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Research Article

Three Solutions for a Discrete Nonlinear Neumann Problem Involving the p-Laplacian

Pasquale Candito[1] and Giuseppina D'Aguì[2]

[1] *DIMET University of Reggio Calabria, Via Graziella (Feo Di Vito), 89100 Reggio Calabria, Italy*
[2] *Department of Mathematics of Messina, DIMET University of Reggio Calabria, 89100 Reggio Calabria, Italy*

Correspondence should be addressed to Giuseppina D'Aguì, dagui@unime.it

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We investigate the existence of at least three solutions for a discrete nonlinear Neumann boundary value problem involving the p-Laplacian. Our approach is based on three critical points theorems.

1. Introduction

In these last years, the study of discrete problems subject to various boundary value conditions has been widely approached by using different abstract methods as fixed point theorems, lower and upper solutions, and Brower degree (see, e.g., [1–3] and the reference given therein). Recently, also the critical point theory has aroused the attention of many authors in the study of these problems [4–12].

The main aim of this paper is to investigate different sets of assumptions which guarantee the existence and multiplicity of solutions for the following nonlinear Neumann boundary value problem

$$\begin{gathered} -\Delta\left(\phi_p(\Delta u_{k-1})\right) + q_k\phi_p(u_k) = \lambda f(k,u_k), \quad k \in [1,N], \\ \Delta u_0 = \Delta u_N = 0, \end{gathered} \tag{P_λ^f}$$

where N is a fixed positive integer, $[1,N]$ is the discrete interval $\{1,\dots,N\}$, $q_k > 0$ for all $k \in [1,N]$, λ is a positive real parameter, $\Delta u_k := u_{k+1} - u_k$, $k = 0,1,\dots,N+1$, is the forward difference operator, $\phi_p(s) := |s|^{p-2}s$, $1 < p < +\infty$, and $f : [1,N] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In particular, for every λ lying in a suitable interval of parameters, at least three solutions are obtained under mutually independent conditions. First, we require that the primitive F of f is p-sublinear at infinity and satisfies appropriate local growth condition (Theorem 3.1). Next, we obtain at least three positive solutions uniformly bounded with respect to λ, under a suitable sign hypothesis on f, an appropriate growth conditions on F in a bounded interval, and without assuming asymptotic condition at infinity on f (Theorem 3.4, Corollary 3.6). Moreover, the existence of at least two nontrivial solutions for problem (P_λ^f) is obtained assuming that F is p-sublinear at zero and p-superlinear at infinity (Theorem 3.5).

It is worth noticing that it is the first time that this type of results are obtained for discrete problem with Neumann boundary conditions; instead of Dirichlet problem, similar results have been already given in [6, 9, 13]. Moreover, in [14], the existence of multiple solutions to problem (P_λ^f) is obtained assuming different hypotheses with respect to our assumptions (see Remark 3.7).

Investigation on the relation between continuous and discrete problems are available in the papers [15, 16]. General references on difference equations and their applications in different fields of research are given in [17, 18]. While for an overview on variational methods, we refer the reader to the comprehensive monograph [19].

2. Critical Point Theorems and Variational Framework

Let X be a real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two functions of class C^1 on X, and let λ be a positive real parameter. In order to study problem (P_λ^f), our main tools are critical points theorems for functional of type $\Phi - \lambda\Psi$ which insure the existence at least three critical points for every λ belonging to well-defined open intervals. These theorems have been obtained, respectively, in [6, 20, 21].

Theorem 2.1 (see [11, Theorem 2.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0. \tag{2.1}$$

Assume that there exist $r > 0$ and $v \in X$, with $r < \Phi(v)$ such that

(a_1) $\sup_{\Phi(u)\le r}\Psi(u)/r < \Psi(v)/\Phi(v)$,

(a_2) *for each $\lambda \in \Lambda_r :=]\Phi(v)/\Psi(v), r/\sup_{\Phi(u)\le r}\Psi(u)[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 (see [7, Corollary 3.1]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0. \tag{2.2}$$

Assume that there exist two positive constants r_1, r_2 *and* $v \in X$, *with* $2r_1 < \Phi(v) < r_2/2$ *such that*

