/2}\langle x\rangle^{-1}\int_{\mathbf{R}}\left(|\phi'(\xi)| + \langle\xi\rangle^{-3}|\phi(\xi)|\right)d\xi \\ &\le Ct^{-1/2}\langle x\rangle^{-1}\left(\|\partial_\xi\phi\|_{\mathbf{L}^1} + C\left\|\langle\xi\rangle^{-3}\phi\right\|_{\mathbf{L}^1}\right).\end{aligned} \tag{2.91}$$

For the case of $1 < |x| \le 2$ we integrate by parts via the identity $e^{it(x\xi-\langle\xi\rangle)} = (1/(\xi/\langle\xi\rangle + it\langle\xi\rangle(x-\xi/\langle\xi\rangle)))\partial_\xi(\langle\xi\rangle e^{it(x\xi-\langle\xi\rangle)})$

$$\begin{aligned}\mathcal{W}(t)\phi &= -\frac{\sqrt{it}}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\langle\xi\rangle)}\frac{\langle\xi\rangle}{\xi/\langle\xi\rangle + it\langle\xi\rangle(x-\xi/\langle\xi\rangle)}\phi'(\xi)d\xi \\ &\quad + \frac{\sqrt{it}}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{it(x\xi-\langle\xi\rangle)}\frac{1/\langle\xi\rangle^2 + it\langle\xi\rangle(\xi x/\langle\xi\rangle - 1)}{(\xi/\langle\xi\rangle + it\langle\xi\rangle(x-\xi/\langle\xi\rangle))^2}\phi(\xi)d\xi.\end{aligned} \tag{2.92}$$

Since $|x-\xi/\langle\xi\rangle| \ge ||x|-1|+|1-|\xi|/\langle\xi\rangle|$ and $|\xi x/\langle\xi\rangle-1| \le |\xi|/\langle\xi\rangle||x|-1|+|1-|\xi|/\langle\xi\rangle| \le |x-\xi/\langle\xi\rangle|$ for $1 < |x|$, therefore

$$\begin{aligned}|\mathcal{W}(t)\phi| &\le C\sqrt{t}\int_{\mathbf{R}}\frac{\langle\xi\rangle|\phi'(\xi)| + |\phi(\xi)|}{|\xi|/\langle\xi\rangle + t\langle\xi\rangle(||x|-1| + |1-|\xi|/\langle\xi\rangle|)}d\xi \\ &\le C\sqrt{t}\int_{\mathbf{R}}\frac{\langle\xi\rangle|\phi'(\xi)| + |\phi(\xi)|}{1 + t\langle\xi\rangle^{-2} + t\langle\xi\rangle||x|-1|}d\xi\end{aligned} \tag{2.93}$$

for $1 < |x| \le 2$. Hence we get

$$\begin{aligned}
\|\mathcal{W}(t)\phi\|_{\mathbf{L}^r} &\le Ct^{-1/2}\left\|\langle x\rangle^{-1}\right\|_{\mathbf{L}^r(|x|\ge 2)}\left(\|\partial_\xi\phi\|_{\mathbf{L}^1}+C\left\|\langle\xi\rangle^{-3}\phi\right\|_{\mathbf{L}^1}\right)\\
&\quad + C\sqrt{t}\int_{\mathbf{R}} d\xi\frac{\langle\xi\rangle|\phi'(\xi)|+|\phi(\xi)|}{1+t\langle\xi\rangle^{-2}}\left\|\frac{1}{1+(t\langle\xi\rangle/((1+t\langle\xi\rangle^{(}-2)\ )\ ))\ ||x|-1|}\right\|_{\mathbf{L}^r(1<|x|\le 2)}\\
&\le Ct^{-1/2}\left(\|\partial_\xi\phi\|_{\mathbf{L}^1}+C\left\|\langle\xi\rangle^{-3}\phi\right\|_{\mathbf{L}^1}\right)\\
&\quad + Ct^{1/2-1/r}\int_{\mathbf{R}}\left(\langle\xi\rangle^{1-1/r}|\phi'(\xi)|+\langle\xi\rangle^{-1/r}|\phi(\xi)|\right)\left(1+t\langle\xi\rangle^{-2}\right)^{1/r-1}d\xi\\
&\le Ct^{-1/2+(1-\delta)(1-1/r)}\left(\left\|\langle\xi\rangle^{(1+2\delta)(1-1/r)}\partial_\xi\phi\right\|_{\mathbf{L}^1}+\left\|\langle\xi\rangle^{(1+2\delta)(1-1/r)-1}\phi\right\|_{\mathbf{L}^1}\right)
\end{aligned} \tag{2.94}$$

for $\delta \in [0,1]$. This yields the estimate of the lemma. Lemma 2.5 is proved. □

We next prove the time decay estimate in terms of the operator $\mathcal{Q}$.

Lemma 2.6. *The estimate is valid*

$$\|u\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1/2}\|u\|_{\mathbf{H}^{3/2}}^{1/2}\left(\|u\|_{\mathbf{H}^{3/2}}^{1/2}+\|\mathcal{Q}u\|_{\mathbf{H}^{1/2}}^{1/2}\right) \tag{2.95}$$

for all $t \ge 0$, provided that the right-hand side is finite.

Proof. Since $\|u\|_{\mathbf{L}^1} \le C\|xu\|_{\mathbf{L}^2}^{1/2}\|u\|_{\mathbf{L}^2}^{1/2}$, by applying the $\mathbf{L}^\infty$-$\mathbf{L}^1$ time decay estimate of the free evolution group $e^{-i\langle i\partial_x\rangle t}$ (see paper of [19, Lemma 1]), we get

$$\begin{aligned}
\|u\|_{\mathbf{L}^\infty} &= \left\|e^{-i\langle i\partial_x\rangle t}e^{i\langle i\partial_x\rangle t}u\right\|_{\mathbf{L}^\infty} \le Ct^{-1/2}\left\|\langle i\partial_x\rangle^{3/2}e^{i\langle i\partial_x\rangle t}u\right\|_{\mathbf{L}^1}\\
&\le Ct^{-1/2}\left\|x\langle i\partial_x\rangle^{3/2}e^{i\langle i\partial_x\rangle t}u\right\|_{\mathbf{L}^2}^{1/2}\left\|\langle i\partial_x\rangle^{3/2}e^{i\langle i\partial_x\rangle t}u\right\|_{\mathbf{L}^2}^{1/2}\\
&\le Ct^{-1/2}\|\mathcal{Q}u\|_{\mathbf{H}^{1/2}}^{1/2}\|u\|_{\mathbf{H}^{3/2}}^{1/2}
\end{aligned} \tag{2.96}$$

for all $t > 0$. Then by the Sobolev inequality we have $\|u\|_{\mathbf{L}^\infty} \le C\|u\|_{\mathbf{H}^1}$. Thus the desired estimate follows. Lemma 2.6 is proved. □

Next we obtain the asymptotics for the integral

$$\int_{\mathbf{R}} e^{itA(\xi,y)}\phi(\xi,y)dy, \tag{2.97}$$

where $A(\xi,y) = \langle\xi\rangle + \langle\zeta\rangle - 2\langle\eta\rangle$, and $\zeta = \xi + 2y, \eta = \xi + y$.

