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Research Article

Existence of Solutions for a Class of Damped Vibration Problems on Time Scales

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We present a recent approach via variational methods and critical point theory to obtain the existence of solutions for a class of damped vibration problems on time scale $\mathbb{T}$, $u^{\Delta^2}(t) + w(t)u^{\Delta}(\sigma(t)) = \nabla F(\sigma(t), u(\sigma(t)))$, Δ-a.e. $t \in [0,T]_{\mathbb{T}}^{\kappa}$, $u(0) - u(T) = 0$, $u^{\Delta}(0) - u^{\Delta}(T) = 0$, where $u^{\Delta}(t)$ denotes the delta (or Hilger) derivative of u at t, $u^{\Delta^2}(t) = (u^{\Delta})^{\Delta}(t)$, σ is the forward jump operator, T is a positive constant, $w \in \mathcal{R}^+([0,T]_{\mathbb{T}}, \mathbb{R})$, $e_w(T,0) = 1$, and $F : [0,T]_{\mathbb{T}} \times \mathbb{R}^N \to \mathbb{R}$. By establishing a proper variational setting, three existence results are obtained. Finally, three examples are presented to illustrate the feasibility and effectiveness of our results.

1. Introduction

Consider the damped vibration problem on time-scale $\mathbb{T}$

$$\begin{gathered} u^{\Delta^2}(t) + w(t)u^{\Delta}(\sigma(t)) = \nabla F(\sigma(t), u(\sigma(t))), \quad \Delta\text{-a.e. } t \in [0,T]_{\mathbb{T}}^{\kappa}, \\ u(0) - u(T) = 0, \qquad u^{\Delta}(0) - u^{\Delta}(T) = 0, \end{gathered} \tag{1.1}$$

where $u^{\Delta}(t)$ denotes the delta (or Hilger) derivative of u at t, $u^{\Delta^2}(t) = (u^{\Delta})^{\Delta}(t)$, σ is the forward jump operator, T is a positive constant, $w \in \mathcal{R}^+([0,T]_{\mathbb{T}}, \mathbb{R})$, $e_w(T,0) = 1$, and $F : [0,T]_{\mathbb{T}} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption.

(A) $F(t,x)$ is Δ-measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for $t \in [0,T]_{\mathbb{T}}$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1_{\Delta}([0,T]_{\mathbb{T}}, \mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \qquad |\nabla F(t,x)| \le a(|x|)b(t) \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and $t \in [0,T]_{\mathbb{T}}$, where $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x.

Problem (1.1) covers the second-order damped vibration problem (for when $\mathbb{T} = \mathbb{R}$)

$$\begin{gathered}\ddot{u}(t) + w(t)\dot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = 0, \qquad \dot{u}(0) - \dot{u}(T) = 0,\end{gathered} \tag{1.3}$$

as well as the second-order discrete damped vibration problem (for when $\mathbb{T} = \mathbb{Z}, T \geq 2$)

$$\begin{gathered}\Delta^2(t) + w(t)\Delta u(t+1) = \nabla F(t+1, u(t+1)), \quad t \in [0, T-1] \cap \mathbb{Z}, \\ u(0) - u(T) = 0, \qquad \Delta u(0) - \Delta u(T) = 0.\end{gathered} \tag{1.4}$$

The calculus of time-scales was initiated by Stefan Hilger in his Ph.D. thesis in 1988 in order to create a theory that can unify discrete and continuous analysis. A time-scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers, which has the topology inherited from the real numbers with the standard topology. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. The time-scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, and social sciences (see [1]). For example, it can model insect populations that are continuous while in season (and may follow a difference scheme with variable step-size), die out in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

In recent years, dynamic equations on time-scales have received much attention. We refer the reader to the books [2–7] and the papers [8–15]. In this century, some authors have begun to discuss the existence of solutions of boundary value problems on time-scales (see [16–22]). There have been many approaches to study the existence and the multiplicity of solutions for differential equations on time-scales, such as methods of lower and upper solutions, fixed-point theory, and coincidence degree theory. In [14], the authors have used the fixed-point theorem of strict-set-contraction to study the existence of positive periodic solutions for functional differential equations with impulse effects on time-scales. However, the study of the existence and the multiplicity of solutions for differential equations on time-scales using variational method has received considerably less attention (see, e.g., [19, 23]). Variational method is, to the best of our knowledge, novel and it may open a new approach to deal with nonlinear problems on time-scales.

When $w(t) \equiv 0$, (1.1) is the second-order Hamiltonian system on time-scale $\mathbb{T}$

$$\begin{gathered}u^{\Delta^2}(t) = \nabla F(\sigma(t), u(\sigma(t))), \quad \Delta\text{-a.e. } t \in [0, T]_{\mathbb{T}}^{\kappa}, \\ u(0) - u(T) = 0, \qquad u^{\Delta}(0) - u^{\Delta}(T) = 0.\end{gathered} \tag{1.5}$$

Zhou and Li in [23] studied the existence of solutions for (1.5) by critical point theory on the Sobolevs spaces on time-scales that they established.

When $w(t) \not\equiv 0$, to the best of our knowledge, the existence of solutions for problems (1.1) have not been studied yet. Our purpose of this paper is to study the variational structure of problem (1.1) in an appropriate space of functions and the existence of solutions for problem (1.1) by some critical point theorems.

This paper is organized as follows. In Section 2, we present some fundamental definitions and results from the calculus on time-scales and Sobolev's spaces on time-scales. In Section 3, we make a variational structure of (1.1). From this variational structure, we can reduce the problem of finding solutions of problem (1.1) to one of seeking the critical points of a corresponding functional. Section 4 is the existence of solutions. Section 5 is the conclusion of this paper.

2. Preliminaries and Statements

In this section, we present some basic definitions and results from the calculus on time-scales and Sobolev's spaces on time-scales that will be required below. We first briefly recall some basic definitions and results concerning time-scales. Further general details can be found in [3–5, 7, 10, 13].

Through this paper, we assume that $0, T \in \mathbb{T}$. We start by the definitions of the forward and backward jump operators.

Definition 2.1 (see [3, Definition 1.1]). Let $\mathbb{T}$ be a time-scale, for $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}, \quad \forall t \in \mathbb{T}, \tag{2.1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T},\ s < t\}, \quad \forall t \in \mathbb{T} \tag{2.2}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set). A point $t \in \mathbb{T}$ is called right scattered, left scattered, if $\sigma(t) > t$, $\rho(t) < t$ hold, respectively. Points that are right scattered and left scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left dense. Points that are right-dense and left dense at the same time are called dense. The set $\mathbb{T}^\kappa$ which is derived from the time-scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left scattered maximum m, the $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{2.3}$$

When $a, b \in \mathbb{T}$, $a < b$, we denote the intervals $[a,b]_\mathbb{T}$, $[a,b)_\mathbb{T}$, and $(a,b]_\mathbb{T}$ in $\mathbb{T}$ by

$$[a,b]_\mathbb{T} = [a,b] \cap \mathbb{T}, \qquad [a,b)_\mathbb{T} = [a,b) \cap \mathbb{T}, \qquad (a,b]_\mathbb{T} = (a,b] \cap \mathbb{T}, \tag{2.4}$$

respectively. Note that $[a,b]_\mathbb{T}^\kappa = [a,b]_\mathbb{T}$ if b is left dense and $[a,b]_\mathbb{T}^\kappa = [a,b)_\mathbb{T} = [a,\rho(b)]_\mathbb{T}$ if b is left scattered. We denote $[a,b]_\mathbb{T}^{\kappa^2} = ([a,b]_\mathbb{T}^\kappa)^\kappa$, therefore $[a,b]_\mathbb{T}^{\kappa^2} = [a,b]_\mathbb{T}$ if b is left dense and $[a,b]_\mathbb{T}^{\kappa^2} = [a,\rho(b)]_\mathbb{T}^\kappa$ if b is left scattered.

Definition 2.2 (see [3, Definition 1.10]). Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s| \quad \forall s \in U. \tag{2.5}$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. The function f is delta (or Hilger) differentiable on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^{\kappa}$.

Definition 2.3 (see [23, Definition 2.3]). Assume that $f : \mathbb{T} \to \mathbb{R}^N$ is a function, $f(t) = (f^1(t), f^2(t), \ldots, f^N(t))$ and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t) = (f^{1^{\Delta}}(t), f^{2^{\Delta}}(t), \ldots, f^{N^{\Delta}}(t))$ (provided it exists). We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. The function f is delta (or Hilger) differentiable provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}^N$ is then called the delta derivative of f on $\mathbb{T}^{\kappa}$.

Definition 2.4 (see [3, Definition 2.7]). For a function $f : \mathbb{T} \to \mathbb{R}$ we will talk about the second derivative f^{Δ^2} provided f^{Δ} is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{\Delta^2} = (f^{\Delta})^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$.

Definition 2.5 (see [23, Definition 2.5]). For a function $f : \mathbb{T} \to \mathbb{R}^N$ we will talk about the second derivative f^{Δ^2} provided f^{Δ} is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{\Delta^2} = (f^{\Delta})^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}^N$.

Definition 2.6 (see [23, Definition 2.6]). A function $f : \mathbb{T} \to \mathbb{R}^N$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left sided limits exist (finite) at left dense points in $\mathbb{T}$.

