Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 789302, 20 pages
doi:10.1155/2010/789302

Research Article

Asymptotic Constancy in Linear Difference Equations: Limit Formulae and Sharp Conditions

István Győri and László Horváth

Department of Mathematics, University of Pannonia, Egyetem u. 10. Veszprém 8200, Hungary

Correspondence should be addressed to István Győri, gyori@almos.uni-pannon.hu

Received 20 January 2010; Accepted 23 March 2010

Academic Editor: Ağacik Zafer

It is found that every solution of a system of linear delay difference equations has finite limit at infinity, if some conditions are satisfied. These are much weaker than the known sufficient conditions for asymptotic constancy of the solutions. When we impose some positivity assumptions on the coefficient matrices, our conditions are also necessary. The novelty of our results is illustrated by examples.

1. Introduction

Consider the nonautonomous linear delay difference system

$$y(n+1) - y(n) = \sum_{i=1}^{m} A_i(n)\big(y(n-\tau_i(n)) - y(n-\sigma_i(n))\big), \quad n \geq 0, \tag{1.1}$$

where the following are considered.

(A_1) $m \geq 1$ is an integer, and $A_i(n) \in \mathbb{R}^{d\times d}(1 \leq i \leq m, n \geq 0)$.

(A_2) $(\tau_i(n))_{n\geq 0}$ and $(\sigma_i(n))_{n\geq 0}$ are sequences of nonnegative integers $(1 \leq i \leq m)$ such that

$$s := \max_{1\leq i\leq m}\left\{\max\left\{\sup_{n\geq 0}\tau_i(n),\ \sup_{n\geq 0}\sigma_i(n)\right\}\right\} \tag{1.2}$$

is finite.

Without loss of generality we may (and do) assume the following.

(A_3) For each $1 \le i \le m$ and $n \ge 0$,

$$\tau_i(n) \le \sigma_i(n), \qquad 0 \le \tau_1(n) \le \cdots \le \tau_m(n). \tag{1.3}$$

Under these conditions, $s = \max_{1\le i\le m}\{\max_{n\ge 0}\sigma_i(n)\}$.

Whenever the delays are constants, we get the system

$$y(n+1) - y(n) = \sum_{i=1}^{m} A_i(n)\big(y(n-k_i) - y(n-l_i)\big), \quad n \ge 0, \tag{1.4}$$

where we suppose that

(A_4) $k_i < l_i (1 \le i \le m)$ are nonnegative integers and

$$0 \le k_1 \le \cdots \le k_m. \tag{1.5}$$

In this case, $s = \max_{1\le i\le m}\{l_1, \ldots, l_m\}$.

Together with the above equations we assume initial conditions of the form

$$y(n) = \varphi(n) \in \mathbb{R}^d, \quad -s \le n \le 0, \tag{1.6}$$

where $\varphi := (\varphi(-s), \ldots, \varphi(-1), \varphi(0)) \in \mathbb{R}^{(s+1)d}$. Clearly, (1.1) with (1.6) (and similarly (1.4) with (1.6)) has a unique solution which exists for any $n \ge 0$. The solution is denoted by $y(\varphi) := (y(\varphi)(n))_{n\ge -s}$.

Driver et al. [1] have shown that if

$$\sum_{i=1}^{m} (l_i - k_i)\|A_i(n)\| \le q < 1, \quad n \ge 0, \tag{1.7}$$

for some matrix norm $\|\cdot\|$ on $\mathbb{R}^{d\times d}$, then every solution $y(\varphi)$ of (1.4) tends to a finite limit at infinity which will be denoted by

$$y(\varphi)(\infty) := \lim_{n\to\infty} y(\varphi)(n). \tag{1.8}$$

In the paper of [2] the same statement has been proved under the condition

$$\sum_{i=1}^{m} \sum_{n=j+k_i}^{j+l_i-1} \|A_i(n)\| \le q < 1, \quad j \ge 0. \tag{1.9}$$

As we will show in Section 4.1 (see Example 4.1), conditions (1.7) and (1.9) are independent if the coefficients are time dependent. In the special case of (1.4) with constant coefficients (each $A_i(n)$ is independent of n)

$$y(n+1) - y(n) = \sum_{i=1}^{m} A_i\big(y(n-k_i) - y(n-l_i)\big), \quad n \geq 0, \tag{1.10}$$

conditions (1.7) and (1.9) coincide and each reduces to

$$\sum_{i=1}^{m} (l_i - k_i)\|A_i\| < 1. \tag{1.11}$$

Moreover, considering (1.10) under the condition (A_4), the existence of the finite limit of each solution (for whatever reason) implies that

$$y(\psi)(\infty) = \left(I - \sum_{i=1}^{m}(l_i - k_i)A_i\right)^{-1}\left(\psi(0) - \sum_{i=1}^{m}\left(A_i \sum_{j=-l_i}^{-k_i-1}\psi(j)\right)\right). \tag{1.12}$$

(See [1].)

In the nonautonomous case with constant delays, it has been proved by Pituk [2] that the value of the limit can be characterized in an implicit formula by using a special solution of the adjoint equation to (1.4) and the initial values.

In this paper we prove similar results for the general delay difference system

$$y(n+1) - y(n) = \sum_{j=n-s}^{n-1} K(n,j)\big(y(j+1) - y(j)\big), \quad n \geq 0, \tag{1.13}$$

where

(A_5) $s \geq 1$ is an integer, and $K(n,j) \in \mathbb{R}^{d\times d}$ $(n \geq 0,\ n-s \leq j \leq n-1)$.