Lemma 2.7. *The following asymptotics is true:*

$$\int_{\mathbf{R}} e^{itA(\xi,y)}\phi(\xi,y)dy = \sqrt{\frac{i\pi}{t}}\langle\xi\rangle^{3/2}\phi(\xi,0) + O\left(t^{-3/4}\left\|\frac{\langle\zeta\rangle^{3/2}\langle\eta\rangle^{3/2}}{(\langle\zeta\rangle+\langle\eta\rangle)^{3/4}}\phi_y(\xi,y)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_y}\right) + O\left(t^{-3/4}\left\|\langle\zeta\rangle^{1/2}\langle\eta\rangle^{1/2}(\langle\zeta\rangle+\langle\eta\rangle)^{1/4}\phi(\xi,y)\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^2_y}\right) \tag{2.98}$$

for $t \geq 1$ uniformly with respect to $\xi \in \mathbf{R}$, provided the right-hand side is finite.

Proof. Denote $\zeta = \xi + 2y$, $\eta = \xi + y$, so that $y = \zeta - \eta = \eta - \xi$. We have the identities

$$\begin{aligned} A(\xi,y) &= 2\frac{1+\langle\eta\rangle\langle\zeta\rangle - \eta\zeta + \langle\xi\rangle\langle\zeta\rangle - \xi\zeta + \langle\xi\rangle\langle\eta\rangle - \xi\eta}{(\langle\xi\rangle+\langle\zeta\rangle)(\langle\xi\rangle+\langle\eta\rangle)(\langle\zeta\rangle+\langle\eta\rangle)}y^2, \\ A_y(\xi,y) &= 2\left(\frac{\zeta}{\langle\zeta\rangle} - \frac{\eta}{\langle\eta\rangle}\right) = 2\frac{1+\langle\eta\rangle\langle\zeta\rangle - \eta\zeta}{\langle\zeta\rangle\langle\eta\rangle(\langle\zeta\rangle+\langle\eta\rangle)}y, \end{aligned} \tag{2.99}$$

and $A_{yy}(\xi,y) = 2(2/\langle\zeta\rangle^3 - 1/\langle\eta\rangle^3)$. We now change $z = \sqrt{A(\xi,y)}$. We denote the inverse functions by $y_j(z)$, so that $y_1(z) : (0,\infty) \to (0,\infty)$ and $y_2(z) : (0,\infty) \to (-\infty,0)$. Thus the stationary point $z = 0$ transforms into $y = 0$. Hence we can write the representation

$$\int_{\mathbf{R}} e^{itA(\xi,y)}\phi(\xi,y)dy = 2\sum_{j=1}^{2}\int_0^\infty e^{-itz^2}\phi(\xi,y_j(z))\frac{\sqrt{A(\xi,y_j(z))}}{|A_y(\xi,y_j(z))|}dz. \tag{2.100}$$

By a direct calculation we find

$$\frac{dz}{dy} = \frac{|A_y(\xi,y)|}{2\sqrt{A(\xi,y)}} = \frac{1}{Q(\xi,\zeta,\eta)}, \tag{2.101}$$

where

$$Q(\xi,\zeta,\eta) \equiv \frac{\sqrt{2}\langle\zeta\rangle\langle\eta\rangle\sqrt{\langle\zeta\rangle+\langle\eta\rangle}\sqrt{1+\langle\eta\rangle\langle\zeta\rangle - \eta\zeta + \langle\xi\rangle\langle\zeta\rangle - \xi\zeta + \langle\xi\rangle\langle\eta\rangle - \xi\eta}}{(1+\langle\eta\rangle\langle\zeta\rangle - \eta\zeta)\sqrt{(\langle\xi\rangle+\langle\zeta\rangle)(\langle\xi\rangle+\langle\eta\rangle)}}. \tag{2.102}$$

Therefore we obtain

$$\int_{\mathbf{R}} e^{itA(\xi,y)}\phi(\xi,y)dy = \int_0^\infty e^{-itz^2}\Phi(\xi,z)dz, \tag{2.103}$$

where

$$\Phi(\xi, z) = \sqrt{2}\sum_{j=1}^{2} \phi(\xi, y) Q(\xi, \zeta, \eta)\big|_{y=y_j(z)}. \tag{2.104}$$

Since $Q(\xi, \xi, \xi) = \langle \xi \rangle^{3/2}$, then the application of Lemma 2.3 yields

$$\int_{\mathbf{R}} e^{itA(\xi,y)} \phi(\xi, y) dy = \sqrt{\frac{i\pi}{t}} \langle \xi \rangle^{3/2} \phi(\xi, 0) + O\left(t^{-3/4} \|\Phi_z(\xi, z)\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_z}\right). \tag{2.105}$$

By a direct calculation we have

$$\begin{aligned}\Phi_z(\xi, z)\sqrt{\frac{dz}{dy}} &= \sum_{j=1}^{2} \phi_y(\xi, y) \left(Q(\xi, \zeta, \eta)\right)^{3/2}\Big|_{y=y_j(z)} \\ &+ \sum_{j=1}^{2} \phi(\xi, y) \sqrt{Q(\xi, \zeta, \eta)} \left(2Q_\zeta(\xi, \zeta, \eta) + Q_\eta(\xi, \zeta, \eta)\right)\Big|_{y=y_j(z)}.\end{aligned} \tag{2.106}$$

By (2.38)

$$\begin{aligned} &Q(\xi, \zeta, \eta) \le C\frac{\langle \zeta \rangle \langle \eta \rangle}{\left(\langle \zeta \rangle + \langle \eta \rangle\right)^{1/2}}, \\ &\left|Q_\zeta(\xi, \zeta, \eta)\right| \le C\langle \zeta \rangle^{-1} Q(\xi, \zeta, \eta)\left(1 + \frac{|\eta - \xi|}{\langle \eta \rangle + \langle \xi \rangle} + \frac{|\zeta - \xi|}{\langle \zeta \rangle + \langle \xi \rangle}\right) \le \frac{C\langle \eta \rangle}{\left(\langle \zeta \rangle + \langle \eta \rangle\right)^{1/2}}, \\ &\left|Q_\eta(\xi, \zeta, \eta)\right| \le \frac{C\langle \zeta \rangle}{\left(\langle \zeta \rangle + \langle \eta \rangle\right)^{1/2}}. \end{aligned} \tag{2.107}$$

Therefore we get the estimate

$$\begin{aligned}\|\Phi_z(\xi, z)\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_z} = \left\|\Phi_z(\xi, z)\sqrt{\frac{dz}{dy}}\right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_y} &\le C\left\|\frac{\langle \zeta \rangle^{3/2} \langle \eta \rangle^{3/2}}{\left(\langle \zeta \rangle + \langle \eta \rangle\right)^{3/4}} \phi_y(\xi, y)\right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_y} \\ &+ C\left\|\langle \zeta \rangle^{1/2} \langle \eta \rangle^{1/2} \left(\langle \zeta \rangle + \langle \eta \rangle\right)^{1/4} \phi(\xi, y)\right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_y}.\end{aligned} \tag{2.108}$$

This yields the estimate of the lemma. Lemma 2.7 is proved. □

In the next lemma we obtain the asymptotics for the nonlinear term $\mathcal{F}\mathcal{U}(-t)\mathcal{G}(\overline{u}, \langle i\partial_x \rangle^{-1} u^2)$ in (1.14).