(b_1) $\sup_{\Phi(u)\le r_1} \Psi(u)/r_1 < (2/3)(\Psi(v)/\Phi(v))$,

(b_2) $\sup_{\Phi(u)\le r_2} \Psi(u)/r_2 < (1/3)(\Psi(v)/\Phi(v))$,

(b_3) *for each* $\lambda \in \Lambda' :=](3/2)(\Phi(v)/\Psi(v)), \min\{r_1/\sup_{\Phi(u)\le r_1}\Psi(v), r_2/2\sup_{\Phi(u)\le r_2}\Psi(u)\}[$ *and for every* $u_1, u_2 \in X$, *which are local minima for the functional* $\Phi - \lambda\Psi$ *such that* $\Psi(u_1) \ge 0$ *and* $\Psi(u_2) \ge 0$, *and one has* $\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \ge 0$.

Then, for each $\lambda \in \Lambda'$, *the functional* $\Phi - \lambda\Psi$ *admits at least three critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

Finally, for all $r > \inf_X \Phi$, we put

$$\varphi(r) = \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{u\in(\Phi^{-1}]-\infty,r[)}\Psi(u)\right) - \Psi(u)}{r - \Phi(u)}, \qquad \lambda^* := \frac{1}{\inf_{\{r>\inf_X\Phi\}}\varphi(r)}, \tag{2.3}$$

where we read $1/0 := +\infty$ if this case occurs.

Theorem 2.3 (see [8, Theorem 2.3]). *Let* X *be a finite dimensional real Banach space. Assume that for each* $\lambda \in]0, \lambda^*[$ *one has*

(e) $\lim_{\|u\|\to\infty} \Phi - \lambda\Psi = -\infty$.

Then, for each $\lambda \in]0, \lambda^*[$, *the functional* $\Phi - \lambda\Psi$ *admits at least three distinct critical points.*

Remark 2.4. It is worth noticing that whenever X is a finite dimensional Banach space, a careful reading of the proofs of Theorems 2.1 and 2.2 shows that regarding to the regularity of the derivative of Φ and Ψ, it is enough to require only that Φ' and Ψ' are two continuous functionals on X^*.

Now, consider the N-dimensional normed space $W = \{u : [0, N+1] \to \mathbb{R} : \Delta u_0 = \Delta u_N = 0\}$ endowed with the norm

$$\|u\| := \left(\sum_{k=1}^{N+1} |\Delta u_{k-1}|^p + \sum_{k=1}^{N} q_k |u_k|^p\right)^{1/p}, \quad \forall u \in W. \tag{2.4}$$

In the sequel, we will use the following inequality:

$$\max_{k\in[0,N+1]} |u_k| \le \frac{\|u\|}{q^{1/p}}, \quad \forall u \in W \text{ where } q := \min_{k\in[1,N]} q_k. \tag{2.5}$$

Moreover, put

$$\Phi(u) := \frac{\|u\|^p}{p}, \qquad \Psi(u) := \sum_{k=1}^{N} F(k, u_k), \quad \forall u \in W, \tag{2.6}$$

where $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t) \in [1,N] \times \mathbb{R}$. It is easy to show that Φ and Ψ are two C^1-functionals on W.

Next lemma describes the variational structure of problem (P_λ^f), for the reader convenience we give a sketch of the proof, see also [14],

Lemma 2.5. $(W, \|\cdot\|)$ *is a Banach space. Let* $u \in W$, *u be a solution of problem* (P_λ^f) *if and only if* u *is a critical point of the functional* $\Phi - \lambda\Psi$.

Proof. Bearing in mind both that a finite dimensional normed space is a Banach space and the following partial sum:

$$-\sum_{k=1}^{N} \Delta(\phi_p(\Delta u_{k-1}))v_k = \sum_{k=1}^{N+1} (\phi_p(\Delta u_{k-1}))\Delta v_{k-1}, \tag{2.7}$$

for every u and $v \in W$, standard variational arguments complete the proof. □

Finally, we point out the following strong maximum principle for problem (P_λ^f).

Lemma 2.6. *Fix* $u \in W$ *such that*

$$-\Delta(\phi_p(\Delta u_{k-1})) + q_k|u_k|^{p-2}u_k \geq 0 \quad \forall k \in [1,N]. \tag{2.8}$$

Then, either $u > 0$ *in* $[1,N]$, *or* $u \equiv 0$.

