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Research Article

On Homoclinic Solutions of a Semilinear p-Laplacian Difference Equation with Periodic Coefficients

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We study the existence of homoclinic solutions for semilinear p-Laplacian difference equations with periodic coefficients. The proof of the main result is based on Brezis-Nirenberg's Mountain Pass Theorem. Several examples and remarks are given.

1. Introduction

This paper is concerned with the study of the existence of homoclinic solutions for the p-Laplacian difference equation

$$\Delta_p^2 u(k-1) - V(k)u(k)|u(k)|^{q-2} + \lambda f(k, u(k)) = 0, \quad u(t) \to 0, \ |t| \to \infty, \tag{1.1}$$

where $u(k), k \in \mathbb{Z}$ is a sequence or real numbers, Δ is the difference operator $\Delta u(k) = u(k+1) - u(k)$,

$$\Delta_p^2 u(k-1) = \Delta u(k)|\Delta u(k)|^{p-2} - \Delta u(k-1)|\Delta u(k-1)|^{p-2} \tag{1.2}$$

is referred to as the p-Laplacian difference operator, and functions $V(k)$ and $f(k,x)$ are T-periodic in k and satisfy suitable conditions.

In the theory of differential equations, a trajectory $x(t)$, which is asymptotic to a constant as $|t| \to \infty$ is called doubly asymptotic or homoclinic orbit. The notion of homoclinic orbit is introduced by Poincaré [1] for continuous Hamiltonian systems.

Recently, there is a large literature on the use of variational methods to the existence of homoclinic or heteroclinic orbits of Hamiltonian systems; see [2–7] and the references therein.

In the recent paper of Li [8] a unified approach to the existence of homoclinic orbits for some classes of ODE's with periodic potentials is presented. It is based on the Brezis and Nirenberg's mountain-pass theorem [9]. In this paper we extend this approach to homoclinic orbits for discrete p-Laplacian type equations.

Discrete boundary value problems have been intensively studied in the last decade. The studies of such kind of problems can be placed at the interface of certain mathematical fields, such as nonlinear differential equations and numerical analysis. On the other hand, they are strongly motivated by their applicability to mathematical physics and biology.

The variational approach to the study of various problems for difference equations has been recently applied in, among others, the papers of Agarwal et al. [10], Cabada et al. [11], Chen and Fang [12], Fang and Zhao [13], Jiang and Zhou [14], Ma and Guo [15], Mihăilescu et al. [16], Kristály et al. [17].

Along the paper, given two integer numbers $a < b$, we will denote $[a,b] = \{a,\dots,b\}$. Moreover, for every $p > 1$, we consider the following function

$$\varphi_p(t) = t|t|^{p-2}, \qquad \Phi_p(t) = \frac{|t|^p}{p}. \tag{1.3}$$

It is obvious that $\Phi_p'(t) = \varphi_p(t)$ for all $t \in \mathbb{R}$ and $p \neq 0$. Moreover

$$\Delta_p^2 u(k-1) = \Delta\left(\varphi_p(\Delta u(k-1))\right). \tag{1.4}$$

Suppose that

$$V : \mathbb{Z} \to \mathbb{R} \text{ is a } T\text{-periodic positive potential} \tag{1.5}$$

$$0 < V_0 = \min\{V(0),\dots,V(T-1)\} \le \max\{V(0),\dots,V(T-1)\} = V_1. \tag{1.6}$$

Denote

$$A(u) = \sum_{k\in\mathbb{Z}} \Phi_p(\Delta u(k-1)) + \sum_{k\in\mathbb{Z}} V(k)\Phi_q(u(k)). \tag{1.7}$$

Let us consider functions f satisfying the following assumptions.

(F_1) The function $f(k,t)$ is continuous in $t \in \mathbb{R}$ and T-periodic in k.

(F_2) The potential function $F(k,t)$ of $f(k,t)$

$$F(k,t) = \int_0^t f(k,s)ds \tag{1.8}$$

satisfies the Rabinowitz's type condition:

There exist $\mu > p \geq q > 1$ and $s > 0$ such that

$$\begin{aligned} &\mu F(k,t) \leq t f(k,t), \qquad k \in \mathbb{Z},\ t \neq 0, \\ &F(k,t) > 0, \qquad \forall k \in \mathbb{Z},\ \text{for } t \geq s > 0. \end{aligned} \tag{1.9}$$

(F_3) $f(k,t) = o(|t|^{q-1})$ as $|t| \to 0$.

Further we consider the semilinear eigenvalue p-Laplacian difference equation

$$\Delta_p^2 u(k-1) - V(k)u(k)|u(k)|^{q-2} + \lambda f(k,u(k)) = 0, \tag{1.10}$$

where $\lambda > 0$ and we are looking for its homoclinic solutions, that is, solutions of (1.10) such that $u(k) \to 0$ as $|k| \to \infty$.

In order to obtain homoclinic solutions of (1.10), we will use variational approach and Brezis-Nirenberg mountain pass theorem [9].

To this end, consider the functional $J : \ell^q \to \mathbb{R}$, defined as

$$J(u) = A(u) - \lambda \sum_{k\in\mathbb{Z}} F(k,u(k)). \tag{1.11}$$

Our main result is the following.