Definition 2.7 (see [3, Definition 2.25]). We we say that a function $w : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)w(t) \neq 0 \quad \forall t \in \mathbb{T}^{\kappa} \tag{2.6}$$

holds. The set of all regressive and rd-continuous functions $w : \mathbb{T} \to \mathbb{R}$ is denoted by

$$\begin{aligned} \mathcal{R} = \mathcal{R}(\mathbb{T}) &= \mathcal{R}(\mathbb{T}, \mathbb{R}), \\ \mathcal{R}^{+}(\mathbb{T}, \mathbb{R}) &= \{w \in \mathcal{R} : 1 + \mu(t)w(t) > 0 \ \forall t \in \mathbb{T}\}. \end{aligned} \tag{2.7}$$

Definition 2.8 (see [7, Definition 8.2.18]). If $w \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the unique solution of the initial value problem

$$y^{\Delta} = w(t)y, \qquad y(t_0) = 1 \tag{2.8}$$

is called the exponential function and denoted by $e_w(\cdot, t_0)$.

The exponential function has some important properties.

Lemma 2.9 (see [3, Theorem 2.36]). *If $w \in \mathcal{R}$, then*

$$e_0(t,s) \equiv 1, \qquad e_w(t,t) \equiv 1. \tag{2.9}$$

Throughout this paper, we will use the following notations:

$$\begin{gathered}
C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}\left(\mathbb{T},\mathbb{R}^N\right) = \left\{f : \mathbb{T} \longrightarrow \mathbb{R}^N : f \text{ is rd-continuous}\right\}, \\
C^1_{\mathrm{rd}}(\mathbb{T}) = C^1_{\mathrm{rd}}\left(\mathbb{T},\mathbb{R}^N\right) = \left\{f : \mathbb{T} \longrightarrow \mathbb{R}^N : f \text{ is differentiable on } \mathbb{T}^\kappa \text{ and } f^\Delta \in C_{\mathrm{rd}}(\mathbb{T}^\kappa)\right\}, \\
C^1_{T,\mathrm{rd}}\left([0,T]_{\mathbb{T}},\mathbb{R}^N\right) = \left\{f \in C^1_{\mathrm{rd}}\left([0,T]_{\mathbb{T}},\mathbb{R}^N\right) : f(0) = f(T)\right\}.
\end{gathered} \tag{2.10}$$

The Δ-measure m_Δ and Δ-integration are defined as those in [10].

Definition 2.10 (see [23, Definition 2.7]). Assume that $f : \mathbb{T} \to \mathbb{R}^N$ is a function, $f(t) = (f^1(t), f^2(t), \dots, f^N(t))$ and let A be a Δ-measurable subset of $\mathbb{T}$. f is integrable on A if and only if $f^i (i = 1,2,\dots,N)$ are integrable on A, and

$$\int_A f(t)\Delta t = \left(\int_A f^1(t)\Delta t, \int_A f^2(t)\Delta t, \dots, \int_A f^N(t)\Delta t\right). \tag{2.11}$$

Definition 2.11 (see [13, Definition 2.3]). Let $B \subset \mathbb{T}$. B is called Δ-null set if $m_\Delta(B) = 0$. Say that a property P holds Δ-almost everywhere (Δ-a.e.) on B, or for Δ-almost all (Δ-a.a.) $t \in B$ if there is a Δ-null set $E_0 \subset B$ such that P holds for all $t \in B \setminus E_0$.

For $p \in \mathbb{R}$, $p \geq 1$, we set the space

$$L^p_\Delta\left([0,T)_{\mathbb{T}},\mathbb{R}^N\right) = \left\{u : [0,T)_{\mathbb{T}} \longrightarrow \mathbb{R}^N : \int_{[0,T)_{\mathbb{T}}} |f(t)|^p \Delta t < +\infty\right\} \tag{2.12}$$

with the norm

$$\|f\|_{L^p_\Delta} = \left(\int_{[0,T)_{\mathbb{T}}} |f(t)|^p \Delta t\right)^{1/p}. \tag{2.13}$$

We have the following lemma.

Lemma 2.12 (see [23, Theorem 2.1]). *Let $p \in \mathbb{R}$ be such that $p \geq 1$. Then the space $L^p_\Delta([0,T)_{\mathbb{T}},\mathbb{R}^N)$ is a Banach space together with the norm $\|\cdot\|_{L^p_\Delta}$. Moreover, $L^2_\Delta([a,b)_{\mathbb{T}},\mathbb{R}^N)$ is a Hilbert space together with the inner product given for every $(f,g) \in L^p_\Delta([a,b)_{\mathbb{T}},\mathbb{R}^N) \times L^p_\Delta([a,b)_{\mathbb{T}},\mathbb{R}^N)$ by*

$$\langle f,g\rangle_{L^2_\Delta} = \int_{[a,b)_{\mathbb{T}}} (f(t),g(t))\Delta t, \tag{2.14}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$.

As we know from general theory of Sobolev spaces, another important class of functions is just the absolutely continuous functions on time-scales.

Definition 2.13 (see [13, Definition 2.9]). A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be absolutely continuous on $[a,b]_{\mathbb{T}}$ (i.e., $f \in AC([a,b]_{\mathbb{T}}, \mathbb{R})$), if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_k, b_k)_{\mathbb{T}}\}_{k=1}^{n}$ is a finite pairwise disjoint family of subintervals of $[a,b]_{\mathbb{T}}$ satisfying $\sum_{k=1}^{n}(b_k - a_k) < \delta$, then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$.

Definition 2.14 (see [23, Definition 2.11]). A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^N$, $f(t) = (f^1(t), f^2(t), \ldots, f^N(t))$. We say that f is absolutely continuous on $[a,b]_{\mathbb{T}}$ (i.e., $f \in AC([a,b]_{\mathbb{T}}, \mathbb{R}^N)$), if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_k, b_k)_{\mathbb{T}}\}_{k=1}^{n}$ is a finite pairwise disjoint family of subintervals of $[a,b]_{\mathbb{T}}$ satisfying $\sum_{k=1}^{n}(b_k - a_k) < \delta$, then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$.

Absolutely continuous functions have the following properties.

Lemma 2.15 (see [23, Theorem 2.2]). *A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^N$ is absolutely continuous on $[a,b]_{\mathbb{T}}$ if and only if f is delta differentiable Δ-a.e. on $[a,b)_{\mathbb{T}}$ and*

$$f(t) = f(a) + \int_{[a,t)_{\mathbb{T}}} f^{\Delta}(s)\Delta s, \quad \forall t \in [a,b]_{\mathbb{T}}. \tag{2.15}$$

Lemma 2.16 (see [23, Theorem 2.3]). *Assume that functions $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}^N$ are absolutely continuous on $[a,b]_{\mathbb{T}}$. Then fg is absolutely continuous on $[a,b]_{\mathbb{T}}$ and the following equality is valid:*

$$\begin{aligned}\int_{[a,b)_{\mathbb{T}}} \Big(\big(f^{\Delta}(t), g(t)\big) + \big(f^{\sigma}(t), g^{\Delta}(t)\big)\Big)\Delta t &= (f(b), g(b)) - (f(a), g(a)) \\ &= \int_{[a,b)_{\mathbb{T}}} \Big(\big(f(t), g^{\Delta}(t)\big) + \big(f^{\Delta}(t), g^{\sigma}(t)\big)\Big)\Delta t.\end{aligned} \tag{2.16}$$

Now, we recall the definition and properties of the Sobolev space on $[0,T]_{\mathbb{T}}$ in [23]. For the sake of convenience, in the sequel, we will let $u^{\sigma}(t) = u(\sigma(t))$.

Definition 2.17 (see [23, Definition 2.12]). Let $p \in \mathbb{R}$ be such that $p \geq 1$ and $u : [0,T]_{\mathbb{T}} \to \mathbb{R}^N$. We say that $u \in W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ if and only if $u \in L_{\Delta}^{p}([0,T)_{\mathbb{T}}, \mathbb{R}^N)$ and there exists $g : [0,T]_{\mathbb{T}}^{\kappa} \to \mathbb{R}^N$ such that $g \in L_{\Delta}^{p}([0,T)_{\mathbb{T}}, \mathbb{R}^N)$ and

$$\int_{[0,T)_{\mathbb{T}}} \big(u(t), \phi^{\Delta}(t)\big)\Delta t = -\int_{[0,T)_{\mathbb{T}}} (g(t), \phi^{\sigma}(t))\Delta t, \quad \forall \phi \in C_{T,\mathrm{rd}}^{1}\big([0,T]_{\mathbb{T}}, \mathbb{R}^N\big). \tag{2.17}$$

For $p \in \mathbb{R}, p \geq 1$, we denote

$$V_{\Delta,T}^{1,p}\big([0,T]_{\mathbb{T}}, \mathbb{R}^N\big) = \Big\{x \in AC\big([0,T]_{\mathbb{T}}, \mathbb{R}^N\big) : x^{\Delta} \in L_{\Delta}^{p}\big([0,T)_{\mathbb{T}}, \mathbb{R}^N\big),\ x(0) = x(T)\Big\}. \tag{2.18}$$

Remark 2.18 (see [23, Remark 2.2]). $V_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}_N) \subset W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ is true for every $p \in \mathbb{R}$ with $p \geq 1$.