Lemma 2.8. *Let $v \in \mathbf{H}^{1,3}$. Then the asymptotic formula holds as follows:*

$$\left|\langle\xi\rangle\left(\mathcal{F}\mathcal{U}(-t)\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right)-\frac{1}{t}\Omega(\xi)|v(t,\xi)|^2v(t,\xi)\right)\right|\leq Ct^{-5/4}\|v\|^3_{\mathbf{H}^{1,3}} \tag{2.109}$$

for $t\geq 1$ uniformly with respect to $\xi\in\mathbf{R}$, where $v(t,\xi)=e^{it\langle\xi\rangle}\widehat{u}(t,\xi)$, $\Omega(\xi)\equiv\langle\xi\rangle^2/\langle 2\xi\rangle(2\langle\xi\rangle+\langle 2\xi\rangle)$.

Proof. We first find the representation for $\mathcal{F}\mathcal{U}(-t)(\langle i\partial_x\rangle^{-1}u^2(t))$. We introduce the operator $\mathcal{Q}(t)=\mathcal{B}\mathcal{V}(t)+\mathcal{W}(t)=\overline{M}\mathfrak{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$, so that the representation for the free evolution group is true $\mathcal{U}(t)\mathcal{F}^{-1}=e^{-it\langle i\partial_x\rangle}\mathcal{F}^{-1}=\mathcal{F}^{-1}e^{-it\langle\xi\rangle}=\mathfrak{D}_tM\mathcal{Q}(t)$. We also have $\mathcal{F}e^{it\langle i\partial_x\rangle}=\mathcal{Q}^{-1}(t)\overline{M}\mathfrak{D}_t^{-1}$, where the right-inverse operator $\mathcal{Q}^{-1}(t)=\mathcal{V}^{-1}(t)\mathcal{B}^{-1}+\mathcal{W}^{-1}(t)=e^{it\langle\xi\rangle}\mathcal{F}\mathfrak{D}_tM$.

Since $\mathfrak{D}_{\omega t}=\sqrt{i}\mathfrak{D}_\omega\mathfrak{D}_t$, $\mathfrak{D}_\omega^{-1}=i\mathfrak{D}_{1/\omega}$, and $\mathcal{F}\mathfrak{D}_{1/\omega}=\mathfrak{D}_\omega\mathcal{F}$, we find

$$\begin{aligned}\mathcal{Q}^{-1}(t)M^{\omega-1}&=e^{it\langle\xi\rangle}\mathcal{F}\mathfrak{D}_tM^\omega=\sqrt{i}e^{it\langle\xi\rangle}\mathfrak{D}_\omega\mathcal{F}\mathfrak{D}_{\omega t}M^\omega\\&=\sqrt{i}e^{it\langle\xi\rangle}\mathfrak{D}_\omega e^{-i\omega t\langle\xi\rangle}\mathcal{Q}^{-1}(\omega t)\end{aligned} \tag{2.110}$$

with $\omega>0$. Applying this formula with $\omega=2$ and putting $u(t)=\mathcal{U}(t)\mathcal{F}^{-1}v=\mathfrak{D}_tM\mathcal{Q}(t)v$, we get

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)\left(\langle i\partial_x\rangle^{-1}u^2(t)\right)&=\langle\xi\rangle^{-1}\mathcal{Q}^{-1}(t)\overline{M}\mathfrak{D}_t^{-1}(\mathfrak{D}_tM\mathcal{Q}(t)v)^2\\&=\frac{1}{\langle\xi\rangle\sqrt{it}}\mathcal{Q}^{-1}(t)M(\mathcal{Q}(t)v)^2=\frac{e^{it\langle\xi\rangle}}{\sqrt{t}}\mathfrak{D}_2\frac{e^{-2it\langle\xi\rangle}}{\langle 2\xi\rangle}\mathcal{Q}^{-1}(2t)(\mathcal{Q}(t)v)^2.\end{aligned} \tag{2.111}$$

Since $\mathcal{F}\mathcal{U}(-t)=\mathcal{F}e^{i\langle i\partial_x\rangle t}=e^{it\langle\xi\rangle}\mathcal{F}$, we can write

$$\begin{aligned}\mathcal{F}_{x\to\eta}\left(\langle i\partial_x\rangle^{-1}u^2(t)\right)&=\frac{1}{\sqrt{t}}\mathfrak{D}_2\frac{e^{-2it\langle\eta\rangle}}{\langle 2\eta\rangle}\mathcal{Q}^{-1}(2t)(\mathcal{Q}(t)v)^2\\&=\frac{1}{\sqrt{2it}}e^{-2it\langle\eta/2\rangle}\Phi\left(t,\frac{\eta}{2}\right),\end{aligned} \tag{2.112}$$

where $\Phi(t,\eta)=\langle 2\eta\rangle^{-1}\mathcal{Q}^{-1}(2t)(\mathcal{Q}(t)v)^2$. Hence we obtain

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right)&=e^{it\langle\xi\rangle}\mathcal{F}\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right)\\&=\frac{e^{it\langle\xi\rangle}}{\sqrt{2\pi}\langle\xi\rangle}\int_{\mathbf{R}}\frac{\overline{\widehat{u}(t,\eta-\xi)}\mathcal{F}_{x\to\eta}\left(\langle i\partial_x\rangle^{-1}u^2(t)\right)d\eta}{\langle\xi\rangle+\langle\xi-\eta\rangle+\langle\eta\rangle}.\end{aligned} \tag{2.113}$$

Since $\overline{\widehat{u}(t,\eta-\xi)} = e^{it\langle\eta-\xi\rangle}\overline{v(t,\eta-\xi)}$, we get

$$\mathcal{F}\mathcal{U}(-t)\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right) = \frac{1}{2\sqrt{\pi it}\langle\xi\rangle}\int_{\mathbf{R}}\frac{e^{it\tilde{A}}\overline{v(t,\eta-\xi)}\Phi(t,\eta/2)}{\langle\xi\rangle+\langle\xi-\eta\rangle+\langle\eta\rangle}d\eta \tag{2.114}$$

with $\tilde{A} = \langle\xi\rangle + \langle\eta-\xi\rangle - 2\langle\eta/2\rangle$.