Proof. Let $j \in [1,N]$ be such that $u_j = \min_{k\in[1,N]} u_k$. An immediate computation gives

$$\Delta u_j \geq 0, \qquad \Delta u_{j-1} \leq 0. \tag{2.9}$$

From this, by (2.8), we obtain

$$q_j|u_j|^{p-2}u_j \geq |\Delta u_j|^{p-2}\Delta u_j - |\Delta u_{j-1}|^{p-2}\Delta u_{j-1} \geq 0, \tag{2.10}$$

so $u_j \geq 0$, that is $u \geq 0$. Moreover, assuming that $u_j = 0$, from the preciding inequality and nonnegativity of u_{j-1}, u_{j+1}, one has

$$0 \leq |\Delta u_j|^{p-2}(u_{j+1}) + |\Delta u_{j-1}|^{p-2}(u_{j-1}) \leq 0, \tag{2.11}$$

so $u_{j-1} = u_{j+1} = 0$. Thus, repeating these arguments, the conclusion follows at once. □

3. Main Results

For each positive constants c and d, we write

$$A(c) := \frac{\sum_{k=1}^{N} \max_{|t|\leq c} F(k,t)}{c^p}, \qquad B(d) := \frac{\sum_{k=1}^{N} F(k,d)}{d^p}, \qquad Q := \sum_{k=1}^{N} q_k. \tag{3.1}$$

Now, we give the main results.

Theorem 3.1. *Assume that there exist three positive constants c, d, and s with $c < d$, and $s < p$ such that*

(i_1) $A(c) < (q/Q)B(d)$,

(i_2) $\max_{k\in[1,N]}\limsup_{|t|\to+\infty}(F(k,t)/|t|^s) < +\infty$.

Then, for every

$$\lambda \in \left]\frac{Q}{p}\frac{1}{B(d)}, \frac{q}{p}\frac{1}{A(c)}\right[, \tag{3.2}$$

problem (P_λ^f) admits at least three solutions.

Proof. We apply Theorem 2.1, by putting Φ and Ψ defined as in (2.6) on the space W. An easy computation ensures the regularity assumptions required on Φ and Ψ; see Remark 2.4. Therefore, it remains to verify assumptions (a_1) and (a_2). To this hand, we put

$$r = \frac{q}{p}c^p, \tag{3.3}$$

and we pick $v \in W$, defined by putting

$$v_k = d \quad \text{for every } k \in [1, N]. \tag{3.4}$$

Clearly, since $c < d$, one has $r < \Phi(v) = (Q/p)d^p$, and in addition, by (2.5), we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} \le \frac{\sup_{\|u\|_\infty\le c}\Psi}{(q/p)c^p} \le \frac{p}{q}A(c). \tag{3.5}$$

On the other hand, we compute

$$\frac{\Psi(v)}{\Phi(v)} = \frac{p}{Q}B(d). \tag{3.6}$$

Therefore, by (i_1), combining (3.5) and (3.6), it is clear that (a_1) holds. Moreover, one has

$$\left]\frac{Q}{p}\frac{1}{B(d)}, \frac{q}{p}\frac{1}{A(c)}\right[\subset \Lambda_r. \tag{3.7}$$

Now, fix λ as in the conclusion; first, we observe that for every $1 \le s \le p$, one has

$$\sum_{k=1}^{N}|u_k|^s \le Nq^{(-s/p)}\|u\|^s, \quad \forall u \in W. \tag{3.8}$$

Next, by (i_2), there exist two positive constants M_1 and M_2 such that

$$F(k,\xi) \leq M_1|\xi|^s + M_2, \quad \forall (k,\xi) \in [1,N] \times \mathbb{R}. \tag{3.9}$$

Hence, for every $u \in W$, we get

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &\geq \frac{\|u\|^p}{p} - \lambda M_1 \sum_1^N |u_k|^s - \lambda N M_2 \\ &\geq \frac{\|u\|^p}{p} - \lambda M_1 \frac{N}{q^{s/p}} \|u\|^s - \lambda N M_2. \end{aligned} \tag{3.10}$$

At this point, since $s < p$, it is clear that the functional $\Phi - \lambda\Psi$ turns out to be coercive. □

Remark 3.2. We note that hypothesis (i_2) can be replaced with the following:

(i_2') $\max_{k\in[1,N]} \limsup_{|t|\to+\infty} F(k,t)/|t|^p < A(c)/N$.