The main novelty of our paper is that we prove the existence of the limit of the solutions of the above equations under much weaker conditions than (1.9). Moreover, utilizing our new limit formula, we show that some of our sufficient conditions are also necessary.

After recalling some preliminary facts on matrices in the next section, we state our main results on the asymptotic constancy of the solutions of (1.13), and derive a generalization of the limit formula (1.12) to the time-dependent case (Section 3). Section 4 is divided into three parts. In Section 4.1 we illustrate the independence of conditions (1.7) and (1.9). The relation between our new conditions is studied in Section 4.2. In the third part of Section 4 we specialize to (1.1), (1.4), and (1.10). The proofs of the main results are included in Section 5.

2. Preliminaries

If $d \geq 1$ is an integer, the space of all $d \times d$ matrices is denoted by $\mathbb{R}^{d\times d}$, the zero matrix by O, and the identity matrix by I. $\mathbb{R}^{d\times d}$ is a lattice under the canonical ordering defined by what follows: $A \leq B$ means that $a_{ij} \leq b_{ij}$ for every $1 \leq i, j \leq d$, where $A = (a_{ij})$ and $B = (b_{ij})$. Of course, the absolute value of $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ is given by $|A| = (|a_{ij}|)$. The spectral radius of a matrix $A \in \mathbb{R}^{d\times d}$ is denoted by $\rho(A)$. It is well known that for any norm $\|\cdot\|$ on $\mathbb{R}^{d\times d}$ we have $\rho(A) = \lim_{n\to\infty}\|A^n\|^{1/n} \leq \|A\|$. We use that $A, B, C \in \mathbb{R}^{d\times d}$, $A \leq B$, and $C \geq O$ imply that $AC \leq BC$.

3. The Main Results

Consider the general delay difference system (1.13) with the initial condition (1.6). This initial value problem has a unique solution which is denoted by $y(\psi) := (y(\psi)(n))_{n\geq 0}$.

In our first theorem we give a new limit formula in terms of the initial values. To this end, we introduce the linear mapping $c : \mathbb{R}^{(s+1)d} \to \mathbb{R}^d$ which is defined by

$$c(\psi) := \psi(0) + \sum_{j=-s+1}^{0}\left(\sum_{l=-s}^{j-1}(K(j-l-1, j-1) - K(j-l, j))\right)\psi(j) - \sum_{l=-s}^{-1}K(0, l)\psi(l) \tag{3.1}$$

for any $\psi := (\psi(-s), \dots, \psi(-1), \psi(0)) \in \mathbb{R}^{(s+1)d}$.

Theorem 3.1. *Assume* (A_5). *For an initial sequence* $\psi \in \mathbb{R}^{(s+1)d}$, *the solution* $y(\psi)$ *of* (1.13) *and* (1.6) *has a finite limit if and only if*

$$\begin{aligned} d(\psi) := \lim_{n\to\infty}\Bigg(&\sum_{l=-s}^{-1}K(n, n+l)y(\psi)(n+l+1) \\ &+ \sum_{j=1}^{n-s-1}\left(\sum_{l=-s}^{-1}(K(j-l-1, j-1) - K(j-l, j))\right)y(\psi)(j) \\ &+ \sum_{j=n-s}^{n-1}\left(\sum_{l=j-n}^{-1}(K(j-l-1, j-1) - K(j-l, j))\right)y(\psi)(j)\Bigg) \end{aligned} \tag{3.2}$$

is finite, and in this case

$$y(\psi)(\infty) := \lim_{n\to\infty} y(\psi)(n) = c(\psi) + d(\psi). \tag{3.3}$$

In the next theorem we prove the convergence of the solutions of (1.13) under a condition much weaker than (1.9), as it is illustrated in Section 4.3.

Theorem 3.2. *Assume* (A_5). *If*

either

$$\limsup_{j\to\infty}\sum_{n=j+1}^{j+s}\|K(n,j)\| < 1, \tag{3.4}$$

for some matrix norm $\|\cdot\|$ *on* $\mathbb{R}^{d\times d}$, *or*

$$B := \limsup_{j\to\infty}\sum_{n=j+1}^{j+s}|K(n,j)| \tag{3.5}$$

is finite with $\rho(B) < 1$, *then for every initial sequence* $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.13) *and* (1.6) *has a finite limit which obeys* (3.3).

For the independence of conditions (3.4) and (3.5), see Section 4.1.

As a corollary, we get the next result.

Corollary 3.3. *Assume* (A_5), *and for each* $l \in \{1,\dots,s\}$ *let the limit*

$$L(l) := \lim_{n\to\infty} K(n+l,n) \tag{3.6}$$

be finite. Then the following are considered.

(a) *If for an initial sequence* $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.13) *and* (1.6) *has a finite limit, then*

$$\left(I-\sum_{l=1}^{s}L(l)\right)y(\psi)(\infty) = c(\psi)+\sum_{j=1}^{\infty}\left(\sum_{l=-s}^{-1}(K(j-l-1,j-1)-K(j-l,j))\right)y(\psi)(j). \tag{3.7}$$

(b) *If either*

$$\sum_{l=1}^{s}\|L(l)\| < 1, \tag{3.8}$$

for some matrix norm $\|\cdot\|$ *on* $\mathbb{R}^{d\times d}$, *or*

$$\rho\left(\sum_{l=1}^{s}|L(l)|\right) < 1, \tag{3.9}$$

then for every initial sequence $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.13) *and* (1.6) *has a finite limit which obeys*

$$y(\psi)(\infty) = \left(I - \sum_{l=1}^{s} L(l)\right)^{-1} \left(c(\psi) + \sum_{j=1}^{\infty} \left(\sum_{l=-s}^{-1} (K(j-l-1, j-1) - K(j-l, j)) \right) y(\psi)(j) \right). \tag{3.10}$$

Now consider the equation

$$y(n+1) - y(n) = \sum_{j=n-s}^{n-1} L(n-j)(y(j+1) - y(j)), \quad n \geq 0, \tag{3.11}$$

where $L(l) \in \mathbb{R}^{d \times d}$ for each $l \in \{1, \ldots, s\}$.