Since $\mathcal{Q}(t) = \mathcal{B}\mathcal{V}(t)$ for $|x| \le 1$ and $\mathcal{Q}(t) = \mathcal{W}(t)$ for $|x| \ge 1$, we then have

$$(\mathcal{Q}(t)v)^2 = (\mathcal{B}\mathcal{V}(t)v)^2 + (\mathcal{W}(t)v)^2 \tag{2.115}$$

for all $x \in \mathbf{R}$. In the same manner applying the identity $\mathcal{B}^{-1}(\mathcal{B}v)^2 = \langle\xi\rangle^{3/2}v^2(\xi)$, we obtain

$$\begin{aligned}\Phi(t,\eta) &= \langle 2\eta\rangle^{-1}\mathcal{V}^{-1}(2t)\mathcal{B}^{-1}(\mathcal{B}\mathcal{V}(t)v)^2 + \langle 2\eta\rangle^{-1}\mathcal{W}^{-1}(2t)(\mathcal{W}(t)v)^2\\ &= \langle 2\eta\rangle^{-1}\mathcal{V}^{-1}(2t)\langle\xi\rangle^{3/2}(\mathcal{V}(t)v)^2 + \langle 2\eta\rangle^{-1}\mathcal{W}^{-1}(2t)(\mathcal{W}(t)v)^2\\ &= \Phi_1(t,\eta) + \Phi_2(t,\eta).\end{aligned} \tag{2.116}$$

After a change $\eta = 2\xi + 2y$, we find

$$\mathcal{F}\mathcal{U}(-t)\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right) = \int_{\mathbf{R}} e^{itA(\xi,y)}\phi_1(\xi,y)dy + \int_{\mathbf{R}} e^{itA(\xi,y)}\phi_2(\xi,y)dy, \tag{2.117}$$

where

$$\begin{aligned}\phi_1(\xi,y) &\equiv \frac{\overline{v(t,\xi+2y)}\Phi_1(t,\xi+y)}{\sqrt{\pi it}\langle\xi\rangle(\langle\xi\rangle+\langle\xi+2y\rangle+\langle 2(\xi+y)\rangle)},\\ \phi_2(\xi,y) &\equiv \frac{\overline{v(t,\xi+2y)}\Phi_2(t,\xi+y)}{\sqrt{\pi it}\langle\xi\rangle(\langle\xi\rangle+\langle\xi+2y\rangle+\langle 2(\xi+y)\rangle)},\end{aligned} \tag{2.118}$$

and $A(\xi,y) = \langle\xi\rangle + \langle\xi+2y\rangle - 2\langle\xi+y\rangle$. Note that

$$\begin{aligned}\left\|\mathcal{W}^{-1}(2t)\phi\right\|_{\mathbf{L}^2} &= \left\|\overline{E}(2t)\mathcal{F}\mathfrak{D}_{2t}(1-\theta)\phi\right\|_{\mathbf{L}^2} = \left\|\mathfrak{D}_{2t}(1-\theta)\phi\right\|_{\mathbf{L}^2}\\ &= \|(1-\theta)\phi\|_{\mathbf{L}^2} \le \|\phi\|_{\mathbf{L}^2}.\end{aligned} \tag{2.119}$$

Hence by Lemma 2.5 with $\delta = 5/6, r = 4$

$$\|\mathcal{W}(t)\phi\|_{\mathbf{L}^4}^2 \le Ct^{-3/4}\left(\left\|\langle\xi\rangle^2\partial_\xi\phi\right\|_{\mathbf{L}^1} + \|\langle\xi\rangle\phi\|_{\mathbf{L}^1}\right) \tag{2.120}$$

for all $t \ge 1$, and

$$\left\|\mathcal{W}^{-1}(2t)(\mathcal{W}(t)v)^2\right\|_{\mathbf{L}^2} \le \|\mathcal{W}(t)v\|_{\mathbf{L}^4}^2 \le Ct^{-3/4}\left(\left\|\langle\xi\rangle^2\partial_\xi v\right\|_{\mathbf{L}^1} + \|\langle\xi\rangle v\|_{\mathbf{L}^1}\right)^2. \tag{2.121}$$

Thus

$$\left\| \langle \xi \rangle \int_{\mathbf{R}} e^{itA(\xi,y)} \phi_2(\xi,y)\,dy \right\|_{\mathbf{L}^\infty} \le Ct^{-1/2} \|v(t)\|_{\mathbf{L}^2} \left\| \langle \xi \rangle^{-2} \mathcal{W}^{-1}(2t)(\mathcal{W}(t)v)^2 \right\|_{\mathbf{L}^2} \le Ct^{-5/4} \|v\|^3_{\mathbf{H}^{1,3}}. \tag{2.122}$$

We apply Lemma 2.7 to find

$$\begin{aligned}
&\left| \langle \xi \rangle \mathcal{F}\mathcal{U}(-t)\mathcal{G}\left(\overline{u}, \langle i\partial_x \rangle^{-1} u^2\right) - t^{-1} \langle \xi \rangle^{-1/2} \Omega(\xi) \overline{v(t,\xi)} \mathcal{V}^{-1}(2t) \langle \xi \rangle^{3/2} (\mathcal{V}(t)v)^2 \right| \\
&\le Ct^{-5/4} \left\| \langle \xi + 2y \rangle^{1/2} \langle \xi + y \rangle^{1/2} \left(\langle \xi + 2y \rangle + \langle \xi + y \rangle \right)^{1/4} \frac{\overline{v(t,\xi+2y)} \Phi_1(t,\xi+y)}{\langle \xi + 2y \rangle + \langle 2(\xi + y) \rangle} \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_y} \\
&\quad + Ct^{-5/4} \left\| \frac{\langle \xi + 2y \rangle^{3/2} \langle \xi + y \rangle^{3/2}}{\left(\langle \xi + 2y \rangle + \langle \xi + y \rangle \right)^{3/4}} \partial_y \frac{\overline{v(t,\xi+2y)} \Phi_1(t,\xi+y)}{\langle \xi \rangle + \langle \xi + 2y \rangle + \langle 2(\xi+y) \rangle} \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^2_y} + Ct^{-5/4} \|v\|^3_{\mathbf{H}^{1,3}} \\
&\le Ct^{-5/4} \big(\|v\|_{\mathbf{L}^\infty} \|\Phi_1\|_{\mathbf{H}^{0,1/4}} + \|\Phi_1\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{H}^{0,1/4}} \\
&\qquad + \|v\|_{\mathbf{H}^{0,1}_\infty} \|\Phi_1\|_{\mathbf{H}^{1,1/4}} + \|v\|_{\mathbf{H}^{1,1}} \|\Phi_1\|_{\mathbf{H}^{0,1/4}_\infty} + \|v\|^3_{\mathbf{H}^{1,3}} \big) \\
&\le Ct^{-5/4} \left(\|v\|_{\mathbf{H}^{1,1}} \|\Phi_1\|_{\mathbf{H}^{1,1/4}} + \|v\|^3_{\mathbf{H}^{1,3}} \right)
\end{aligned} \tag{2.123}$$

since

$$\langle \xi \rangle \phi_1(\xi,y) \equiv \frac{\overline{v(t,\xi+2y)} \Phi_1(t,\xi+y)}{\sqrt{\pi i t} \left(\langle \xi \rangle + \langle \xi + 2y \rangle + \langle 2(\xi+y) \rangle \right)}. \tag{2.124}$$

By Lemma 2.2 we have

$$\begin{aligned}
\|\Phi_1(t)\|_{\mathbf{H}^{1,1/4}} &\le C \left\| \langle \xi \rangle^{-3/4} \partial_\xi \left(\mathcal{V}^{-1}(2t) \langle \xi \rangle^{3/2} (\mathcal{V}(t)v)^2 \right) \right\|_{\mathbf{L}^2} \\
&\le C \left\| \langle \xi \rangle^{3/2+1/4+\gamma} \partial_\xi (\mathcal{V}(t)v)^2 \right\|_{\mathbf{L}^2} + C \left\| \langle \xi \rangle^{3/2-3/4+\gamma} (\mathcal{V}(t)v)^2 \right\|_{\mathbf{L}^2} \le C \|v\|^2_{\mathbf{H}^{1,2}}.
\end{aligned} \tag{2.125}$$