Arguing as before, there exist two constant $L_1 < A(c)/N$ and L_2 such that

$$F(k,\xi) \leq L_1|\xi|^p + L_2, \quad \forall (k,\xi) \in [1,N] \times \mathbb{R}. \tag{3.11}$$

Hence, for every $u \in W$, it easy to see that

$$\Phi(u) - \lambda\Psi(u) \geq \frac{\|u\|^p}{p} - \frac{q}{p}\frac{1}{A(c)} L_1 \frac{N}{q} \|u\|^p - \lambda N L_2 \geq \frac{1}{p}\left(1 - \frac{NL_1}{A(c)}\right)\|u\|^p - \lambda N L_2, \tag{3.12}$$

with $(1 - NL_1/A(c)) > 0$.

Remark 3.3. It is worth noticing that a careful reading of the proof of Theorem 3.1 shows that, provided that $A(c) = 0$ and under the only condition (i_2), problem (P_λ^f) admits at least one solution for every $\lambda > 0$ and at least three solutions for every $\lambda \in](Q/p)(1/B(d)), +\infty[$, whenever there exists $d > 0$ for which $B(d) > 0$.

Theorem 3.4. *Let f be a continuous function in $[1,N] \times [0,+\infty[$ such that $f(k,0) \neq 0$ for some $k \in [1,N]$. Assume that there exist three positive constants c_1, d, and c_2 with $(2q/Q)^{1/p} c_1 < d < ((1/2)(q/Q))^{1/p} c_2$ such that*

(j_1) $f(k,\xi) \geq 0$ for each $(k,\xi) \in [1,N] \times [0,c_2]$,

(j_2) $\max\{B(c_1), 2B(c_2)\} < (2/3)(q/Q)B(d)$.

Then, for each $\lambda \in](3/2)(Q/p)(1/B(d)), (q/p)\min\{1/B(c_1), 1/2B(c_2)\}[$, problem (P_λ^f) admits at least three positive solutions u^i, $i = 1,2,3$, such that

$$u_k^i < c_2, \tag{3.13}$$

for all $k \in [1,N]$, $i = 1,2,3$.

Proof. Consider the auxiliary problem

$$\begin{gathered} -\Delta\left(\phi_p(\Delta u_{k-1})\right) + q_k\phi_p(u_k) = \lambda \widehat{f}(k,u_k), \quad k \in [1,N], \\ \Delta u_0 = \Delta u_N = 0, \end{gathered} \tag{$P_\lambda^{\widehat{f}}$}$$

where $\widehat{f} : [1,N] \times \mathbb{R} \to \mathbb{R}$ is a continuous function defined putting

$$\widehat{f}(k,\xi) = \begin{cases} f(k,0), & \text{if } \xi < 0, \\ f(k,\xi), & \text{if } 0 \le \xi \le c_2, \\ f(k,c_2), & \text{if } \xi > c_2. \end{cases} \tag{3.14}$$

From (j_1), owing to Lemma 2.6, any solution of problem (P_λ^f) is positive. In addition, if it satisfies also the condition $0 \le u_k \le c_2$, and for every $k \in [1,N]$, clearly it turns to be also a positive solution of (P_λ^f). Therefore, for our goal, it is enough to show that our conclusion holds for (P_λ^f). In this connection, our aim is to apply Theorem 2.2. Fix λ in $](3/2)(Q/p)(1/B(d)),(q/p)\min\{1/B(c_1),1/2B(c_2)\}[$ and let Φ, Ψ and W as before. Now, take

$$r_1 = \frac{q}{p}c_1^p, \qquad r_2 = \frac{q}{p}c_2^p. \tag{3.15}$$

From (2.5), arguing as before, we obtain

$$\max_{k\in[1,T]} |u_k| \le c_1, \tag{3.16}$$

for all $u \in W$ such that $\|u\| \le (pr_1)^{1/p}$, and

$$\max_{k\in[1,T]} |u_k| \le c_2, \tag{3.17}$$

for all $u \in W$ such that $\|u\| \le (pr_2)^{1/p}$.