Lemma 2.19 (see [23, Theorem 2.5]). *Suppose that* $u \in W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ *for some* $p \in \mathbb{R}$ *with* $p \geq 1$, *and that* (2.17) *holds for* $g \in L^p_{\Delta}([0,T)_{\mathbb{T}}, \mathbb{R}^N)$. *Then, there exists a unique function* $x \in V^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ *such that the equalities*

$$x = u, \quad x^{\Delta} = g \quad \Delta\text{-a.e.} \quad \text{on } [0,T)_{\mathbb{T}} \tag{2.19}$$

are satisfied and

$$\int_{[0,T)_{\mathbb{T}}} g(t)\Delta t = 0. \tag{2.20}$$

Lemma 2.20 (see [3, Theorem 1.16]). *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^{\kappa}$. *Then, one has the following.*

(i) *If* f *is differentiable at* t, *then* f *is continuous at* t.

(ii) *If* f *is differentiable at* t, *then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.21}$$

By identifying $u \in W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ with its absolutely continuous representative $x \in V^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ for which (2.19) holds, the set $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ can be endowed with the structure of Banach space.

Theorem 2.21. *Assume* $p \in \mathbb{R}$ *and* $p \geq 1$. *The set* $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ *is a Banach space together with the norm defined as*

$$\|u\|_{W^{1,p}_{\Delta,T}} = \left(\int_{[0,T)_{\mathbb{T}}} |u^{\sigma}(t)|^p \Delta t + \int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(t) \right|^p \Delta t \right)^{1/p} \quad \forall u \in W^{1,p}_{\Delta,T}\left([0,T]_{\mathbb{T}}, \mathbb{R}^N\right). \tag{2.22}$$

Moreover, the set $H^1_{\Delta,T} = W^{1,2}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ *is a Hilbert space together with the inner product*

$$\langle u, v \rangle_{H^1_{\Delta,T}} = \int_{[0,T)_{\mathbb{T}}} (u^{\sigma}(t), v^{\sigma}(t))\Delta t + \int_{[0,T)_{\mathbb{T}}} \left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t \quad \forall u, v \in H^1_{\Delta,T}. \tag{2.23}$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$. That is, $\{u_n\}_{n\in\mathbb{N}} \subset L^p_{\Delta}([0,T)_{\mathbb{T}}, \mathbb{R}^N)$ and there exist $g_n : [0,T]^{\kappa} \to \mathbb{R}^N$ such that $\{g_n\}_{n\in\mathbb{N}} \subset L^p_{\Delta}([0,T)_{\mathbb{T}}, \mathbb{R}^N)$ and

$$\int_{[0,T)_{\mathbb{T}}} \left(u_n(t), \phi^{\Delta}(t)\right)\Delta t = -\int_{[0,T)_{\mathbb{T}}} (g_n(t), \phi^{\sigma}(t))\Delta t, \quad \forall \phi \in C^1_{T,\mathrm{rd}}\left([0,T]_{\mathbb{T}}, \mathbb{R}^N\right). \tag{2.24}$$

Thus, by Lemma 2.19, there exists $\{x_n\}_{n\in\mathbb{N}} \subset V^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ such that

$$x_n = u_n, \quad x_n^{\Delta} = g_n \quad \Delta\text{-a.e.} \quad \text{on } [0,T)_{\mathbb{T}}. \tag{2.25}$$

Combining (2.24) and (2.25), we obtain

$$\int_{[0,T)_{\mathbb{T}}} \left(x_n(t), \phi^{\Delta}(t)\right)\Delta t = -\int_{[0,T)_{\mathbb{T}}} \left(x_n^{\Delta}(t), \phi^{\sigma}(t)\right)\Delta t, \quad \forall \phi \in C^1_{T,\mathrm{rd}}\left([0,T]_{\mathbb{T}}, \mathbb{R}^N\right). \tag{2.26}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$, by (2.22), one has

$$\int_{[0,T)_{\mathbb{T}}} |u_n^{\sigma}(t) - u_m^{\sigma}(t)|^2 \Delta t \longrightarrow 0 \quad (m,n \longrightarrow \infty), \tag{2.27}$$

$$\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t) - u_m^{\Delta}(t)\right|^2 \Delta t \longrightarrow 0 \quad (m,n \longrightarrow \infty). \tag{2.28}$$

It follows from Lemma 2.20, (2.27), and (2.28) that

$$\begin{aligned}
\int_{[0,T)_{\mathbb{T}}} |u_n(t) - u_m(t)|^2 \Delta t &= \int_{[0,T)_{\mathbb{T}}} \left|(u_n^{\sigma}(t) - u_m^{\sigma}(t)) - \mu(t)\left(u_n^{\Delta}(t) - u_m^{\Delta}(t)\right)\right|^2 \Delta t \\
&\leq 2\int_{[0,T)_{\mathbb{T}}} |u_n^{\sigma}(t) - u_m^{\sigma}(t)|^2 \Delta t + 2(\sigma(T))^2 \int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t) - u_m^{\Delta}(t)\right|^2 \Delta t \\
&\longrightarrow 0 \quad (m, n\infty).
\end{aligned} \tag{2.29}$$

By Lemma 2.12, (2.28) and (2.29), there exist $u, g \in L^p_{\Delta}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ such that

$$\|u_n - u\|_{L^p_{\Delta}} \longrightarrow 0 \quad (n \longrightarrow \infty), \qquad \left\|u_n^{\Delta} - g\right\|_{L^p_{\Delta}} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.30}$$

From (2.26) and (2.30), one has

$$\int_{[0,T)_{\mathbb{T}}} \left(u(t), \phi^{\Delta}(t)\right)\Delta t = -\int_{[0,T)_{\mathbb{T}}} (g(t), \phi^{\sigma}(t))\Delta t, \quad \forall \phi \in C^1_{T,\mathrm{rd}}\left([0,T]_{\mathbb{T}}, \mathbb{R}^N\right). \tag{2.31}$$

From (2.31), we conclude that $u \in W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$. Moreover, by Lemma 2.20 and (2.30), one has

$$\begin{aligned}\int_{[0,T)_{\mathbb{T}}} |u_n^{\sigma}(t) - u^{\sigma}(t)|^2 \Delta t &= \int_{[0,T)_{\mathbb{T}}} \left|(u_n(t) - u(t)) + \mu(t)\left(u_n^{\Delta}(t) - u^{\Delta}(t)\right)\right|^2 \Delta t \\ &= \int_{[0,T)_{\mathbb{T}}} \left|(u_n(t) - u(t)) + \mu(t)\left(u_n^{\Delta}(t) - g(t)\right)\right|^2 \Delta t \\ &\leq 2\int_{[0,T)_{\mathbb{T}}} |u_n(t) - u(t)|^2 \Delta t + 2(\sigma(T))^2 \int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t) - g(t)\right|^2 \Delta t \\ &\longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{2.32}$$

Thereby, it follows from Remark 2.18, (2.30), (2.32), and Lemma 2.19 that there exists $x \in V_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N) \subset W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ such that

$$\|u_n - x\|_{W_{\Delta,T}^{1,p}} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.33}$$

Obviously, the set $H_{\Delta,T}^1$ is a Hilbert space together with the inner product

$$\langle u, v\rangle_{H_{\Delta,T}^1} = \int_{[0,T)_{\mathbb{T}}} (u^{\sigma}(t), v^{\sigma}(t))\Delta t + \int_{[0,T)_{\mathbb{T}}} \left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t \quad \forall u, v \in H_{\Delta,T}^1. \tag{2.34}$$

□

We will derive some properties of the Banach space $W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$.

Lemma 2.22 (see [10, Theorem A.2]). *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a continuous function on $[a,b]_{\mathbb{T}}$ which is delta differentiable on $[a,b)_{\mathbb{T}}$. Then there exist $\xi, \tau \in [a,b)_{\mathbb{T}}$ such that*

$$f^{\Delta}(\tau) \leq \frac{f(b) - f(a)}{b - a} \leq f^{\Delta}(\xi). \tag{2.35}$$

Theorem 2.23. *There exists $K = K(p) > 0$ such that the inequality*

$$\|u\|_{\infty} \leq K\|u\|_{W_{\Delta,T}^{1,p}} \tag{2.36}$$

holds for all $u \in W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$, where $\|u\|_{\infty} = \max_{t\in[0,T]_{\mathbb{T}}}|u(t)|$.