We now write the representation for $\Psi(t,\xi)$ as

$$\begin{aligned}
\Psi(t,\xi) &= \frac{\langle \xi \rangle^{3/2}}{\langle 2\xi \rangle (2\langle \xi \rangle + \langle 2\xi \rangle)} \overline{v(t,\xi)} \mathcal{V}^{-1}(2t) \langle \xi \rangle^{3/2} (\mathcal{V}(t)v)^2 \\
&= \langle \xi \rangle \Omega(\xi) |v|^2 v + R,
\end{aligned} \tag{2.126}$$

where the remainder is

$$
\begin{aligned}
R &= \langle \xi \rangle^{-1/2} \langle 2\xi \rangle \Omega(\xi) \overline{v(t,\xi)} \langle 2\xi \rangle^{-1} \\
&\quad \times \left(\left(\mathcal{V}^{-1}(2t) - 1 \right) \langle \xi \rangle^{3/2} (\mathcal{V}(t) v)^2 + \langle \xi \rangle^{3/2} \left((\mathcal{V}(t) v)^2 - v^2 \right) \right).
\end{aligned} \tag{2.127}
$$

By the second estimate of Lemma 2.4 with $\beta = 5/4$ and the first estimate of Lemma 2.4 with $\beta = 3/4$, we obtain

$$
\begin{aligned}
&\left\| \langle 2\xi \rangle^{-1} \left(\mathcal{V}^{-1}(2t) - 1 \right) \langle \xi \rangle^{3/2} (\mathcal{V}(t) v)^2 \right\|_{\mathbf{L}^\infty} \\
&\quad \le C t^{-1/4} \left(\left\| \langle \xi \rangle^{3/4} \mathcal{V}(t) v \right\|_{\mathbf{L}^\infty} \left\| \langle \xi \rangle^2 \partial_\xi \mathcal{V}(t) v \right\|_{\mathbf{L}^2} + \| \langle \xi \rangle \mathcal{V}(t) v \|_{\mathbf{L}^4}^2 \right) \\
&\quad \le C t^{-1/4} \left\| \langle \xi \rangle^{3/4} (\mathcal{V}(t) - 1) \phi \right\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{H}^{1,2}} + C t^{-1/4} \|v\|_{\mathbf{H}^{1,2}}^2 \le C t^{-1/4} \|v\|_{\mathbf{H}^{1,2}}^2
\end{aligned} \tag{2.128}
$$

for all $t \ge 1$. Also

$$
\begin{aligned}
\left\| \langle \xi \rangle^{1/2} \left((\mathcal{V}(t) v)^2 - v^2 \right) \right\|_{\mathbf{L}^\infty} &\le C \left(\left\| \langle \xi \rangle^{1/4} (\mathcal{V}(t) - 1) v \right\|_{\mathbf{L}^\infty} + \left\| \langle \xi \rangle^{1/4} v \right\|_{\mathbf{L}^\infty} \right) \left\| \langle \xi \rangle^{1/4} (\mathcal{V}(t) - 1) v \right\|_{\mathbf{L}^\infty} \\
&\le C t^{-1/4} \|v\|_{\mathbf{H}^{1,2}}^2 .
\end{aligned} \tag{2.129}
$$

Thus we find the estimate for the remainder $R(t)$ as

$$
\begin{aligned}
\|R(t)\|_{\mathbf{L}^\infty} &\le C \left\| \langle \xi \rangle^{-1/2} \langle 2\xi \rangle \Omega(\xi) v \right\|_{\mathbf{L}^\infty} \left\| \langle 2\xi \rangle^{-1} \left(\mathcal{V}^{-1}(2t) - 1 \right) \langle \xi \rangle^{3/2} (\mathcal{V}(t) v)^2 \right\|_{\mathbf{L}^\infty} \\
&\quad + C \left\| \langle \xi \rangle^{-1/2} \langle 2\xi \rangle \Omega(\xi) v \right\|_{\mathbf{L}^\infty} \left\| \langle \xi \rangle^{1/2} \left((\mathcal{V}(t) v)^2 - v^2 \right) \right\|_{\mathbf{L}^\infty} \le C t^{-1/4} \|v\|_{\mathbf{H}^{1,2}}^3 .
\end{aligned} \tag{2.130}
$$

Hence

$$
\Psi(t,\xi) = \langle \xi \rangle \Omega(\xi) |v|^2 v + O\left(t^{-1/4} \|v\|_{\mathbf{H}^{1,2}}^3 \right). \tag{2.131}
$$

Therefore the asymptotics of the lemma is true. Lemma 2.8 is proved. □

3. Proof of Theorem 1.1

We introduce the function space

$$\mathbf{X}_T = \left\{ \phi \in \mathbf{C}\left([0,T];\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_T} < \infty \right\}, \tag{3.1}$$

where

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[0,T]} \left(\langle t\rangle^{-\gamma} \|\phi(t)\|_{\mathbf{H}^3} + \langle t\rangle^{-\gamma} \|\mathcal{J}\phi(t)\|_{\mathbf{H}^1} + \langle t\rangle^{-3\gamma} \|\mathcal{J}\phi(t)\|_{\mathbf{H}^2} + \langle t\rangle^{1/2} \|\phi(t)\|_{\mathbf{H}^1_\infty} \right), \tag{3.2}$$

and $\gamma > 0$ is small.

The local existence in the function space $\mathbf{X}_T$ can be proved by a standard contraction mapping principle. We state it without a proof.

Theorem 3.1. *Let* $u_0 \in \mathbf{H}^{3,1}$ *and the norm* $\|u_0\|_{\mathbf{H}^{3,1}} = \varepsilon$. *Then there exist* $\varepsilon_0 > 0$ *and* $T > 1$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the initial value problem* (1.1) *has a unique local solution* $u \in \mathbf{C}([0,T];\mathbf{H}^{3,1})$ *with the estimate* $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$.

Let us prove that the existence time T can be extended to infinity which then yields the result of Theorem 1.1. By contradiction, we assume that there exists a minimal time $T > 0$ such that the a priori estimate $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$ does not hold; namely, we have $\|u\|_{\mathbf{X}_T} \le \sqrt{\varepsilon}$.

We apply the energy method to (1.14) $\mathcal{L}(u + i\lambda \mathcal{G}(\overline{u},\overline{u})) = -2i|\lambda|^2 \mathcal{G}(\overline{u}, \langle i\partial_x\rangle^{-1}u^2)$, (i.e., multiplying both sides of the above equation by $\langle i\partial_x\rangle^6 \overline{(u + i\lambda\mathcal{G}(\overline{u},\overline{u}))}$, taking the real part, and integrating over the space) to obtain

$$\begin{aligned} \frac{d}{dt}\|u + i\lambda\mathcal{G}(\overline{u},\overline{u})\|_{\mathbf{H}^3} &\le C\left\|\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^3} \\ &\le C\|u\|^2_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^3} \le C\varepsilon^{3/2}\langle t\rangle^{\gamma-1} \end{aligned} \tag{3.3}$$