Based on the above results we give a necessary and sufficient condition for the solutions of (3.11) to have a finite limit.

Theorem 3.4. *Consider* (3.11).

(a) *If for every initial sequence* $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (3.11) *and* (1.6) *has a finite limit, then*

$$y(\psi)(\infty) = \left(I - \sum_{l=1}^{s} L(l)\right)^{-1} \left(\psi(0) - \sum_{l=-s}^{-1} L(-l)\psi(l)\right). \tag{3.12}$$

(b) *Assume that* $L(l) \geq O$ *for each* $l \in \{1, \ldots, s\}$. *Then the next two statements are equivalent.*

  (i) *For every initial sequence* $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (3.11) *and* (1.6) *has a finite limit.*

  (ii) *And*

$$\rho\left(\sum_{l=1}^{s} L(l)\right) < 1. \tag{3.13}$$

4. Discussion and Applications

4.1. Comparison of Conditions (1.7) *and* (1.9)

The independence of conditions (1.7) and (1.9) is illustrated by the next example.

Example 4.1. Let $m = 2$, $k_1 = k_2 = 0$, $l_1 = 1$, and $l_2 = 2$. Elementary considerations show the following.

(a) If

$$\|A_1(n)\| = \begin{cases} \dfrac{1}{2}, & \text{if } n \geq 0 \text{ is even,} \\ 0, & \text{if } n \geq 0 \text{ is odd,} \end{cases} \qquad \|A_2(n)\| = \begin{cases} \dfrac{5}{24}, & \text{if } n \geq 0 \text{ is even,} \\ \dfrac{1}{3}, & \text{if } n \geq 0 \text{ is odd,} \end{cases} \tag{4.1}$$

then condition (1.7) is satisfied, but condition (1.9) does not hold.

(b) If

$$\|A_1(n)\| = \begin{cases} \dfrac{1}{2}, & \text{if } n \geq 0 \text{ is even,} \\ \dfrac{1}{4}, & \text{if } n \geq 0 \text{ is odd,} \end{cases} \qquad \|A_2(n)\| = \begin{cases} \dfrac{9}{20}, & \text{if } n \geq 0 \text{ is even,} \\ 0, & \text{if } n \geq 0 \text{ is odd,} \end{cases} \tag{4.2}$$

then condition (1.7) does not hold, but condition (1.9) is satisfied.

4.2. Independence of Conditions (3.4) *and* (3.5)

It is illustrated by the following two examples that condition (3.4) does not generally imply condition (3.5) and conversely.

Example 4.2. Let the matrices $K(n, n-1)$ and $K(n, n-2)(n \geq 0)$ be defined by

$$K(n, n-1) := \begin{pmatrix} \dfrac{4n+5}{5(n+1)} & 0 \\ 0 & \dfrac{1}{n+1} \end{pmatrix}, \qquad K(n, n-2) := \begin{pmatrix} \dfrac{1}{n+1} & 0 \\ 0 & \dfrac{2n+3}{5(n+1)} \end{pmatrix}. \tag{4.3}$$

Since

$$\rho(K(n, n-1)) = \frac{4n+5}{5(n+1)} \geq \frac{4}{5}, \qquad \rho(K(n, n-2)) = \frac{2n+3}{5(n+1)} \geq \frac{2}{5}, \quad n \geq 1, \tag{4.4}$$

yield that

$$\|K(n, n-1)\| \geq \frac{4}{5}, \qquad \|K(n, n-2)\| \geq \frac{2}{5}, \quad n \geq 1, \tag{4.5}$$

for every matrix norm $\|\cdot\|$ on $\mathbb{R}^{2\times 2}$, hence

$$\lim_{j \to \infty} \sum_{n=j+1}^{j+2} \|K(n, j)\| \geq \frac{6}{5} > 1 \tag{4.6}$$

for every matrix norm on $\mathbb{R}^{2\times 2}$.

On the other hand

$$\rho\left(\lim_{j\to\infty}\sum_{n=j+1}^{j+2}\left|K(n,j)\right|\right)=\rho\left(\begin{pmatrix}\frac{4}{5} & 0\\ 0 & \frac{2}{5}\end{pmatrix}\right)=\frac{4}{5}<1. \tag{4.7}$$

We can see that there are situations in which (3.5) is satisfied but (3.4) is not.