Moreover, if $\int_{[0,T)_{\mathbb{T}}} u(t)\Delta t = 0$, then

$$\|u\|_{\infty} \leq K\left\|u^{\Delta}\right\|_{L_{\Delta}^p}. \tag{2.37}$$

Proof. Going to the components of u, we can assume that $N = 1$. If $u \in W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R})$, by Lemma 2.19, $U(t) = \int_{[0,t)_{\mathbb{T}}} u(s)\Delta s$ is absolutely continuous on $[a,b]_{\mathbb{T}}$. It follows from Lemma 2.22 that there exists $\zeta \in [a,b)_{\mathbb{T}}$ such that

$$u(\zeta) \leq \frac{U(T) - U(0)}{T} = \frac{1}{T}\int_{[0,T)_{\mathbb{T}}} u(s)\Delta s. \tag{2.38}$$

Hence, for $t \in [a,b]_{\mathbb{T}}$, using Lemma 2.15, (2.38), and Hölder's inequality, one has that

$$\begin{aligned}
|u(t)| &= \left| u(\zeta) + \int_{[\zeta,t)_{\mathbb{T}}} u^{\Delta}(s)\Delta s \right| \\
&\leq |u(\zeta)| + \int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right| \Delta s \\
&\leq \frac{1}{T}\left| \int_{[0,T)_{\mathbb{T}}} u(s)\Delta s \right| + T^{1/q}\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p},
\end{aligned} \tag{2.39}$$

where $1/p + 1/q = 1$. If $\int_{[0,T)_{\mathbb{T}}} u(t)\Delta t = 0$, by (2.39), we obtain (2.37). In the general case, for $t \in [a,b]_{\mathbb{T}}$, by Lemma 2.20 and Hölder's inequality, we get

$$\begin{aligned}
|u(t)| &\leq \frac{1}{T}\left| \int_{[0,T)_{\mathbb{T}}} u(s)\Delta s \right| + T^{1/q}\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p} \\
&\leq \frac{1}{T}\int_{[0,T)_{\mathbb{T}}} |u(s)|\Delta s + T^{1/q}\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p} \\
&= \frac{1}{T}\int_{[0,T)_{\mathbb{T}}} \left| u^{\sigma}(s) - \mu(s)u^{\Delta}(s) \right| \Delta s + T^{1/q}\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p} \\
&= \frac{1}{T}\int_{[0,T)_{\mathbb{T}}} |u^{\sigma}(s)|\Delta s + \frac{1}{T}\sigma(T)\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right| \Delta s + T^{1/q}\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p} \\
&\leq T^{(-1/p)}\left(\int_{[0,T)_{\mathbb{T}}} |u^{\sigma}(s)|^{p} \Delta s \right)^{1/p} + T^{(-1/p)}\sigma(T)\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p} \\
&\quad + T^{1/q}\left(\int_{[0,T)_{\mathbb{T}}} \left| u^{\Delta}(s) \right|^{p} \Delta s \right)^{1/p} \\
&\leq \left(T^{(-1/p)} + T^{(-1/p)}\sigma(T) + T^{1/q} \right)\|u\|_{W^{1,p}_{\Delta,T}}.
\end{aligned} \tag{2.40}$$

From (2.40), (2.36) holds. □

Remark 2.24. It follows from Theorem 2.23 that $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^{N})$ is continuously immersed into $C([0,T]_{\mathbb{T}}, \mathbb{R}^{N})$ with the norm $\|\cdot\|_{\infty}$.

Theorem 2.25. *If the sequence* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *converges weakly to* u *in* $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$*, then* $\{u_k\}_{k\in\mathbb{N}}$ *converges strongly in* $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *to* u*.*

Proof. Since $u_k \rightharpoonup u$ in $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ and, hence, in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$. It follows from Remark 2.24 that $u_k \rightharpoonup u$ in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$. For $t_1,t_2\in[0,T]_{\mathbb{T}}$, $t_1\le t_2$, there exists $C_1>0$ such that

$$\begin{aligned}|u_k(t_2)-u_k(t_1)| &\le \int_{[t_1,t_2]_{\mathbb{T}}}\left|u_k^{\Delta}(s)\right|\Delta s\\ &\le (t_2-t_1)^{1/q}\left(\int_{[t_1,t_2]_{\mathbb{T}}}\left|u_k^{\Delta}(s)\right|^p\Delta s\right)^{1/p}\\ &\le (t_2-t_1)^{1/q}\|u_k\|_{W^{1,p}_{\Delta,T}}\\ &\le C_1(t_2-t_1)^{1/q}.\end{aligned}\tag{2.41}$$

Hence, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is equicontinuous. By Ascoli-Arzela theorem, $\{u_k\}_{k\in\mathbb{N}}$ is relatively compact in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$. By the uniqueness of the weak limit in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$, every uniformly convergent subsequence of $\{u_k\}_{k\in\mathbb{N}}$ converges to u. Thus, $\{u_k\}_{k\in\mathbb{N}}$ converges strongly in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$ to u. □

Remark 2.26. By Theorem 2.25, the immersion $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N) \hookrightarrow C([0,T]_{\mathbb{T}},\mathbb{R}^N)$ is compact.

Theorem 2.27. *Let* $L:[0,T]_{\mathbb{T}}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}, (t,x,y)\to L(t,x,y)$ *be Lebesgue* Δ*-measurable in* t *for each* $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ *and continuously differentiable in* (x,y) *for every* $t\in[0,T]_{\mathbb{T}}$*. If there exist* $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b,c\in[0,T]_{\mathbb{T}}\to\mathbb{R}^+$, $b^{\sigma}\in L^1_{\Delta}([0,T)_{\mathbb{T}},\mathbb{R}^+)$ *and* $c^{\sigma}\in L^q_{\Delta}([0,T)_{\mathbb{T}},\mathbb{R}^+)(1<q<+\infty)$ *such that for* Δ*-almost* $t\in[0,T]_{\mathbb{T}}$ *and every* $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$*, one has*

$$\begin{aligned}\left|L(t,x,y)\right| &\le a(|x|)\left(b(t)+|y|^p\right),\\ \left|L_x(t,x,y)\right| &\le a(|x|)\left(b(t)+|y|^p\right),\\ \left|L_y(t,x,y)\right| &\le a(|x|)\left(c(t)+|y|^{p-1}\right),\end{aligned}\tag{2.42}$$

where $1/p+1/q=1$*, then the functional* $\Phi: W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)\to\mathbb{R}$ *defined as*

$$\Phi(u)=\int_{[0,T)_{\mathbb{T}}} L\left(\sigma(t),u^{\sigma}(t),u^{\Delta}(t)\right)\Delta t\tag{2.43}$$

is continuously differentiable on $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *and*

$$\langle \Phi'(u),v\rangle = \int_{[0,T)_{\mathbb{T}}}\left[\left(L_x\left(\sigma(t),u^{\sigma}(t),u^{\Delta}(t)\right),v^{\sigma}(t)\right)+\left(L_y\left(\sigma(t),u^{\sigma}(t),u^{\Delta}(t)\right),v^{\Delta}(t)\right)\right]\Delta t. \tag{2.44}$$

Proof. It suffices to prove that Φ has at every point u a directional derivative $\Phi'(u) \in (W^{1,p}_{\Delta,T}([0,T)_{\mathbb{T}},\mathbb{R}^N))^*$ given by (2.44) and that the mapping

$$\Phi' : W^{1,p}_{\Delta,T}\left([0,T]_{\mathbb{T}},\mathbb{R}^N\right) \longrightarrow \left(W^{1,p}_{\Delta,T}\left([0,T]_{\mathbb{T}},\mathbb{R}^N\right)\right)^* \tag{2.45}$$

is continuous.

Firstly, it follows from (2.42) that Φ is everywhere finite on $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$. We define, for u and v fixed in $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$, $t\in[0,T)_{\mathbb{T}}$, $\lambda\in[-1,1]$,

$$\begin{gathered} G(\lambda,t) = L\left(\sigma(t),u^{\sigma}(t)+\lambda v^{\sigma}(t),u^{\Delta}(t)+\lambda v^{\Delta}(t)\right),\\ \Psi(\lambda) = \int_{[0,T)_{\mathbb{T}}} G(\lambda,t)\Delta t = \Phi(u+\lambda v). \end{gathered} \tag{2.46}$$

From (2.42), one has

$$\begin{aligned} |D_{\lambda}G(\lambda,t)| &\le \left|\left(D_xL\left(\sigma(t),u^{\sigma}(t)+\lambda v^{\sigma}(t),u^{\Delta}(t)+\lambda v^{\Delta}(t)\right),v^{\sigma}(t)\right)\right|\\ &\quad+\left|\left(D_yL\left(\sigma(t),u^{\sigma}(t)+\lambda v^{\sigma}(t),u^{\Delta}(t)+\lambda v^{\Delta}(t)\right),v^{\Delta}(t)\right)\right|\\ &\le a(|u^{\sigma}(t)+\lambda v^{\sigma}(t)|)\left(b^{\sigma}(t)+\left|u^{\Delta}(t)+\lambda v^{\Delta}(t)\right|^{p}\right)|v^{\sigma}(t)|\\ &\quad+a(|u^{\sigma}(t)+\lambda v^{\sigma}(t)|)\left(c^{\sigma}(t)+\left|u^{\Delta}(t)+\lambda v^{\Delta}(t)\right|^{p-1}\right)\left|v^{\Delta}(t)\right|\\ &\le \overline{a}\left(b^{\sigma}(t)+\left(\left|u^{\Delta}(t)\right|+\left|v^{\Delta}(t)\right|\right)^{p}\right)|v^{\sigma}(t)|\\ &\quad+\overline{a}\left(c^{\sigma}(t)+\left(\left|u^{\Delta}(t)\right|+\left|v^{\Delta}(t)\right|\right)^{p-1}\right)\left|v^{\Delta}(t)\right|\\ &\triangleq d(t), \end{aligned} \tag{2.47}$$

where

$$\overline{a} = \max_{(\lambda,t)\in[-1,1]\times[0,T]_{\mathbb{T}}} a(|u(t)+\lambda v(t)|), \tag{2.48}$$

thus, $d \in L^1_\Delta([0,T)_\mathbb{T},\mathbb{R}^+)$. Since $b^\sigma \in L^1_\Delta([0,T)_\mathbb{T},\mathbb{R}^+)$, $(|u^\Delta|+|v^\Delta|)^p \in L^1_\Delta([0,T)_\mathbb{T},\mathbb{R})$, $c^\sigma \in L^q_\Delta([0,T)_\mathbb{T},\mathbb{R}^+)$, one has

$$|D_\lambda G(\lambda,t)| \le d(t), \tag{2.49}$$

$$\begin{aligned}\Psi'(0) &= \int_{[0,T)_\mathbb{T}} D_\lambda G(0,t)\Delta t \\ &= \int_{[0,T)_\mathbb{T}} \left[\left(D_x L\left(\sigma(t),u^\sigma(t),u^\Delta(t)\right),v^\sigma(t)\right) + \left(D_y L\left(\sigma(t),u^\sigma(t),u^\Delta(t)\right),v^\Delta(t)\right)\right]\Delta t.\end{aligned} \tag{2.50}$$