since by Lemma 2.1

$$\begin{aligned} \left\|\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^3} &\le C\left\|\mathcal{G}\left(\partial_x^3\overline{u}, u^2\right)\right\|_{\mathbf{L}^2} + C\left\|\mathcal{G}\left(\overline{u}, \partial_x^2 u^2\right)\right\|_{\mathbf{L}^2} + C\left\|\mathcal{G}\left(\partial_x^2\overline{u}, u^2\right)\right\|_{\mathbf{L}^2} \\ &\quad + C\left\|\mathcal{G}\left(\partial_x\overline{u}, u^2\right)\right\|_{\mathbf{L}^2} \le C\|u\|^2_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^3} \end{aligned} \tag{3.4}$$

and $\|u\|_{\mathbf{H}^1_\infty} \le \sqrt{\varepsilon}\langle t\rangle^{-1/2}$, $\|u\|_{\mathbf{H}^3} \le \sqrt{\varepsilon}\langle t\rangle^{\gamma}$ by the estimate $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$. Hence by Theorem 3.1 we have $\|u + i\lambda\mathcal{G}(\overline{u},\overline{u})\|_{\mathbf{H}^3} \le C\varepsilon\langle t\rangle^{\gamma}$ and

$$\begin{aligned} \|u\|_{\mathbf{H}^3} &\le \|u + i\mathcal{G}(\overline{u},\overline{u})\|_{\mathbf{H}^3} + \|\mathcal{G}(\overline{u},\overline{u})\|_{\mathbf{H}^3} \\ &\le C\varepsilon\langle t\rangle^{\gamma} + C\|u\|_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^3} \le C\varepsilon\langle t\rangle^{\gamma}. \end{aligned} \tag{3.5}$$

Next we use the commutator relations $[x, \langle i\partial_x\rangle^\alpha] = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x, [\mathcal{P}\langle i\partial_x\rangle^\alpha] = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x\partial_t, \partial_t u = -i\langle i\partial_x\rangle u + i\langle i\partial_x\rangle^{-1}(\overline{u}^2)$, and $\mathcal{L}\mathcal{P} = (\mathcal{P} - i\langle i\partial_x\rangle^{-1}\partial_x)\mathcal{L}$, to get $\mathcal{L}(u + i\mathcal{G}(\overline{u}, \overline{u})) = -2i|\lambda|^2\mathcal{G}(\overline{u}, \langle i\partial_x\rangle^{-1}u^2)$. Hence

$$\mathcal{L}\mathcal{P}(u + i\lambda\mathcal{G}(\overline{u}, \overline{u})) = -2i|\lambda|^2\mathcal{P}\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right) + 2|\lambda|^2\langle i\partial_x\rangle^{-1}\partial_x\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right). \tag{3.6}$$

Then by the energy method (i.e., multiplying both sides of the above equation by $\langle i\partial_x\rangle^4\overline{\mathcal{P}(u - \mathcal{G}(\overline{u}, \overline{u}))}$, taking the real part, and integrating over the space), by Lemma 2.1, using $\partial_t u = -i\langle i\partial_x\rangle u + i\langle i\partial_x\rangle^{-1}(\overline{u}^2)$, we obtain

$$\begin{aligned}\left\|\mathcal{P}\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^2} &\le C(\|\mathcal{P}u\|_{\mathbf{H}^2} + \|\partial_t u\|_{\mathbf{H}^2})\|u\|^2_{\mathbf{H}^1_\infty} \\ &\le C(\|\mathcal{P}u\|_{\mathbf{H}^2} + \|u\|_{\mathbf{H}^3})\|u\|^2_{\mathbf{H}^1_\infty}.\end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned}\frac{d}{dt}\|\mathcal{P}(u + i\lambda\mathcal{G}(\overline{u}, \overline{u}))\|_{\mathbf{H}^2} &\le C\left\|\mathcal{P}\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^2} + C\left\|\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^2} \\ &\le C(\|\mathcal{P}u\|_{\mathbf{H}^2} + \|u\|_{\mathbf{H}^3})\|u\|^2_{\mathbf{H}^1_\infty} \le C\varepsilon\langle t\rangle^{-1}\|\mathcal{P}u\|_{\mathbf{H}^2} + C\varepsilon^2\langle t\rangle^{\gamma-1}.\end{aligned} \tag{3.8}$$

Therefore by Theorem 3.1 it follows that

$$\|\mathcal{P}u\|_{\mathbf{H}^2} \le C\varepsilon\langle t\rangle^{\gamma}. \tag{3.9}$$

The energy estimate and the identity $\mathcal{L}x = x\mathcal{L} - i\langle i\partial_x\rangle^{-1}\partial_x$ imply that

$$\begin{aligned}\frac{d}{dt}\|x(u + i\lambda\mathcal{G}(\overline{u}, \overline{u}))\|_{\mathbf{H}^2} &\le C\left\|x\mathcal{G}\left(\overline{u}, \langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^2} + \|u + i\mathcal{G}(\overline{u}, \overline{u})\|_{\mathbf{H}^2} \\ &\le C\|u\|^2_{\mathbf{H}^1_\infty}\|xu\|_{\mathbf{H}^2} + C\|u\|^2_{\mathbf{H}^1_\infty}\|u\|_{\mathbf{H}^2} + C\varepsilon\langle t\rangle^{\gamma} \\ &\le C\varepsilon\langle t\rangle^{-1}\|xu\|_{\mathbf{H}^2} + C\varepsilon\langle t\rangle^{\gamma},\end{aligned} \tag{3.10}$$

which yields

$$\|xu\|_{\mathbf{H}^2} \le C\varepsilon\langle t\rangle^{\gamma+1}. \tag{3.11}$$

Then by the identity $\mathcal{J} = i\mathcal{P} - ix\mathcal{L} - \langle i\partial_x\rangle^{-1}\partial_x$ we obtain

$$\begin{aligned}
\|\mathcal{J}(u + i\lambda\mathcal{G}(\overline{u},\overline{u}))\|_{\mathbf{H}^2} &\leq \|\mathcal{P}(u + i\lambda\mathcal{G}(\overline{u},\overline{u}))\|_{\mathbf{H}^2} + \|x\mathcal{L}(u + i\lambda\mathcal{G}(\overline{u},\overline{u}))\|_{\mathbf{H}^2} + \|u + i\lambda\mathcal{G}(\overline{u},\overline{u})\|_{\mathbf{H}^2} \\
&\leq C\varepsilon\langle t\rangle^{\gamma} + C\left\|x\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right)\right\|_{\mathbf{H}^2} \\
&\leq C\varepsilon\langle t\rangle^{\gamma} + C\|u\|_{\mathbf{H}^1_\infty}^2\|xu\|_{\mathbf{H}^2} + C\|u\|_{\mathbf{H}^1_\infty}^2\|u\|_{\mathbf{H}^2} \leq C\varepsilon\langle t\rangle^{\gamma}.
\end{aligned} \tag{3.12}$$