Therefore, one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} = \frac{\sup_{\|u\|<(pr_1)^{1/p}}\sum_{k=1}^N F(k,u(k))}{r_1} \le \frac{\sum_{k=1}^N F(k,c_1)}{r_1} = \frac{p}{q}B(c_1), \tag{3.18}$$

as well as

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} \le \frac{p}{q}B(c_2). \tag{3.19}$$

On the other hand, pick $v \in W$, defined as in (3.4), bearing in mind (3.6), and from $(2q/Q)^{1/p}c_1 < d < ((1/2)(q/Q))^{1/p}c_2$, we obtain $2r_1 < \Phi(v) < c_2/2$ Moreover, taking into account (3.18), (3.19), from (j_1), assumptions (b_1) and (b_2) follow. Further, again from (3.18), (3.19), and (3.6), one has that

$$\lambda \in \left]\frac{3}{2}\frac{Q}{p}\frac{1}{B(d)}, \frac{q}{p}\min\left\{\frac{1}{B(c_1)}, \frac{1}{2B(c_2)}\right\}\right[\subset \Lambda'. \tag{3.20}$$

Now, let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$. Owing to Lemmas 2.5 and 2.6, they are two positive solutions for (P_λ^f) so $tu_k^1 + (1-t)u_k^2 \geq 0$, for all $k \in [1, N]$ and for all $t \in [0,1]$. Hence, since one has $\Psi(tu^1 + (1-t)u^2) \geq 0$ for all $t \in [0,1]$, (b_3) is verified.

Therefore, the functional $\Phi - \lambda\Psi$ admits at least three critical points u^i, $i = 1,2,3$, which are three positive solutions of (P_λ^f). Finally, from (2.5), for $i = 1,2,3$, one has

$$\max_{k\in[1,N]} \left|u_k^i\right| \leq c_2, \tag{3.21}$$

and the proof is completed. □

Theorem 3.5. *Let $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(k, 0) \neq 0$ for some $k \in [1, N]$. Assume that there exist four constants M_1, M_2, s, and α, with $M_1 > 0$, $s > p$ and $0 \leq \alpha < s$ such that*

(l) $F(k, \xi) \geq M_1|\xi|^s - M_2|\xi|^\alpha$, for all $(k, \xi) \in [1, N] \times \mathbb{R}$.

Then, for each $\lambda \in]0, \lambda^*[$, where

$$\lambda^* := \frac{q}{p}\frac{1}{\sup_{c>0} A(c)}, \tag{3.22}$$

problem (P_λ^f) admits at least three nontrivial solutions.

Proof. Our aim is to apply Theorem 2.3 with Φ and Ψ as above. Fix $\lambda \in]0, \overline{\lambda}[$, and there is $c > 0$ such that $\lambda < (q/p)(1/A(c))$. Setting $r = (q/p)c^p$ and arguing as in the proof of Theorem 3.1, one has

$$\frac{1}{\lambda^*} \leq \varphi(r) \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leq \frac{p}{q}A(c) < \frac{1}{\lambda}, \tag{3.23}$$

that is $\lambda < \lambda^*$. Moreover, denote

$$\overline{q} = \max_{k\in[1,N]} q_k, \tag{3.24}$$

it is a simple matter to show that for each $u \in W$, one has

$$\sum_{k=1}^{N} |u(k)|^{s} \geq \frac{\|u\|^{s}}{\left[(N+1)2^{p}+\overline{q}\right]^{s/p} N^{(s-p)/p}}, \qquad \sum_{k=1}^{N} |u(k)|^{\alpha} \leq N q^{-\alpha/p} \|u\|^{\alpha}. \tag{3.25}$$

Hence, from (l), for each $u \in W$, we get

$$\Phi(u) - \lambda\Psi(u) \leq \frac{\|u\|^{p}}{p} - \frac{\lambda M_1}{\left[(N+1)2^{p}+\overline{q}\right]^{s/p} N^{(s-p)/p}} \|u\|^{s} + \lambda M_2 N q^{-\alpha/p} \|u\|^{\alpha}. \tag{3.26}$$

Therefore, since $s > p$ and $s > \alpha$, condition (e) is verified. Hence, from Theorem 2.3, the functional $\Phi - \lambda\Psi$ admits three critical points, which are three solutions for (P_{λ}^{f}). Since $f(k,0) \neq 0$ for some $k \in [1,N]$, they are nontrivial solutions, and the conclusion is proved. □

Corollary 3.6. *Let $f : [1,N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(k,0) \neq 0$ for some $k \in [1,N]$. Assume that there exist four constants M_1, M_2, c, and α with $M_1 > 0$ and $0 \leq \alpha < p$ such that*

(l_1) $A(c) < qM_1/[(N+1)2^{p}+\overline{q}]$,

(l_2) $F(k,\xi) \geq M_1|\xi|^{p} - M_2|\xi|^{\alpha}$, *for all* $(k,\xi) \in [1,N] \times \mathbb{R}$.