On the other hand, it follows from (2.42) that

$$\begin{aligned}\left|D_x L\left(\sigma(t),u^\sigma(t),u^\Delta(t)\right)\right| &\le a(|u^\sigma(t)|)\left(b^\sigma(t)+\left|u^\Delta(t)\right|^p\right) \triangleq \psi_1(t),\\ \left|D_y L\left(\sigma(t),u^\sigma(t),u^\Delta(t)\right)\right| &\le a(|u^\sigma(t)|)\left(c^\sigma(t)+\left|u^\Delta(t)\right|^{p-1}\right) \triangleq \psi_2(t),\end{aligned} \tag{2.51}$$

thus $\psi_1 \in L^1_\Delta([0,T)_\mathbb{T},\mathbb{R}^+)$, $\psi_2 \in L^q_\Delta([0,T)_\mathbb{T},\mathbb{R}^+)$. Thereby, by Theorem 2.23, (2.50), and (2.51), there exist positive constants C_2, C_3, C_4 such that

$$\begin{aligned}&\int_{[0,T)_\mathbb{T}} \left[\left(D_x L\left(\sigma(t),u^\sigma(t),u^\Delta(t)\right),v^\sigma(t)\right) + \left(D_y L\left(\sigma(t),u^\sigma(t),u^\Delta(t)\right),v^\Delta(t)\right)\right]\Delta t\\ &\quad\le C_2\|v\|_\infty + C_3\left\|v^\Delta\right\|_{L^p_\Delta}\\ &\quad\le C_4\|v\|_{W^{1,p}_{\Delta,T}}\end{aligned} \tag{2.52}$$

and Φ has a directional derivative at u and $\Phi'(u) \in (W^{1,p}_{\Delta,T}([0,T]_\mathbb{T},\mathbb{R}^N))^*$ given by (2.44).

Moreover, (2.42) implies that the mapping from $W^{1,p}_{\Delta,T}([0,T]_\mathbb{T},\mathbb{R}^N)$ into $L^1_\Delta([0,T)_\mathbb{T},\mathbb{R}^N) \times L^q_\Delta([0,T)_\mathbb{T},\mathbb{R}^N)$ defined by

$$u \longrightarrow \left(D_x L\left(\cdot,u^\sigma,u^\Delta\right), D_y L\left(\cdot,u^\sigma,u^\Delta\right)\right) \tag{2.53}$$

is continuous, so that Φ' is continuous from $W^{1,p}_{\Delta,T}([0,T]_\mathbb{T},\mathbb{R}^N)$ into $(W^{1,p}_{\Delta,T}([0,T]_\mathbb{T},\mathbb{R}^N))^*$. □

3. Variational Setting

In this section, in order to apply the critical point theory, we make a variational structure. From this variational structure, we can reduce the problem of finding solutions of problem (1.1) to one of seeking the critical points of a corresponding functional.

By Theorem 2.21, the space $H^1_{\Delta,T} = W^{1,2}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ with the inner product

$$\langle u, v\rangle_{H^1_{\Delta,T}} = \int_{[0,T)_{\mathbb{T}}} (u^{\sigma}(t), v^{\sigma}(t))\Delta t + \int_{[0,T)_{\mathbb{T}}} \left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t \tag{3.1}$$

and the induced norm

$$\|u\|_{H^1_{\Delta,T}} = \left(\int_{[0,T)_{\mathbb{T}}} |u^{\sigma}(t)|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} \left|u^{\Delta}(t)\right|^2\Delta t\right)^{1/2} \tag{3.2}$$

is Hilbert space.

Since $w \in \mathcal{R}^{+}([0,T]_{\mathbb{T}}, \mathbb{R})$, by Theorem 2.44 in [3], one has that

$$e_w(t,0) > 0 \quad \forall t \in [0,T]_{\mathbb{T}}, \tag{3.3}$$

in $H^1_{\Delta,T}$, we also consider the inner product

$$\langle u, v\rangle = \int_{[0,T)_{\mathbb{T}}} e_w(t,0)(u^{\sigma}(t), v^{\sigma}(t))\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t \tag{3.4}$$

and the induced norm

$$\|u\| = \left(\int_{[0,T)_{\mathbb{T}}} e_w(t,0)|u^{\sigma}(t)|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u^{\Delta}(t)\right|^2\Delta t\right)^{1/2}, \tag{3.5}$$

we prove the following theorem.

Theorem 3.1. *The norm $\|\cdot\|$ and $\|\cdot\|_{H^1_{\Delta,T}}$ are equivalent.*

Proof. Since $e_w(\cdot,0)$ is continuous on $[0,T]_{\mathbb{T}}$ and

$$e_w(t,0) > 0 \quad \forall t \in [0,T]_{\mathbb{T}}, \tag{3.6}$$

there exist two positive constants M_1 and M_2 such that

$$M_1 = \min_{t\in[0,T]_{\mathbb{T}}} e_w(t,0), \qquad M_2 = \max_{t\in[0,T]_{\mathbb{T}}} e_w(t,0). \tag{3.7}$$

Hence, one has

$$\sqrt{M_1}\|u\|_{H^1_{\Delta,T}} \le \|u\| \le \sqrt{M_2}\|u\|_{H^1_{\Delta,T}}, \quad \forall u \in H^1_{\Delta,T}. \tag{3.8}$$

Consequently, the norm $\|\cdot\|$ and $\|\cdot\|_{H^1_{\Delta,T}}$ are equivalent. ☐

Consider the functional $\varphi : H^1_{\Delta,T} \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u^{\Delta}(t)\right|^2 \Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u^{\sigma}(t))\Delta t. \tag{3.9}$$

We have the following facts.

Theorem 3.2. *The functional φ is continuously differentiable on $H^1_{\Delta,T}$ and*

$$\langle \varphi'(u), v\rangle = \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)(\nabla F(\sigma(t),u^{\sigma}(t)), v^{\sigma}(t))\Delta t \tag{3.10}$$

for all $v \in H^1_{\Delta,T}$.

Proof. Let $L(t,x,y) = e_w(\rho(t),0)\ [(1/2)|y|^2 + F(t,x)]$ for all $x,y \in \mathbb{R}^N$ and $t \in [0,T]_{\mathbb{T}}$. Then, by condition (A), $L(t,x,y)$ satisfies all assumptions of Theorem 2.27. Hence, by Theorem 2.27, we know that the functional φ is continuously differentiable on $H^1_{\Delta,T}$ and

$$\langle \varphi'(u), v\rangle = \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)(\nabla F(\sigma(t),u^{\sigma}(t)), v^{\sigma}(t))\Delta t \tag{3.11}$$

for all $v \in H^1_{\Delta,T}$. □

Theorem 3.3. *If $u \in H^1_{\Delta,T}$ is a critical point of φ in $H^1_{\Delta,T}$, that is, $\varphi'(u) = 0$, then u is a solution of problem* (1.1).

Proof. Since $\varphi'(u) = 0$, it follows from Theorem 3.2 that

$$\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)(\nabla F(\sigma(t),u^{\sigma}(t)), v^{\sigma}(t))\Delta t = 0 \tag{3.12}$$

for all $v \in H^1_{\Delta,T}$, that is,

$$\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t = -\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(\nabla F(\sigma(t),u^{\sigma}(t)), v^{\sigma}(t))\Delta t \tag{3.13}$$

for all $v \in H^1_{\Delta,T}$. From condition (A) and Definition 2.17, one has that $e_w(\cdot,0)u^{\Delta} \in H^1_{\Delta,T}$. By Lemma 2.19 and (2.20), there exists a unique function $x \in V^{1,2}_{\Delta,T}([0,T]_{\mathbb{T}}, \mathbb{R}^N)$ such that

$$x = u, \quad \left(e_w(t,0)x^{\Delta}(t)\right)^{\Delta} = e_w(t,0)\nabla F(\sigma(t),u^{\sigma}(t)) \quad \Delta\text{-a.e.} \quad \text{on } [0,T]^{\kappa}_{\mathbb{T}}, \tag{3.14}$$

$$\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\nabla F(\sigma(t),u^{\sigma}(t))\Delta t = 0. \tag{3.15}$$

By (3.14), one has

$$e_w(t,0)x^{\Delta^2}(t) + w(t)e_w(t,0)x^{\Delta}(\sigma(t)) = e_w(t,0)\nabla F(\sigma(t),u^{\sigma}(t)) \quad \Delta\text{-a.e.} \quad \text{on } [0,T]^{\kappa}_{\mathbb{T}}. \tag{3.16}$$

Combining (3.14), (3.15), (3.16), and Lemma 2.19, we obtain

$$\begin{gathered} x^{\Delta^2}(t) + w(t)x^{\Delta}(\sigma(t)) = \nabla F(\sigma(t),u^{\sigma}(t)) \quad \Delta\text{-a.e.} \quad \text{on } [0,T]^{\kappa}_{\mathbb{T}}, \\ x(0) - x(T) = 0, \qquad x^{\Delta}(0) - x^{\Delta}(T) = 0. \end{gathered} \tag{3.17}$$