We use Lemma 2.8 to obtain for the new variable $v(t) = e^{it\langle\xi\rangle}\widehat{u}(t,\xi)$

$$\langle\xi\rangle\mathcal{F}\mathcal{U}(-t)\mathcal{G}\left(\overline{u},\langle i\partial_x\rangle^{-1}u^2\right) = \frac{1}{t}\langle\xi\rangle\Omega(\xi)|v(t,\xi)|^2 v(t,\xi) + O\left(t^{-5/4}\|v\|_{\mathbf{H}^{1,3}}^3\right) \tag{3.13}$$

for $t \geq 1$ uniformly with respect to $\xi \in \mathbf{R}$, where $\Omega(\xi) \equiv \langle\xi\rangle^2/\langle 2\xi\rangle(2\langle\xi\rangle + \langle 2\xi\rangle)$. Hence for the new function $\phi = e^{it\langle\xi\rangle}\mathcal{F}(u + i\lambda\mathcal{G}(\overline{u},\overline{u}))$ we get

$$\partial_t\langle\xi\rangle\phi = -2i|\lambda|^2\frac{1}{t}\langle\xi\rangle\Omega(\xi)\left|\phi(t,\xi)\right|^2\phi(t,\xi) + O\left(t^{-5/4}\varepsilon^3\langle t\rangle^{3\gamma}\right) \tag{3.14}$$

in $\mathbf{L}^r$, $2 \leq r \leq \infty$. Then integrating, we obtain $\sup_{t\geq 1}\|\langle\xi\rangle\phi(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon$. Hence

$$\sup_{t\geq 1}\|v(t)\|_{\mathbf{H}^{0,1}_\infty} \leq C\varepsilon. \tag{3.15}$$

By the decomposition of the free evolutions group we have the identity

$$\begin{aligned}
\langle i\partial_x\rangle u(t) &= (\mathfrak{D}_t M(t)\mathcal{B}\mathcal{V}(t) + \mathfrak{D}_t\mathcal{W}(t))\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) \\
&= \mathfrak{D}_t M(t)\mathcal{B}\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) \\
&\quad + \left(\mathfrak{D}_t M(t)\mathcal{B}\langle\xi\rangle^{-3/2}\langle\xi\rangle^{3/2}(\mathcal{V}(t) - 1) + \mathfrak{D}_t\mathcal{W}(t)\right)\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t).
\end{aligned} \tag{3.16}$$

By Lemmas 2.4 and 2.5 we find that the last term of the right-hand side of (3.16) is a remainder. Indeed we have the estimate

$$\begin{aligned}
\|u(t)\|_{\mathbf{H}^1_\infty} &\leq C\left\|\mathfrak{D}_t M(t)\mathcal{B}\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)\right\|_{\mathbf{L}^\infty} + C\varepsilon\langle t\rangle^{-5/4}\left\|\langle\xi\rangle\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)\right\|_{\mathbf{H}^{1,2}} \\
&\leq C\langle t\rangle^{-1/2}\sup_{t\geq 1}\|v(t)\|_{\mathbf{H}^{0,1}_\infty} + C\langle t\rangle^{-5/4}\|\mathcal{J}u(t)\|_{\mathbf{H}^2},
\end{aligned} \tag{3.17}$$

which yields

$$\sup_{t\geq 1}\langle t\rangle^{1/2}\|u(t)\|_{\mathbf{H}^1_\infty} \leq C\varepsilon. \tag{3.18}$$

From (3.15) and (3.18) it follows that

$$\|u\|_{\mathbf{X}_T} \le C\varepsilon < \sqrt{\varepsilon}, \tag{3.19}$$

which implies the desired contradiction. Thus there exists a unique global solution $u \in \mathbf{C}([0,\infty); \mathbf{H}^{3,1})$ to (1.1) with the time decay estimate.

We now prove the asymptotics of solutions. By (3.14) as in the proof of Lemma 2.7 we have

$$\left\|\varphi(t) - \varphi(s)\right\|_{\mathbf{H}^{0,1}_\infty} + \left\|\varphi(t) - \varphi(s)\right\|_{\mathbf{H}^{0,1}} \le C\varepsilon^{3/2}\int_s^t \tau^{\gamma-5/4} d\tau \le C\varepsilon^{3/2} s^{\gamma-1/4} \tag{3.20}$$

with $\gamma \in (0, 1/4)$, where

$$\varphi(t) = ve^{-2i|\lambda|^2\Omega|v|^2\log t} = \mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)e^{-2i|\lambda|^2\Omega|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)|^2\log t}. \tag{3.21}$$

Thus we see that there exists a unique final state $\varphi_+ \in \mathbf{H}^{0,1}_\infty \cap \mathbf{H}^{0,1}$ such that

$$\left\|\varphi(t) - \varphi_+\right\|_{\mathbf{H}^{0,1}_\infty} + \left\|\varphi(t) - \varphi_+\right\|_{\mathbf{H}^{0,1}} \le C\varepsilon^{3/2} t^{\gamma-1/4}. \tag{3.22}$$

We consider the asymptotics of the phase function

$$\Phi(t) = i\Omega\int_1^t \left(|v(\tau)|^2 - |v(t)|^2\right)\frac{d\tau}{\tau} = i\Omega\int_1^t \left(\left|\varphi(\tau)\right|^2 - \left|\varphi(t)\right|^2\right)\frac{d\tau}{\tau}. \tag{3.23}$$

By a direct calculation we have

$$\Phi(t) - \Phi(s) = i\Omega\left(\int_s^t \left(\left|\varphi(\tau)\right|^2 - \left|\varphi(t)\right|^2\right)\frac{d\tau}{\tau} + \left(\left|\varphi(t)\right|^2 - \left|\varphi(s)\right|^2\right)\log s\right), \tag{3.24}$$

where $1 < s < \tau < t$. Hence

$$\begin{aligned} \|\Phi(t) - \Phi(s)\|_{\mathbf{H}^{0,1}_\infty} \le{}& C\int_s^t \left\|\varphi(\tau) - \varphi(t)\right\|_{\mathbf{H}^{0,1}_\infty}\left(\left\|\varphi(\tau)\right\|_{\mathbf{L}^\infty} + \left\|\varphi(t)\right\|_{\mathbf{L}^\infty}\right)\frac{d\tau}{\tau} \\ &+ C\left\|\varphi(s) - \varphi(t)\right\|_{\mathbf{H}^{0,1}_\infty}\left(\left\|\varphi(s)\right\|_{\mathbf{L}^\infty} + \left\|\varphi(t)\right\|_{\mathbf{L}^\infty}\right)\log s \\ \le{}& C\varepsilon^{5/2}\int_s^t \tau^{\gamma-5/4} d\tau + C\varepsilon^{5/2} s^{\gamma-1/4}\log s, \end{aligned} \tag{3.25}$$

from which it follows that there exists a unique real valued function Φ_+ such that

$$\|\Phi(t) - i\Phi_+\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^{3/2} t^{\gamma-1/4}. \tag{3.26}$$

Similarly, we find $\|\Phi(t) - i\Phi_+\|_{\mathbf{H}^{0,1}} \le C\varepsilon^{3/2}t^{\gamma-1/4}$. Therefore we have the asymptotics of the phase function

$$
\begin{aligned}
i\Omega\int_1^t |\psi(\tau)|^2 \frac{d\tau}{\tau} &= i\Omega\int_1^t \left|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(\tau)\right|^2 \frac{d\tau}{\tau} \\
&= i\Phi_+ + i\Omega|\psi_+|^2 \log t + (\Phi(t) - i\Phi_+) + i\Omega\left(\left|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t)\right|^2 - |\psi_+|^2\right)\log t.
\end{aligned}
\tag{3.27}
$$