Example 4.3. Let the matrices $K(n,n-1)$ and $K(n,n-2)(n\geq 0)$ be defined by

$$K(n,n-1):=\begin{pmatrix}\dfrac{3n}{10n+1} & \dfrac{3n}{10n+1}\\ -\dfrac{3n}{10n+1} & \dfrac{3n}{10n+1}\end{pmatrix},\qquad K(n,n-2):=\begin{pmatrix}\dfrac{3}{10} & \dfrac{3}{10}\\ -\dfrac{3}{10} & \dfrac{3}{10}\end{pmatrix}. \tag{4.8}$$

Observe that

$$\rho\left(\begin{pmatrix}\dfrac{3}{10} & \dfrac{3}{10}\\ -\dfrac{3}{10} & \dfrac{3}{10}\end{pmatrix}\right)=\frac{3\cdot 2^{1/2}}{10}<\frac{9}{20}, \tag{4.9}$$

and therefore there exists a matrix norm $\|\cdot\|$ on $\mathbb{R}^{2\times 2}$ such that

$$\left\|\begin{pmatrix}\dfrac{3}{10} & \dfrac{3}{10}\\ -\dfrac{3}{10} & \dfrac{3}{10}\end{pmatrix}\right\|<\frac{9}{20}. \tag{4.10}$$

From

$$\lim_{n\to\infty}K(n,n-1)=\begin{pmatrix}\dfrac{3}{10} & \dfrac{3}{10}\\ -\dfrac{3}{10} & \dfrac{3}{10}\end{pmatrix} \tag{4.11}$$

and from (4.10), it follows that there is an integer $n_0\geq 0$ such that

$$\|K(n,n-1)\|<\frac{9}{20},\quad n\geq n_0. \tag{4.12}$$

This together with (4.10) gives that

$$\lim_{j\to\infty}\sum_{n=j+1}^{j+2}\|K(n,j)\|<\frac{9}{10}<1. \tag{4.13}$$

Finally,

$$\rho\left(\lim_{j\to\infty}\sum_{n=j+1}^{j+2}\left|K(n,j)\right|\right)=\rho\left(\begin{pmatrix}\frac{3}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{3}{5}\end{pmatrix}\right)=\frac{6}{5}>1. \tag{4.14}$$

We can see that (3.4) does not imply (3.5) in general.

Suppose that $K(n,j)\geq O\ (n\geq 0,\ n-s\leq j\leq n-1)$ and the limit

$$\lim_{n\to\infty}K(n+l,n) \tag{4.15}$$

is finite for each $l\in\{1,\dots,s\}$. In this case condition (3.4) guarantees that condition (3.5) also holds. Really,

$$\begin{aligned}1&>\lim_{j\to\infty}\sum_{n=j+1}^{j+s}\|K(n,j)\|=\sum_{l=1}^{s}\lim_{j\to\infty}\|K(j+l,j)\|=\sum_{l=1}^{s}\left\|\lim_{j\to\infty}K(j+l,j)\right\|\\&\geq\left\|\sum_{l=1}^{s}\lim_{j\to\infty}K(j+l,j)\right\|=\left\|\lim_{j\to\infty}\sum_{n=j+1}^{j+s}K(n,j)\right\|\geq\rho\left(\lim_{j\to\infty}\sum_{n=j+1}^{j+s}K(n,j)\right).\end{aligned} \tag{4.16}$$

However, the implication discussed above may be lost if (4.15) is not satisfied, even if the matrices $K(n,j)$ are nonnegative, as the following example shows.

Example 4.4. Let the matrix $K(n,n-1)\ (n\geq 0)$ be defined by

$$\begin{aligned}K(n,n-1)&:=\begin{pmatrix}\frac{2}{3} & 0\\ 0 & \frac{2}{3}\end{pmatrix},\quad\text{if } n \text{ is even},\\K(n,n-1)&:=\begin{pmatrix}0 & \frac{2}{3}\\ \frac{2}{3} & 0\end{pmatrix},\quad\text{if } n \text{ is odd}.\end{aligned} \tag{4.17}$$

Using the l^1-norm $\|\cdot\|_1$ on $\mathbb{R}^{2\times 2}$, we have

$$\limsup_{j\to\infty}\|K(j+1,j)\|_1=\frac{2}{3}<1, \tag{4.18}$$

while

$$\rho\left(\limsup_{j\to\infty} K(j+1,j)\right) = \rho\left(\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}\right) = \frac{4}{3} > 1. \tag{4.19}$$

4.3. Application to Delay Difference Equations

For every $1 \le i \le m$ and $n \ge 0$ let the function $\chi_i(n,\cdot)$ be defined on the set of integers by

$$\chi_i(n,j) := \begin{cases} 1, & n-\sigma_i(n) \le j \le n-\tau_i(n)-1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.20}$$

Lemma 4.5. *Assume* (A_1)–(A_3)*. Then the delay difference* (1.1) *is equivalent to* (1.13) *if for every* $n \ge 0$ $K(n,\cdot)$ *is defined by*

$$K(n,j) := \sum_{i=1}^{m} \chi_i(n,j) A_i(n), \quad n-s \le j \le n-1. \tag{4.21}$$

Proof. It is easy to see that

$$y(n-\tau_i(n)) - y(n-\sigma_i(n)) = \sum_{j=n-\sigma_i(n)}^{n-\tau_i(n)-1} (y(j+1)-y(j)), \quad n \ge 0. \tag{4.22}$$

By using (4.20) we get

$$y(n-\tau_i(n)) - y(n-\sigma_i(n)) = \sum_{j=n-s}^{n-1} (\chi_i(n,j)(y(j+1)-y(j))), \quad n \ge 0. \tag{4.23}$$

Thus (1.1) can be written in the form

$$\begin{aligned} y(n+1)-y(n) &= \sum_{i=1}^{m}\left(A_i(n)\sum_{j=n-s}^{n-1}(\chi_i(n,j)(y(j+1)-y(j)))\right) \\ &= \sum_{j=n-s}^{n-1}\left(\sum_{i=1}^{m}\chi_i(n,j)A_i(n)\right)(y(j+1)-y(j)), \quad n \ge 0. \end{aligned} \tag{4.24}$$

The proof is complete. □

The following result is an immediate consequence of Theorem 3.2 and Lemma 4.5, and it gives sufficient conditions for the convergence of the solutions of (1.1).