We identify $u \in H^1_{\Delta,T}$ with its absolutely continuous representative $x \in V^{1,2}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ for which (3.14) holds. Thus u is a solution of problem (1.1). □

Theorem 3.4. *The functional φ is weakly lower semicontinuous on $H^1_{\Delta,T}$.*

Proof. Let

$$\begin{aligned} \varphi_1(u) &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u^{\Delta}(t)\right|^2 \Delta t, \\ \varphi_2(u) &= \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u^{\sigma}(t))\Delta t. \end{aligned} \tag{3.18}$$

Then, φ_1 is continuous and convex. Hence, φ_1 is weakly lower semicontinuous. On the other hand, let $\{u_n\}_{n\in\mathbb{N}} \subset H^1_{\Delta,T}, u_n \rightharpoonup u$ in $H^1_{\Delta,T}$. By Theorem 2.25, $\{u_k\}_{k\in\mathbb{N}}$ converges strongly in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$ to u. By condition (A), one has

$$\begin{aligned} \left|\varphi_2(u_n) - \varphi_2(u)\right| &= \left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u_n^{\sigma}(t))\Delta t - \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u^{\sigma}(t))\Delta t\right| \\ &\le M_2 \int_{[0,T)_{\mathbb{T}}} |F(\sigma(t),u_n^{\sigma}(t)) - F(\sigma(t),u^{\sigma}(t))|\Delta t \\ &\longrightarrow 0. \end{aligned} \tag{3.19}$$

Thus, φ_2 is weakly continuous. Consequently, $\varphi = \varphi_1+\varphi_2$ is weakly lower semicontinuous. □

To prove the existence of solutions for problem (1.1), we need the following definitions.

Definition 3.5 (see [23, page 81]). Let X be a real Banach space, $I \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. I is said to satisfy $(\mathrm{PS})_c$-condition on X if the existence of a sequence $\{x_n\} \subseteq X$ such that $I(x_n) \to c$ and $I'(x_n) \to 0$ as $n \to \infty$, implies that c is a critical value of I.

Definition 3.6 (see [23, page 81]). Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. I is said to satisfy P.S. condition on X if any sequence $\{x_n\} \subseteq X$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.

Remark 3.7. It is clear that the P.S. condition implies the $(\mathrm{PS})_c$-condition for each $c \in \mathbb{R}$.

We also need the following result to prove our main results of this paper.

Lemma 3.8 (see [24, Theorem 1.1]). *If φ is weakly lower semicontinuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 3.9 (see [24, Theorem 4.7]). *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$, $R > 0$. Assume that X splits into a direct sum of closed subspace $X = X^- \oplus X^+$ with $\dim X^- < \infty$ and $\sup_{S_R^-} J < \inf_{X^+} J$, where $S_R^- = \{u \in X^- : \|u\| = R\}$. Let $B_R^- = \{u \in X^- : \|u\| \leq R\}$, $M = \{h \in C(B_R^-, X) : h(s) = s \text{ if } s \in S_R^-\}$ and $c = \inf_{h \in M} \max_{s \in B_R^-} J(h(s))$. Then, if J satisfies the $(PS)_c$ condition, c is a critical value of J.*

Lemma 3.10 (see [24, Proposition 1.4]). *Let $G \in C^1(\mathbb{R}^N, \mathbb{R})$ be a convex function. Then, for all $x, y \in \mathbb{R}^N$ one has*

$$G(x) \geq G(y) + (\nabla G(y), x - y). \tag{3.20}$$

4. Existence of Solutions

For $u \in H^1_{\Delta,T}$, let $\overline{u} = (1/T) \int_{[0,T)_{\mathbb{T}}} u(t) \Delta t$ and $\widetilde{u}(t) = u(t) - \overline{u}$, then $\int_{[0,T)_{\mathbb{T}}} \widetilde{u}(t) \Delta t = 0$. We have the following existence results.

Theorem 4.1. *Assume that (A) and the following conditions are satisfied.*

(i) *There exist $f, g : [0,T]_{\mathbb{T}} \to \mathbb{R}^+$ and $\alpha \in [0,1)$ such that $f^\sigma, g^\sigma \in L^1_\Delta([0,T)_{\mathbb{T}}, \mathbb{R}^+)$ and*

$$|\nabla F(t,x)| \leq f(t)|x|^\alpha + g(t) \tag{4.1}$$

for all $x \in \mathbb{R}^N$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$.

(ii) $|x|^{-2\alpha} \int_{[0,T)_{\mathbb{T}}} e_w(t,0) F(\sigma(t), x) \Delta t \to +\infty$ *as* $|x| \to \infty$.

Then problem (1.1) *has at least one solution which minimizes the function φ.*

Proof. By Theorem 2.23, there exists $C_5 > 0$ such that

$$\|\widetilde{u}\|_\infty^2 \leq C_5 \int_{[0,T)_{\mathbb{T}}} \left|u^\Delta(t)\right|^2 \Delta t. \tag{4.2}$$

It follows from (i), Theorem 2.23 and (4.2) that

$$\begin{aligned}
&\left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u^\sigma(t))-F(\sigma(t),\overline{u}))\Delta t\right| \\
&\quad\le \left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(\int_0^1 (\nabla F(\sigma(t),\overline{u}+s\widetilde{u}^\sigma(t)),\widetilde{u}^\sigma(t))\mathrm{d}s\right)\Delta t\right| \\
&\quad\le M_2\int_{[0,T)_{\mathbb{T}}}\left(\int_0^1 f^\sigma(t)|\overline{u}+s\widetilde{u}^\sigma(t)|^\alpha|\widetilde{u}^\sigma(t)|\mathrm{d}s\right)\Delta t+M_2\int_{[0,T)_{\mathbb{T}}}\left(\int_0^1 g^\sigma(t)|\widetilde{u}^\sigma(t)|\mathrm{d}s\right)\Delta t \\
&\quad\le 2M_2\left(|\overline{u}|^\alpha+\|\widetilde{u}\|_\infty^\alpha\right)\|\widetilde{u}\|_\infty\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\Delta t+M_2\|\widetilde{u}\|_\infty\int_{[0,T)_{\mathbb{T}}} g^\sigma(t)\Delta t \\
&\quad\le \frac{M_1}{4C_5}\|\widetilde{u}\|_\infty^2+\frac{4M_2^2C_5}{M_1}|\overline{u}|^{2\alpha}\left(\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\Delta t\right)^2 \\
&\qquad+2M_2\|\widetilde{u}\|_\infty^{\alpha+1}\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\mathrm{d}t+M_2\|\widetilde{u}\|_\infty\int_{[0,T)_{\mathbb{T}}} g^\sigma(t)\Delta t \\
&\quad\le \frac{M_1}{4}\int_{[0,T)_{\mathbb{T}}}\left|u^\Delta(t)\right|^2\Delta t+C_6|\overline{u}|^{2\alpha}+C_7\left(\int_{[0,T)_{\mathbb{T}}}\left|u^\Delta(t)\right|^2\Delta t\right)^{(\alpha+1)/2} \\
&\qquad+C_8\left(\int_{[0,T)_{\mathbb{T}}}\left|u^\Delta(t)\right|^2\Delta t\right)^{1/2}
\end{aligned}\tag{4.3}$$

for all $u\in H^1_{\Delta,T}$, where $C_6=(4M_2^2C_5/M_1)(\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\mathrm{d}t)^2$, $C_7=2M_2(C_5)^{(\alpha+1)/2}\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\Delta t$, $C_8=M_2(C_5)^{1/2}\int_{[0,T)_{\mathbb{T}}} g^\sigma(t)\Delta t$. Therefore, one has

$$\begin{aligned}
\varphi(u)&=\frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u^\Delta(t)\right|^2\Delta t+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u^\sigma(t))\Delta t \\
&=\frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u^\Delta(t)\right|^2\Delta t+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u^\sigma(t))-F(\sigma(t),\overline{u}))\Delta t \\
&\quad+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{u})\Delta t \\
&\ge \frac{1}{4}M_1\int_{[0,T)_{\mathbb{T}}}\left|u^\Delta(t)\right|^2\Delta t+|\overline{u}|^{2\alpha}\left(|\overline{u}|^{-2\alpha}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{u})\Delta t-C_6\right) \\
&\quad-C_7\left(\int_{[0,T)_{\mathbb{T}}}\left|u^\Delta(t)\right|^2\Delta t\right)^{(\alpha+1)/2}-C_8\left(\int_{[0,T)_{\mathbb{T}}}\left|u^\Delta(t)\right|^2\Delta t\right)^{1/2}
\end{aligned}\tag{4.4}$$

for all $u \in H^1_{\Delta,T}$. As $\|u\| \to \infty$ if and only if $\left(|\overline{u}|^2 + \int_0^T |u^{\Delta}(t)|^2 dt\right)^{1/2} \to \infty$, (4.4) and (ii) imply that

$$\varphi(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty. \tag{4.5}$$

By Lemma 3.8 and Theorem 3.4, φ has a minimum point on $H^1_{\Delta,T}$, which is a critical point of φ. From Theorem 3.3, problem (1.1) has at least one solution. □

Example 4.2. Let $\mathbb{T} = \mathbb{R}, T = 2\pi, N = 3$. Consider the damped vibration problem on time-scale $\mathbb{T}$

$$\begin{gathered} \ddot{u}(t) + \cos t\dot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, 2\pi], \\ u(0) - u(2\pi) = \dot{u}(0) - \dot{u}(2\pi) = 0, \end{gathered} \tag{4.6}$$

where $F(t,x) = (4/3 + t)|x|^{3/2}$.