Then, for every

$$\lambda \in \left] \frac{\left[(N+1)2^{p}+\overline{q}\right]}{pM_1}, \frac{q}{p}\frac{1}{A(c)} \right[, \tag{3.27}$$

problem (P_{λ}^{f}) admits at least three solutions.

Proof. Our claim is to prove that condition (e) of Theorem 2.3 holds for every $\lambda \in][(N+1)2^{p}+\overline{q}]/pM_1, (q/p)(1/A(c))[\subset]0,\lambda^{*}[$. Indeed, from (l_1), arguing as in (3.23), one has that $\lambda < \lambda^{*}$. Moreover, by (l_2), from (3.26) with $s = p$, for every $u \in W$, we have

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &\leq \frac{\|u\|^{p}}{p} - \frac{\lambda M_1}{\left[(N+1)2^{p}+\overline{q}\right]} \|u\|^{p} + \lambda M_2 N q^{-\alpha/p} \|u\|^{\alpha} \\ &\leq \left(\frac{1}{p} - \frac{\lambda M_1}{\left[(N+1)2^{p}+\overline{q}\right]} \right) \|u\|^{p} + \lambda M_2 N q^{-\alpha/p} \|u\|^{\alpha}, \end{aligned} \tag{3.28}$$

where $(1/p - \lambda M_1/[(N+1)2^{p}+\overline{q}]) < 0$, which implies condition (e). □

Remark 3.7. In [14], by Mountain Pass Theorem, the authors established the existence of at least one solution for problem (P_{λ}^{f}) requiring the following conditions:

(θ_1) $f(k,t) = \circ(|t|^{p-1})$ for $t \to 0$ uniformly in $k \in [1,N]$,

(θ_2) there exist two positive constants ρ and s with $s > p$ such that

$$0 < sF(k,t) \leq tf(k,t), \tag{3.29}$$

for every $|t| > \rho$ and $(k,\xi) \in [1,N] \times \mathbb{R}$.

Moreover, they remember that the above conditions imply, respectively, the following:

(θ_3) $F(k,t) = \circ(|t|^p)$ for $t \to 0$ uniformly in $k \in [1,N]$,

(θ_4) there exist two positive constants M_1 and M_2 such that

$$F(k,\xi) \geq M_1|\xi|^s - M_2, \quad \forall (k,\xi) \in [1,N] \times \mathbb{R}. \tag{3.30}$$

Next result shows that under more general conditions than (θ_3) and (θ_4), problem (P_1^f) has at least two nontrivial solutions.

Theorem 3.8. *Assume that (l_2) holds and in addition*

(θ_5) $\max_{k\in[1,N]} \limsup_{|t|\to 0}(F(k,t)/|t|^p) < +\infty$.

Then, problem (P_1^f) has at least two nontrivial solutions.

Proof. We claim that the functional $\Phi - \Psi$ admits a local minimum at zero and a local nonzero maximum. To this end, we observe that by (θ_5), there exist $M > 0$ and $\rho > 0$ such that

$$F(k,t) \leq M_1|t|^p, \quad \text{for every } |t| \leq \rho, \ k \in [1,N]. \tag{3.31}$$

Hence, bearing in mind Lemma 2.5 and (3.25), with $s = p$, for every $u \in W$ with $\|u\| \leq \rho\sqrt[p]{q}$, we get

$$\Phi(u) - \Psi(u) \geq \left(\frac{1}{p} - \frac{MN}{q}\right)\frac{\|u\|^p}{p} \geq 0 = \Phi(0) - \Psi(0), \tag{3.32}$$

that is, 0 is a local minimum. Moreover, by (l_2), by now, it is evident that the functional $\Phi - \Psi$ is anticoercive in W. Hence, by the regularity of $\Phi - \Psi$, there exists $\overline{u} \in W$ which is a global maximum for the functional. Therefore, since it is not restrictive to suppose that $\overline{u} \neq 0$ (otherwise, there are infinitely many critical points), our conclusion follows: if $\dim(X) \geq 2$, from Corollary 2.11 of [22] which ensures a third critical point different from 0 and $\overline{u}$ and by standards arguments if $\dim(X) = 1$. □

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