Theorem 4.6. *Assume* (A_1)–(A_3). *If either*

$$\limsup_{j\to\infty}\sum_{n=j+1}^{j+s}\left\|\sum_{i=1}^{m}\chi_i(n,j)A_i(n)\right\|<1, \tag{4.25}$$

for some matrix norm $\|\cdot\|$ *on* $\mathbb{R}^{d\times d}$, *or*

$$B:=\limsup_{j\to\infty}\sum_{n=j+1}^{j+s}\left|\sum_{i=1}^{m}\chi_i(n,j)A_i(n)\right| \tag{4.26}$$

is finite with $\rho(B)<1$, *then for every initial sequence* $\psi\in\mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.1) *and* (1.6) *has a finite limit which obeys* (3.3).

Now consider the constant delay equation (1.4). For every $1\le i\le m$, let the function χ_i^c be defined on the set of integers by

$$\chi_i^c(l):=\begin{cases}1, & k_i+1\le l\le l_i,\\ 0, & \text{otherwise.}\end{cases} \tag{4.27}$$

In (1.4) $\tau_i(n)=k_i$ and $\sigma_i(n)=l_i$ for every $1\le i\le m$ and $n\ge 0$; thus the function $\chi_i(n,\cdot)$ defined in (4.20) satisfies

$$\chi_i(n,j)=\chi_i^c(n-j),\quad 1\le i\le m,\ n\ge 0, \tag{4.28}$$

for each integer j. So, in the constant delay case, from Theorem 4.6 we get the next result.

Theorem 4.7. *Assume* (A_1) *and* (A_4). *If either*

$$\limsup_{j\to\infty}\sum_{n=j+1}^{j+s}\left\|\sum_{i=1}^{m}\chi_i^c(n-j)A_i(n)\right\|<1, \tag{4.29}$$

for some matrix norm $\|\cdot\|$ *on* $\mathbb{R}^{d\times d}$, *or*

$$B:=\limsup_{j\to\infty}\sum_{n=j+1}^{j+s}\left|\sum_{i=1}^{m}\chi_i^c(n-j)A_i(n)\right| \tag{4.30}$$

is finite with $\rho(B)<1$, *then for every initial sequence* $\psi\in\mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.4) *and* (1.6) *has a finite limit which obeys* (3.3).

Remark 4.8. Our condition (4.29) is weaker than condition (1.9). In fact

$$\begin{aligned}\sum_{n=j+1}^{j+s}\left\|\sum_{i=1}^{m}\chi_i^c(n-j)A_i(n)\right\| &\le \sum_{n=j+1}^{j+s}\left(\sum_{i=1}^{m}\chi_i^c(n-j)\|A_i(n)\|\right)\\ &= \sum_{i=1}^{m}\left(\sum_{n=j+1}^{j+s}\chi_i^c(n-j)\|A_i(n)\|\right)\\ &= \sum_{i=1}^{m}\left(\sum_{n=j+k_i+1}^{j+l_i}\|A_i(n)\|\right), \quad j\ge -s.\end{aligned} \tag{4.31}$$

Therefore

$$\begin{aligned}\limsup_{j\to\infty}\sum_{n=j+1}^{j+s}\left\|\sum_{i=1}^{m}\chi_i^c(n-j)A_i(n)\right\| &\le \limsup_{j\to\infty}\sum_{i=1}^{m}\left(\sum_{n=j+k_i+1}^{j+l_i}\|A_i(n)\|\right)\\ &= \limsup_{j\to\infty}\sum_{i=1}^{m}\left(\sum_{n=j+k_i}^{j+l_i-1}\|A_i(n)\|\right) < 1,\end{aligned} \tag{4.32}$$

assuming that (1.9) holds.

In the next example our condition (4.29) holds, but neither condition (1.9) nor condition (1.7) can be applied.

Example 4.9. Consider

$$y(n+1)-y(n) = \frac{n+1}{4n}\big(y(n)-y(n-2)\big) - \frac{n+1}{3n}\big(y(n-1)-y(n-3)\big), \quad n\ge 0. \tag{4.33}$$

An elementary calculation shows that

$$\sum_{n=j+1}^{j+3}\left|\sum_{i=1}^{2}\chi_i(n,j)A_i(n)\right| = \frac{j+2}{4(j+1)} + \left|\frac{j+3}{4(j+2)} - \frac{j+3}{3(j+2)}\right| + \frac{j+4}{3(j+3)} \longrightarrow \frac{2}{3} < 1, \tag{4.34}$$

while

$$\begin{aligned}\sum_{i=1}^{2}(l_i-k_i)|A_i(n)| &= 2\frac{n+1}{4n} + 2\frac{n+1}{3n} \longrightarrow \frac{7}{6} > 1,\\ \sum_{i=1}^{2}\left(\sum_{n=j+k_i+1}^{j+l_i}|A_i(n)|\right) &= \frac{j+2}{4(j+1)} + \frac{j+3}{4(j+2)} + \frac{j+3}{3(j+2)} + \frac{j+4}{3(j+3)} \longrightarrow \frac{7}{6} > 1.\end{aligned} \tag{4.35}$$

By applying Theorem 4.7 and Theorem 3.4(b), we give sufficient and also necessary conditions for the solutions of (1.4) to be asymptotically constant, if in addition each matrix $A_i(n)$ is constant (independent of n).