Since, $F(t,x) = (4/3+t)|x|^{3/2}$, $w(t) = \cos t$, $\nabla F(t,x) = (3/2)(4/3+t)x|x|^{-1/2}$, $\alpha = 1/2$,

$$\begin{aligned} |\nabla F(t,x)| &\le \frac{3}{2}\left(\frac{4}{3}+t\right)|x|^{1/2}, \\ |x|^{-2\times 1/2}\int_0^{2\pi} e^{\sin t}\left(\frac{4}{3}+t\right)|x|^{3/2}dt &\ge |x|^{1/2}e^{-1}\int_0^{2\pi}\left(\frac{4}{3}+t\right)dt \\ &= \left(\frac{8}{3}\pi + 2\pi^2\right)e^{-1}|x|^{1/2} \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty, \end{aligned} \tag{4.7}$$

all conditions of Theorem 4.1 hold. According to Theorem 4.1, problem (4.6) has at least one solution. Moreover, 0 is not the solution of problem (4.6). Thus, problem (4.6) has at least one nontrivial solution.

Theorem 4.3. *Suppose that assumption* (A) *and the condition* (i) *of Theorem 4.1 hold. Assume that*

(iii) $|x|^{-2\alpha}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),x)\Delta t \to -\infty$ *as* $|x| \to \infty$.

Then problem (1.1) *has at least one solution.*

Firstly, we prove the following lemma.

Lemma 4.4. *Suppose that the conditions of Theorem 4.3 hold. Then φ satisfies P.S. condition.*

Proof. Let $\{u_n\} \subseteq H^1_{\Delta,T}$ be a P.S. sequence for φ, that is, $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. It follows from (i), Theorem 2.23 and (4.2) that

$$\begin{aligned}
&\left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u_n^\sigma(t)) - F(\sigma(t),\overline{u}_n))\Delta t\right| \\
&\quad\le \left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(\int_0^1 (\nabla F(\sigma(t),\overline{u}_n + s\widetilde{u}_n^\sigma(t)),\widetilde{u}_n^\sigma(t))\mathrm{d}s\right)\Delta t\right| \\
&\quad\le M_2\int_{[0,T)_{\mathbb{T}}}\left(\int_0^1 f^\sigma(t)|\overline{u}_n + s\widetilde{u}_n^\sigma(t)|^\alpha|\widetilde{u}_n^\sigma(t)|\mathrm{d}s\right)\Delta t + M_2\int_{[0,T)_{\mathbb{T}}}\left(\int_0^1 g^\sigma(t)|\widetilde{u}_n^\sigma(t)|\mathrm{d}s\right)\Delta t \\
&\quad\le 2M_2\left(|\overline{u}_n|^\alpha + \|\widetilde{u}_n\|_\infty^\alpha\right)\|\widetilde{u}_n\|_\infty\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\Delta t + M_2\|\widetilde{u}_n\|_\infty\int_{[0,T)_{\mathbb{T}}} g^\sigma(t)\Delta t \\
&\quad\le \frac{M_1}{4C_5}\|\widetilde{u}_n\|_\infty^2 + \frac{4M_2^2C_5}{M_1}|\overline{u}_n|^{2\alpha}\left(\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\Delta t\right)^2 \\
&\qquad + 2M_2\|\widetilde{u}_n\|_\infty^{\alpha+1}\int_{[0,T)_{\mathbb{T}}} f^\sigma(t)\mathrm{d}t + M_2\|\widetilde{u}_n\|_\infty\int_{[0,T)_{\mathbb{T}}} g^\sigma(t)\Delta t \\
&\quad\le \frac{M_1}{4}\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t + C_6|\overline{u}_n|^{2\alpha} \\
&\qquad + C_7\left(\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t\right)^{(\alpha+1)/2} + C_8\left(\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t\right)^{1/2}
\end{aligned} \tag{4.8}$$

for all n. By (4.8) and (i), one has

$$\begin{aligned}
\|\widetilde{u}_n\| &\ge \langle\varphi'(u_n),\widetilde{u}_n\rangle \\
&= \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_n^\Delta(t)\right|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\nabla F(\sigma(t),u_n^\sigma(t),\widetilde{u}_n(t))\Delta t \\
&\ge \frac{3M_1}{4}\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t - C_6|\overline{u}_n|^{2\alpha} - C_7\left(\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t\right)^{(\alpha+1)/2} \\
&\quad - C_8\left(\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t\right)^{1/2}
\end{aligned} \tag{4.9}$$

for all large n. It follows from (3.5) and (4.2) that

$$M_1\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\Delta t \le \|\widetilde{u}_n\|^2 \le M_2(1+TC_5)\int_{[0,T)_{\mathbb{T}}}\left|u_n^\Delta(t)\right|^2\mathrm{d}t. \tag{4.10}$$

The inequalities (4.9) and (4.10) imply that

$$C_9|\overline{u}_n|^{\alpha} \geq \left(\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \Delta t\right)^{1/2} - C_{10} \tag{4.11}$$

for all large n and some positive constants C_9 and C_{10}. Similar to the proof of Theorem 4.1, one has

$$\begin{aligned}
&\left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u_n^{\sigma}(t)) - F(\sigma(t),\overline{u}_n))\Delta t\right| \\
&\quad\leq \frac{M_1}{4}\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \mathrm{d}t + C_6|\overline{u}_n|^{2\alpha} \\
&\qquad + C_7\left(\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \Delta t\right)^{(\alpha+1)/2} + C_8\left(\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \Delta t\right)^{1/2}
\end{aligned} \tag{4.12}$$

for all n. By the boundedness of $\{\varphi(u_n)\}$, (4.11) and (4.12), there exists constant C_{11} such that

$$\begin{aligned}
C_{11} &\leq \varphi(u_n) \\
&= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_n^{\Delta}(t)\right|^2 \mathrm{d}t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u_n^{\sigma}(t)) - F(\sigma(t),\overline{u}_n))\Delta t \\
&\quad + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{u}_n)\Delta t \\
&\leq \frac{3}{4}M_2\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \Delta t + C_6|\overline{u}_n|^{2\alpha} + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{u}_n)\Delta t \\
&\quad + C_7\left(\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \Delta t\right)^{(\alpha+1)/2} + C_8\left(\int_{[0,T)_{\mathbb{T}}} \left|u_n^{\Delta}(t)\right|^2 \Delta t\right)^{1/2} \\
&\leq |\overline{u}_n|^{2\alpha}\left(|\overline{u}_n|^{-2\alpha}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{u}_n)\Delta t + C_{12}\right)
\end{aligned} \tag{4.13}$$

for all large n and some constant C_{12}. It follows from (4.13) and (iii) that $\{|\overline{u}_n|\}$ is bounded. Hence $\{u_n\}$ is bounded in $H^1_{\Delta,T}$ by (4.10) and (4.11). Therefore, there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that

$$u_n \rightharpoonup u \quad \text{in } H^1_{\Delta,T}. \tag{4.14}$$

By Theorem 2.25, one has

$$u_n \longrightarrow u \quad \text{in } C\left([0,T]_{\mathbb{T}},\mathbb{R}^N\right). \tag{4.15}$$

On the other hand, one has

$$\begin{aligned}\langle\varphi'(u_n)-\varphi'(u),u_n-u\rangle &= \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_n^{\Delta}(t)-u^{\Delta}(t)\right|^2\Delta t \\ &\quad+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(\nabla F(\sigma(t),u_n^{\sigma}(t))-\nabla F(\sigma(t),u^{\sigma}(t)),u_n^{\sigma}(t)-u^{\sigma}(t))\Delta t.\end{aligned} \tag{4.16}$$

From (4.14), (4.15), (4.16), and (A), it follows that $u_n \to u$ in $H^1_{\Delta,T}$. Thus, φ satisfies P.S. condition. □

Now, we prove Theorem 4.3.

Proof. Let W be the subspace of $H^1_{\Delta,T}$ given by

$$W=\left\{u\in H^1_{\Delta,T}:\int_{[0,T)_{\mathbb{T}}} u(t)\Delta t=0\right\}, \tag{4.17}$$

then, $H^1_{\Delta,T}=\mathbb{R}^N\oplus W$. We show that

$$\varphi(u)\longrightarrow+\infty\quad\text{as } u\in W,\ \|u\|\longrightarrow\infty. \tag{4.18}$$

Indeed, for $u\in W$, then $\overline{u}=0$, similar to the proof of Theorem 4.1, one has

$$\begin{aligned}&\left|\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u^{\sigma}(t))-F(\sigma(t),0))\Delta t\right| \\ &\quad\le\frac{M_1}{4}\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^2\Delta t+C_7\left(\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^2\Delta t\right)^{(\alpha+1)/2}+C_8\left(\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^2\Delta t\right)^{1/2}.\end{aligned} \tag{4.19}$$

It follows from (4.19) that

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u^{\Delta}(t)\right|^2\Delta t+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)(F(\sigma(t),u^{\sigma}(t))-F(\sigma(t),0))\Delta t \\ &\quad+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),0)\Delta t \\ &\ge\frac{M_1}{4}\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^2\Delta t-C_7\left(\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^2\Delta t\right)^{(\alpha+1)/2}-C_8\left(\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^2\Delta t\right)^{1/2} \\ &\quad+\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),0)\Delta t\end{aligned} \tag{4.20}$$

for all $u \in W$. By Theorem 2.23 and Theorem 3.1, one has

$$\|u\| \longrightarrow \infty \Longleftrightarrow \left\|u^{\Delta}\right\|_{L^2} \longrightarrow \infty \tag{4.21}$$

on W. Hence (4.18) follows from (4.20).