We also find

$$
\begin{aligned}
&\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \psi_+ e^{2|\lambda|^2(i\Phi_+ + i\Omega|\psi_+|^2\log t)} \\
&\quad= \mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \psi_+ e^{2|\lambda|^2 i\Omega\int_1^t|\psi(\tau)|^2(d\tau/\tau)} \\
&\qquad+ \psi_+\left(e^{2|\lambda|^2 i\Omega\int_1^t|\psi(\tau)|^2(d\tau/\tau)} - e^{2|\lambda|^2 i\Phi_+ + 2|\lambda|^2 i\Omega|\psi_+|^2\log t}\right) \\
&\quad= \psi(t)e^{2|\lambda|^2 i\Omega\int_1^t|\psi(t)|^2(d\tau/\tau)} - \psi_+ e^{2|\lambda|^2 i\Omega\int_1^t|\psi(\tau)|^2(d\tau/\tau)} \\
&\qquad+ \psi_+\left(e^{2|\lambda|^2 i\Omega\int_1^t|\psi(\tau)|^2(d\tau/\tau)} - e^{(2|\lambda|^2 i\Phi_+ + 2|\lambda|^2 i\Omega|\psi_+|^2\log t)}\right).
\end{aligned}
\tag{3.28}
$$

Collecting these estimates, we obtain

$$
\begin{aligned}
&\left\|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \psi_+ e^{2|\lambda|^2 i\Phi_+ + 2|\lambda|^2 i\Omega|\psi_+|^2\log t}\right\|_{\mathbf{H}^{0,1}} \\
&\quad\le C\|\psi(t) - \psi_+\|_{\mathbf{H}^{0,1}} + C\|\psi_+\|_{\mathbf{H}^{0,1}}\|\Phi(t) - i\Phi_+\|_{\mathbf{L}^2} \\
&\qquad+ \|\psi_+\|_{\mathbf{H}^{0,1}}\|\psi(t) - \psi_+\|_{\mathbf{L}^\infty}\left(\|\psi_+\|_{\mathbf{L}^2} + \|\psi(t)\|_{\mathbf{L}^2}\right)\log t \\
&\quad\le C\varepsilon^{3/2}t^{\gamma-1/4},
\end{aligned}
\tag{3.29}
$$

and similarly

$$
\left\|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \psi_+ e^{2|\lambda|^2 i\Phi_+ + 2|\lambda|^2 i\Omega|\psi_+|^2\log t}\right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^{3/2}t^{\gamma-1/4}.
\tag{3.30}
$$

Therefore we have

$$
\begin{aligned}
&\left\|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \widehat{W}_+ e^{i\Omega|\widehat{W}_+|^2\log t}\right\|_{\mathbf{H}^{0,1}} \le C\varepsilon^{3/2}t^{\gamma-1/4}, \\
&\left\|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \widehat{W}_+ e^{i\Omega|\widehat{W}_+|^2\log t}\right\|_{\mathbf{H}^{0,1}_\infty} \le C\varepsilon^{3/2}t^{\gamma-1/4},
\end{aligned}
\tag{3.31}
$$

where $\widehat{W}_+ = \psi_+ \exp(2|\lambda|^2 i\Phi_+)$. Estimate (3.31) means that

$$\left\| u(t) - e^{-i\langle i\partial_x\rangle t}\mathcal{F}^{-1}\widehat{W}_+ e^{2|\lambda|^2 i\Omega|\widehat{W}_+|^2 \log t} \right\|_{\mathbf{H}^{1,0}} \le C\varepsilon^{3/2} t^{\gamma-1/4}. \tag{3.32}$$

Theorem 1.1 is proved.

References

[1] J.-M. Delort, "Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1," *Annales Scientifiques de l'École Normale Supérieure*, vol. 34, no. 1, pp. 1–61, 2001.

[2] S. Klainerman, "Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions," *Communications on Pure and Applied Mathematics*, vol. 38, no. 5, pp. 631–641, 1985.

[3] V. Georgiev and B. Yordanov, "Asymptotic behavior of the one dimensional Klein-Gordon equation with a cubic nonlinearity," preprint.

[4] S. Katayama, "A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension," *Journal of Mathematics of Kyoto University*, vol. 39, no. 2, pp. 203–213, 1999.

[5] K. Moriyama, "Normal forms and global existence of solutions to a class of cubic nonlinear Klein-Gordon equations in one space dimension," *Differential and Integral Equations*, vol. 10, no. 3, pp. 499–520, 1997.

[6] K. Moriyama, S. Tonegawa, and Y. Tsutsumi, "Almost global existence of solutions for the quadratic semilinear Klein-Gordon equation in one space dimension," *Funkcialaj Ekvacioj*, vol. 40, no. 2, pp. 313–333, 1997.

[7] J. Shatah, "Normal forms and quadratic nonlinear Klein-Gordon equations," *Communications on Pure and Applied Mathematics*, vol. 38, no. 5, pp. 685–696, 1985.

[8] H. Sunagawa, "On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension," *Journal of Differential Equations*, vol. 192, no. 2, pp. 308–325, 2003.

[9] H. Lindblad and A. Soffer, "A remark on asymptotic completeness for the critical nonlinear Klein-Gordon equation," *Letters in Mathematical Physics*, vol. 73, no. 3, pp. 249–258, 2005.

[10] H. Lindblad and A. Soffer, "A remark on long range scattering for the nonlinear Klein-Gordon equation," *Journal of Hyperbolic Differential Equations*, vol. 2, no. 1, pp. 77–89, 2005.

[11] N. Hayashi and P. I. Naumkin, "The initial value problem for the cubic nonlinear Klein-Gordon equation," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 59, no. 6, pp. 1002–1028, 2008.

[12] N. Hayashi and P. I. Naumkin, "Final state problem for the cubic nonlinear Klein-Gordon equation," *Journal of Mathematical Physics*, vol. 50, no. 10, Article ID 103511, 14 pages, 2009.

[13] H. Sunagawa, "Remarks on the asymptotic behavior of the cubic nonlinear Klein-Gordon equations in one space dimension," *Differential and Integral Equations*, vol. 18, no. 5, pp. 481–494, 2005.

[14] V. Georgiev and S. Lucente, "Weighted Sobolev spaces applied to nonlinear Klein-Gordon equation," *Comptes Rendus de l'Académie des Sciences*, vol. 329, no. 1, pp. 21–26, 1999.

[15] N. Hayashi and P. I. Naumkin, "Scattering operator for nonlinear Klein-Gordon equations," *Communications in Contemporary Mathematics*, vol. 11, no. 5, pp. 771–781, 2009.

[16] T. Ozawa, K. Tsutaya, and Y. Tsutsumi, "Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions," *Mathematische Zeitschrift*, vol. 222, no. 3, pp. 341–362, 1996.

[17] V. Georgiev, "Decay estimates for the Klein-Gordon equation," *Communications in Partial Differential Equations*, vol. 17, no. 7-8, pp. 1111–1139, 1992.

[18] N. Hayashi and P. I. Naumkin, "Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations," *American Journal of Mathematics*, vol. 120, no. 2, pp. 369–389, 1998.

[19] B. Marshall, W. Strauss, and S. Wainger, "$\mathbf{L}^p - \mathbf{L}^q$ estimates for the Klein-Gordon equation," *Journal de Mathématiques Pures et Appliquées*, vol. 59, no. 4, pp. 417–440, 1980.