Theorem 4.10. *Assume* (A_1) *and* (A_4) *with* $A_i(n) = A_i$ *for each* $1 \leq i \leq m$ *and* $n \geq 0$. *Then the following are considered.*

(a) *If either*

$$\sum_{l=1}^{s} \left\| \sum_{i=1}^{m} \chi_i^c(l) A_i \right\| < 1, \tag{4.36}$$

for some matrix norm $\|\cdot\|$ *on* $\mathbb{R}^{d\times d}$, *or*

$$\rho\left(\sum_{l=1}^{s} \left| \sum_{i=1}^{m} \chi_i^c(l) A_i \right| \right) < 1, \tag{4.37}$$

then for every initial sequence $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.10) *and* (1.6) *has a finite limit.*

(b) *Assume that*

$$\sum_{i=1}^{m} \chi_i^c(l) A_i \geq O \tag{4.38}$$

for each $l \in \{1, \ldots, s\}$. *Then the next two statements are equivalent.*

(i) *For every initial sequence* $\psi \in \mathbb{R}^{(s+1)d}$ *the solution* $y(\psi)$ *of* (1.10) *and* (1.6) *has a finite limit.*

(ii) *And*

$$\rho\left(\sum_{l=1}^{s} \left(\sum_{i=1}^{m} \chi_i^c(l) A_i \right) \right) < 1. \tag{4.39}$$

Remark 4.11. Condition (4.38) does not require the positivity of the coefficient matrices $A_i (i = 1, \ldots, m)$. To illustrate this, see the following example. To the best of our knowledge, no similar result has been proved for (1.10) with both positive and negative coefficients.

Example 4.12. Consider the scalar difference equation

$$y(n+1) - y(n) = A_1\big(y(n) - y(n-1)\big) + A_2\big(y(n) - y(n-2)\big), \quad n \geq 0, \tag{4.40}$$

which is a special case of (1.10). Then

$$\chi_1^c(1)A_1 + \chi_2^c(1)A_2 = A_1 + A_2, \qquad \chi_1^c(2)A_1 + \chi_2^c(2)A_2 = A_2. \tag{4.41}$$

Consequently, the conditions in (4.38) have the form

$$A_1 + A_2 \geq 0, \quad A_2 \geq 0, \tag{4.42}$$

showing clearly that A_1 may be negative.

Remark 4.13. Of course, we have from Theorem 3.4(a) (using that $L(l) = \sum_{i=1}^m \chi_i^c(l)A_i$, $l = 1,\ldots,s$) that if for every initial sequence $\varphi \in \mathbb{R}^{(s+1)d}$ the solution $y(\varphi)$ of (1.10) and (1.6) has a finite limit, then (1.12) holds.

5. Proofs of the Main Results

Proof of Theorem 3.1. Since

$$y(\varphi)(n+1) = \varphi(0) + \sum_{i=0}^{n}\big(y(\varphi)(i+1) - y(\varphi)(i)\big), \quad n \geq 0, \tag{5.1}$$

it comes from (1.13) that

$$\begin{aligned}
y(\varphi)(n+1) &= \varphi(0) + \sum_{i=0}^{n}\left(\sum_{j=i-s}^{i-1} K(i,j)\big(y(\varphi)(j+1) - y(\varphi)(j)\big)\right) \\
&= \varphi(0) + \sum_{i=0}^{n}\left(\sum_{l=-s}^{-1} K(i,i+l)\big(y(\varphi)(i+l+1) - y(\varphi)(i+l)\big)\right) \\
&= \varphi(0) + \sum_{l=-s}^{-1}\left(\sum_{i=0}^{n} K(i,i+l)\big(y(\varphi)(i+l+1) - y(\varphi)(i+l)\big)\right), \quad n \geq 0.
\end{aligned} \tag{5.2}$$

Now a simple calculation confirms that

$$
\begin{aligned}
y(\varphi)(n+1) &= \varphi(0) + \sum_{l=-s}^{-1}\left(\sum_{j=l+1}^{n+l+1} K(j-l-1,j-1)y(\varphi)(j) - \sum_{j=l}^{n+l} K(j-l,j)y(\varphi)(j)\right) \\
&= \varphi(0) + \sum_{l=-s}^{-1} K(n,n+l)y(\varphi)(n+l+1) - \sum_{l=-s}^{-1} K(0,l)y(\varphi)(l) \\
&\quad + \sum_{l=-s}^{-1}\left(\sum_{j=l+1}^{n+l}(K(j-l-1,j-1) - K(j-l,j))y(\varphi)(j)\right) \\
&= \varphi(0) - \sum_{l=-s}^{-1} K(0,l)\varphi(l) + \sum_{l=-s}^{-1} K(n,n+l)y(\varphi)(n+l+1) \\
&\quad + \sum_{j=-s+1}^{0}\left(\sum_{l=-s}^{j-1}(K(j-l-1,j-1) - K(j-l,j))\right)\varphi(j) \\
&\quad + \sum_{j=1}^{n-s-1}\left(\sum_{l=-s}^{-1}(K(j-l-1,j-1) - K(j-l,j))\right)y(\varphi)(j) \\
&\quad + \sum_{j=n-s}^{n-1}\left(\sum_{l=j-n}^{-1}(K(j-l-1,j-1) - K(j-l,j))\right)y(\varphi)(j) \\
&= c(\varphi) + \sum_{l=-s}^{-1} K(n,n+l)y(\varphi)(n+l+1) \\
&\quad + \sum_{j=1}^{n-s-1}\left(\sum_{l=-s}^{-1}(K(j-l-1,j-1) - K(j-l,j))\right)y(\varphi)(j) \\
&\quad + \sum_{j=n-s}^{n-1}\left(\sum_{l=j-n}^{-1}(K(j-l-1,j-1) - K(j-l,j))\right)y(\varphi)(j), \quad n \geq s+2.
\end{aligned}
\tag{5.3}
$$

From (5.3) the assertion and the desired relation (3.3) follow, bringing the proof to an end. □

In order to prove Theorem 3.2, we need the following Lemma from [3, Corollary 10(b)].