On the other hand, by (iii), one has

$$\begin{aligned}\varphi(u) &= \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u)\Delta t \\ &\leq |u|^{2\alpha}\left(|u|^{-2\alpha}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),u)\Delta t\right) \longrightarrow -\infty\end{aligned} \tag{4.22}$$

as $u \in \mathbb{R}^N$ and $|u| \to \infty$. By Theorem 3.3, Lemmas 3.9 and 4.4, problem (1.1) has at least one solution. □

Example 4.5. Let $\mathbb{T} = \mathbb{Z}, T = 20, N = 5$. Consider the damped vibration problem on time-scale $\mathbb{T}$

$$\begin{gathered}\Delta^2(t) + w(t)\Delta u(t+1) = \nabla F(t+1, u(t+1)), \quad t \in [0,19] \cap \mathbb{Z}, \\ u(0) - u(20) = \Delta u(0) - \Delta u(20) = 0,\end{gathered} \tag{4.23}$$

where $F(t,x) = -|x|^{5/3} + ((1,1,2,1,0),x)$ and

$$w(t) = \begin{cases} -\dfrac{1}{2}, & t \in [0,18] \cap \mathbb{Z}, \\ 2^{18} - 1, & t = 19. \end{cases} \tag{4.24}$$

Since, $F(t,x) = -|x|^{5/3} + ((1,1,2,1,0),x)$, $\nabla F(t,x) = -(5/3)x|x|^{-1/3} + (1,1,2,1,0)$, $\alpha = 2/3$, $e_w(t,0) = \prod_{s=0}^{t-1}(1 + w(s))$, $e_w(20,0) = 1$,

$$\begin{aligned}&|\nabla F(t,x)| \leq \frac{5}{3}|x|^{2/3} + \sqrt{7}, \\ &|x|^{-2\times(2/3)}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),x)\Delta t \\ &\quad = |x|^{-4/3}\left(-|x|^{5/3} + ((1,1,2,1,0),x)\right)\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\Delta t \\ &\quad \leq -|x|^{1/3}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\Delta t + \sqrt{7}|x|^{-1/3}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\Delta t \longrightarrow -\infty \quad \text{as } |x| \longrightarrow \infty,\end{aligned} \tag{4.25}$$

all conditions of Theorem 4.3 hold. According to Theorem 4.3, problem (4.23) has at least one solution. Moreover, 0 is not the solution of problem (4.23). Thus, problem (4.23) has at least one nontrivial solution.

Theorem 4.6. *Suppose that assumption* (A) *and the following condition are satisfied.*

(iv) $F(t,\cdot)$ *is convex for* Δ*-a.e.* $t \in [0,T]_{\mathbb{T}}$ *and that*

$$\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),x)\Delta t \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty. \tag{4.26}$$

Then problem (1.1) *has at least one solution which minimizes the function* φ.

Proof. By assumption, the function $G : \mathbb{R}^N \to \mathbb{R}$ defined by

$$G(x) = \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),x)\Delta t \tag{4.27}$$

has a minimum at some point $\overline{x}$ for which

$$\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\nabla F(\sigma(t),\overline{x})\Delta t = 0. \tag{4.28}$$

Let $\{u_k\}$ be a minimizing sequence for φ. From Lemma 3.10 and (4.28), one has

$$\begin{aligned}
\varphi(u_k) &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_k^{\Delta}(t)\right|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(F\left(\sigma(t),u_k^{\sigma}(t)\right) - F(\sigma(t),\overline{x})\right)\Delta t \\
&\quad + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{x})\Delta t \\
&\geq \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_k^{\Delta}(t)\right|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{x})\Delta t \\
&\quad + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(\nabla F(\sigma(t),\overline{x}), u_k^{\sigma}(t) - \overline{x}\right)\Delta t \\
&= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_k^{\Delta}(t)\right|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{x})\Delta t \\
&\quad + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left(\nabla F(\sigma(t),\overline{x}), \tilde{u}_k^{\sigma}(t)\right)\Delta t,
\end{aligned} \tag{4.29}$$

where $\widetilde{u}_k(t) = u_k(t) - \overline{u}_k$, $\overline{u}_k = (1/T)\int_{[0,T)_{\mathbb{T}}} u_k(t)\Delta t$. By (4.29), (A) and Theorem 2.23, we obtain

$$\begin{aligned}\varphi(u_k) &\geq \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_k^{\Delta}(t)\right|^2\Delta t + \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F(\sigma(t),\overline{x})\Delta t \\ &\quad - M_2\left(\int_{[0,T)_{\mathbb{T}}} |\nabla F(\sigma(t),\overline{x})|\Delta t\right)\|\widetilde{u}_k\|_\infty \\ &\geq \frac{1}{2}M_1\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_k^{\Delta}(t)\right|^2\Delta t - C_{13} - C_{14}\left(\int_{[0,T)_{\mathbb{T}}}\left|u_k^{\Delta}(t)\right|^2\Delta t\right)^{1/2}\end{aligned} \tag{4.30}$$

for some positive constants C_{13} and C_{14}. Thus, by (4.30), there exists $C_{15} > 0$ such that

$$\int_{[0,T)_{\mathbb{T}}}\left|u_k^{\Delta}(t)\right|^2\Delta t \leq C_{15}. \tag{4.31}$$

Theorem 2.23 and (4.31) imply that there exists $C_{16} > 0$ such that

$$\|\widetilde{u}_k\|_\infty \leq C_{16}. \tag{4.32}$$

By (iv), one has

$$\begin{aligned}F\left(\sigma(t),\frac{\overline{u}_k}{2}\right) &= F\left(\sigma(t),\frac{u_k^{\sigma}(t) - \widetilde{u}_k^{\sigma}(t)}{2}\right) \\ &\leq \frac{1}{2}F\left(\sigma(t),u_k^{\sigma}(t)\right) + \frac{1}{2}F\left(\sigma(t),-\widetilde{u}_k^{\sigma}(t)\right)\end{aligned} \tag{4.33}$$

for Δ-a.e. $t \in [0,T)_{\mathbb{T}}$ and all $k \in \mathbb{N}$. It follows from (3.9) and (4.33) that

$$\begin{aligned}\varphi(u_k) &\geq \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} e_w(t,0)\left|u_k^{\Delta}(t)\right|^2\Delta t + 2\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F\left(\sigma(t),\frac{\overline{u}_k}{2}\right)\Delta t \\ &\quad - \int_{[0,T)_{\mathbb{T}}} e_w(t,0)F\left(\sigma(t),-\widetilde{u}_k^{\sigma}(t)\right)\Delta t.\end{aligned} \tag{4.34}$$

Combining (4.32) and (4.34), there exists $C_{17} > 0$ such that

$$\varphi(u_k) \geq 2\int_{[0,T)_{\mathbb{T}}} e_w(t,0)F\left(\sigma(t),\frac{\overline{u}_k}{2}\right)\Delta t - C_{17}. \tag{4.35}$$

Therefore, by (4.35) and (iv), $\{\overline{u}_k\}$ is bounded. Hence $\{u_k\}$ is bounded in $H^1_{\Delta,T}$ by Theorem 2.23 and (4.31). By Lemma 3.8 and Theorem 3.4, φ has a minimum point on $H^1_{\Delta,T}$, which is a critical point of φ. Hence, problem (1.1) has at least one solution which minimizes the function φ. □

Example 4.7. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{T} = \{2^k : k \in \mathbb{N}_0\}, T = 64, N = 1$. Consider the damped vibration problem on time-scale $\mathbb{T}$

$$u^{\Delta^2}(t) + w(t)u^{\Delta}(2t) = \nabla F(2t, u(2t)), \quad t \in \{1, 2, 4, 8, 16, 32\},$$
$$u(0) - u(64) = u^{\Delta}(0) - u^{\Delta}(64) = 0, \tag{4.36}$$

where $F(t, x) = x^2 + 2x$ and

$$w(t) = \begin{cases} -\dfrac{1}{2t}, & t \in \{1, 2, 4, 8, 16\}, \\ \dfrac{31}{32}, & t = 32. \end{cases} \tag{4.37}$$

Since, $F(t, x) = x^2 + 2x, e_w(t, 0) = \prod_{s \in \mathbb{T} \cap (0,t)}(1 + sw(s)), e_w(64, 0) = 1$, all conditions of Theorem 4.6 hold. According to Theorem 4.6, problem (4.36) has at least one solution. Moreover, 0 is not the solution of problem (4.36). Thus, problem (4.36) has at least one nontrivial solution.

5. Conclusion

In this paper, we present a new approach via variational methods and critical point theory to obtain the existence of solutions for a class of damped vibration problems on time-scales. Three existence results are obtained. Three examples are presented to illustrate the feasibility and effectiveness of our results.

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