Lemma 5.1. *Consider the initial value problem*

$$
\begin{aligned}
x(n) &= \sum_{j=n-s}^{n-1} B(n,j)x(j), \quad n \geq 0, \\
x(n) &= \varphi(n), \quad -s \leq n \leq -1,
\end{aligned}
\tag{5.4}
$$

where $s \geq 1$ is a given integer, $B(n,j) \in \mathbb{R}^{d\times d}$ $(n \geq 0, n-s \leq j \leq n-1)$, and $\varphi(n) \in \mathbb{R}^d$ $(-s \leq n \leq -1)$. The unique solution of (5.4) *is denoted by $x(\varphi)$. Let $\|\cdot\|_d$ be a norm on $\mathbb{R}^d$. If*

$$\limsup_{j\to\infty} \sum_{n=j+1}^{j+s} \|B(n,j)\| < 1 \tag{5.5}$$

is satisfied for some matrix norm $\|\cdot\|$ on $\mathbb{R}^{d\times d}$, then there are numbers $c := c(\|\cdot\|_d, \varphi) \geq 0$ and $0 < q < 1$ such that

$$\|x(\varphi)(n)\|_d \leq cq^n, \quad n \geq 0. \tag{5.6}$$

Proof of Theorem 3.2. Fix an initial value $\psi := (\psi(-s), \ldots, \psi(-1), \psi(0)) \in \mathbb{R}^{(s+1)d}$.

Since

$$y(\psi)(N+1) = \psi(0) + \sum_{n=0}^{N} (y(\psi)(n+1) - y(\psi)(n)), \quad N \geq 0, \tag{5.7}$$

it is enough to show that the series

$$\sum_{n=0}^{\infty} (y(\psi)(n+1) - y(\psi)(n)) \tag{5.8}$$

is convergent.

Suppose (3.4). Let $\|\cdot\|_d$ be a norm on $\mathbb{R}^d$. According to Lemma 5.1,

$$\sum_{n=0}^{\infty} \|y(\psi)(n+1) - y(\psi)(n)\|_d < \infty, \tag{5.9}$$

so the series (5.8) is convergent.

Now suppose (3.5). Obviously, the convergence of the series (5.8) comes from the convergence of the series

$$\sum_{n=0}^{\infty} |y(\psi)(n+1) - y(\psi)(n)|. \tag{5.10}$$

Moreover, the members of the previous series are nonnegative, so it suffices to prove that the sequence

$$\left(\sum_{n=0}^{N} |y(\psi)(n+1) - y(\psi)(n)|\right)_{N\geq 0} \tag{5.11}$$

is bounded.

Let $E = (e_{ij})$ be the matrix in $\mathbb{R}^{d\times d}$, where $e_{ij} := 1$ for each pair (i, j). By the definition of the matrix B, for each positive number ε there exists a nonnegative integer $j(\varepsilon)$ such that

$$\sum_{n=j+1}^{j+s} \left|K(n,j)\right| \leq B + \varepsilon E, \quad j \geq j(\varepsilon). \tag{5.12}$$

The property $\rho(B) < 1$ insures that we can choose a positive number ε_0 such that $\rho(B+\varepsilon_0 E) < 1$. We set

$$C := B + \varepsilon_0 E, \qquad j_0 := j(\varepsilon_0). \tag{5.13}$$

Equation (1.13) implies that

$$\begin{aligned}
\sum_{n=0}^{N}\left|y(\varphi)(n+1) - y(\varphi)(n)\right| &\leq \sum_{n=0}^{N}\left(\sum_{j=n-s}^{n-1}\left|K(n,j)\right|\left|y(\varphi)(j+1) - y(\varphi)(j)\right|\right) \\
&\leq \sum_{j=-s}^{-1}\left(\sum_{n=0}^{j+s}\left|K(n,j)\right|\right)\left|\varphi(j+1) - \varphi(j)\right| \\
&\quad + \sum_{j=0}^{N}\left(\sum_{n=j+1}^{j+s}\left|K(n,j)\right|\right)\left|y(\varphi)(j+1) - y(\varphi)(j)\right| \\
&= \sum_{j=-s}^{-1}\left(\sum_{n=0}^{j+s}\left|K(n,j)\right|\right)\left|\varphi(j+1) - \varphi(j)\right| \\
&\quad + \sum_{j=0}^{j_0}\left(\sum_{n=j+1}^{j+s}\left|K(n,j)\right|\right)\left|y(\varphi)(j+1) - y(\varphi)(j)\right| \\
&\quad + \sum_{j=j_0+1}^{N}\left(\sum_{n=j+1}^{j+s}\left|K(n,j)\right|\right) \\
&\quad \times \left|y(\varphi)(j+1) - y(\varphi)(j)\right|, \quad N \geq j_0 + 1 + s.
\end{aligned} \tag{5.14}$$

Introducing the notation

$$\begin{aligned}
b(\varphi) := &\sum_{j=-s}^{-1}\left(\sum_{n=0}^{j+s}\left|K(n,j)\right|\right)\left|\varphi(j+1) - \varphi(j)\right| \\
&+ \sum_{j=0}^{j_0}\left(\sum_{n=j+1}^{j+s}\left|K(n,j)\right|\right)\left|y(\varphi)(j+1) - y(\varphi)(j)\right|,
\end{aligned} \tag{5.15}$$

and using (5.12) and (5.13), we calculate

$$\begin{aligned}\sum_{n=0}^{N}\left|y(\varphi)(n+1)-y(\varphi)(n)\right| &\le b(\varphi)+C\sum_{j=j_0+1}^{N}\left|y(\varphi)(j+1)-y(\varphi)(j)\right| \\ &\le b(\varphi)+C\sum_{j=0}^{N}\left|y(\varphi)(j+1)-y(\varphi)(j)\right|, \quad N\ge j_0+1+s.\end{aligned} \tag{5.16}$$

Hence

$$(I-C)\sum_{n=0}^{N}\left|y(\varphi)(n+1)-y(\varphi)(n)\right| \le b(\varphi), \quad N\ge j_0+1+s. \tag{5.17}$$

Because the matrix C was chosen to satisfy $\rho(C)<1$ and C is nonnegative, $(I-C)$ is invertible and $(I-C)^{-1}$ is nonnegative too. Therefore, (5.16) yields that

$$\sum_{n=0}^{N}\left|y(\varphi)(n+1)-y(\varphi)(n)\right| \le (I-C)^{-1}b(\varphi), \quad N\ge j_0+1+s, \tag{5.18}$$

and this gives the boundedness of the sequence (5.11).

The proof is complete. □

Proof of Corollary 3.3. (a) By (3.6),

$$\lim_{n\to\infty}\sum_{l=-s}^{-1}K(n,n+l)y(\varphi)(n+l+1)=\left(\sum_{l=-s}^{-1}L(l)\right)y(\varphi)(\infty). \tag{5.19}$$

From (3.6) it also follows that

$$\begin{aligned}&\lim_{n\to\infty}\sum_{j=n-s}^{n-1}\left(\sum_{l=j-n}^{-1}\left(K(j-l-1,j-1)-K(j-l,j)\right)\right)y(\varphi)(j) \\ &\quad=\lim_{n\to\infty}\sum_{k=-s}^{-1}\left(\sum_{l=k}^{-1}\left(K(n+k-l-1,n+k-1)-K(n+k-l,n+k)\right)\right) \\ &\qquad\times y(\varphi)(n+k)=0.\end{aligned} \tag{5.20}$$

Equations (5.19) and (5.20) together with Theorem 3.1 give the result.

(b) Since conditions (3.8) and (3.9) imply that the matrix

$$I-\sum_{l=1}^{s}L(l) \tag{5.21}$$

is invertible, we can apply Theorem 3.2 and (3.7). □

Proof of Theorem 3.4. (a) Equations (3.7) in Corollary 3.3 and (3.1) imply that

$$\left(I-\sum_{l=1}^{s}L(l)\right)y(\varphi)(\infty)=\varphi(0)-\sum_{l=-s}^{-1}L(-l)\varphi(l). \tag{5.22}$$

Our goal is to prove that the matrix

$$I-\sum_{l=1}^{s}L(l) \tag{5.23}$$

is invertible. To this end, we choose initial sequences $\varphi=(\varphi(-s),\dots,\varphi(-1),\varphi(0))$ of the form $\varphi(-s)=\cdots=\varphi(-1)=0$. Then (5.22) shows that the linear mapping

$$x\longrightarrow\left(I-\sum_{l=1}^{s}L(l)\right)x,\quad x\in\mathbb{R}^d \tag{5.24}$$

is surjective, whence it is an isomorphism. Consequently, (5.23) is invertible. Now the result follows from (5.22).

(b) Suppose (i). We have proved that the matrix (5.23) is invertible. If

$$\left(I-\sum_{l=1}^{s}L(l)\right)^{-1}\geq O \tag{5.25}$$

is also satisfied, then we have (ii) (see [4]). To prove this, choose initial sequences $\varphi=(\varphi(-s),\dots,\varphi(-1),\varphi(0))$ of the form $\varphi(-s)=\cdots=\varphi(-1)=0$. Then, by (5.22)

$$y(\varphi)(\infty)=\left(I-\sum_{l=1}^{s}L(l)\right)^{-1}\varphi(0). \tag{5.26}$$

Therefore, we have only to observe that $\varphi(0)\geq 0$ implies that $y(\varphi)(\infty)\geq 0$. It is enough to show that $\varphi(0)\geq 0$ yields

$$y(\varphi)(n+1)-y(\varphi)(n)\geq 0,\quad n\geq 0, \tag{5.27}$$

but this follows from (3.11) by an easy induction argument.

Now, suppose (ii). Then (i) comes from Corollary 3.3(b) (see the second condition).

The proof is complete. □

Acknowledgment

This paper is supported by Hungarian National Foundations for Scientific Research Grant no. K73274.

References

[1] R. D. Driver, G. Ladas, and P. N. Vlahos, "Asymptotic behavior of a linear delay difference equation," *Proceedings of the American Mathematical Society*, vol. 115, no. 1, pp. 105–112, 1992.
[2] M. Pituk, "The limits of the solutions of a nonautonomous linear delay difference equation," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 543–550, 2001.
[3] I. Győri and L. Horváth, "A new view of the l^p-theory of system of higher order difference equations," *Computers and Mathematics with Applications*, vol. 59, no. 8, pp. 2918–2932, 2010.
[4] H. H. Schaefer, *Banach Lattices and Positive Operators*, vol. 215 of *Die Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1974.