Theorem 1.1. *Suppose that the function $V : \mathbb{Z} \to \mathbb{R}$ is positive and T-periodic and the functions $f(k,\cdot) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ satisfy assumptions* (F_1)–(F_3). *Then, for each $\lambda > 0$,* (1.10) *has a nonzero homoclinic solution $u \in \ell^q$, which is a critical point of the functional $J : \ell^q \to \mathbb{R}$.*

Moreover, given a nontrivial solution u of problem (1.10)*, there exist $k_\pm$ two integer numbers such that for all $k > k_+$ and $k < k_-$, the sequence $u(k)$ is strictly monotone.*

The paper is organized as follows. In Section 2, we present the proof of the main result and discuss the optimality of the condition (F_2). In Section 3, we give some examples of equations modeled by this kind of problems and present some additional remarks.

2. Proof of the Main Result

Let $u = \{u(k) : k \in \mathbb{Z}\}$ be a sequence, $q > 1$ and

$$\ell^q = \left\{ u : |u|_q^q = \sum_{k\in\mathbb{Z}} |u(k)|^q < \infty \right\},$$
$$\ell^\infty = \left\{ u : |u|_\infty = \sup_{k\in\mathbb{Z}} |u(k)| < \infty \right\}. \tag{2.1}$$

It is well known that if $0 < q \le p$, then $\ell^q \subseteq \ell^p$. Indeed, if $\sum_{k\in\mathbb{Z}} |u(k)|^q < \infty$, there exists a positive integer number R, such that for all k satisfying $|k| > R$ it is verified that $|u(k)|^q < 1$ and, as consequence, $|u(k)|^p \le |u(k)|^q$ and the series $\sum_{k\in\mathbb{Z}} |u(k)|^p$ is convergent too.

Consider now the functional $J : \ell^q \to \mathbb{R}$, defined as

$$J(u) = A(u) - \lambda \sum_{k\in\mathbb{Z}} F(k, u(k)), \tag{2.2}$$

with A given in (1.7) and F defined in (1.8).

We have the following result.

Lemma 2.1. *The functional $J : \ell^q \to \mathbb{R}$ is well defined, C^1-differentiable, and its critical points are solutions of* (1.10).

Proof. By using the inequality for nonnegative a and b and $p > 1$

$$\left(\frac{a+b}{2} \right)^p \le \frac{a^p + b^p}{2}, \tag{2.3}$$

and the inclusion $\ell^q \subseteq \ell^p$ for $1 < q \le p$, it follows that

$$\sum_{k\in\mathbb{Z}} |\Delta u(k-1)|^p \le 2^{p-1} \sum_{k\in\mathbb{Z}} \left(|u(k)|^p + |u(k-1)|^p \right) = 2^p \sum_{k\in\mathbb{Z}} |u(k)|^p < \infty. \tag{2.4}$$

Now, let us see that the series $\sum_{k\in\mathbb{Z}} F(k, u(k))$ is convergent: by using (F_3), it follows that there exist $\delta \in (0,1)$ and sufficiently large N such that

$$F(k, u(k)) < |u(k)|^q \quad \text{for } |u(k)|^q < \delta < 1, \ \ |k| > N. \tag{2.5}$$

Then, the series $\sum_{k\in\mathbb{Z}} F(k, u(k))$ is convergent and the functional J is well defined on ℓ^q.

It is Gâteaux differentiable and for $v \in \ell^q$:

$$\begin{aligned}\langle J'(u), v\rangle &= \lim_{t\to 0}\frac{J(u+tv)-J(u)}{t}\\ &= \sum_{k\in\mathbb{Z}}\Delta u(k-1)|\Delta u(k-1)|^{p-2}\Delta v(k-1)\\ &\quad+\sum_{k\in\mathbb{Z}}V(k)u(k)|u(k)|^{q-2}v(k)-\lambda\sum_{k\in\mathbb{Z}}f(k,u(k))v(k)\end{aligned} \tag{2.6}$$

and partial derivatives

$$\frac{\partial J(u)}{\partial u(k)} = -\Delta_p^2 u(k-1) + V(k)u(k)|u(k)|^{q-2} - \lambda f(k,u(k)), \tag{2.7}$$

are continuous functions.

Moreover the functional J is continuously Fréchet-differentiable in ℓ^q. It is clear, by (2.7), that the critical points of J are solutions of (1.10).

To obtain homoclinic solutions of (1.10) we will use mountain-pass theorem of Brezis and Nirenberg [9]. Recall its statement. Let X be a Banach space with norm $\|\cdot\|$, and $I : X \to \mathbb{R}$ be a C^1-functional. I satisfies the $(PS)_c$ *condition* if every sequence (x_k) of X such that

$$I(x_k) \longrightarrow c\,, \qquad I'(x_k) \longrightarrow 0, \tag{2.8}$$

has a convergent subsequence. A sequence $(x_k) \subset X$ such that (2.8) holds is referred to as $(PS)_c$*-sequence*. □

Theorem 2.2 (mountain-pass theorem, Brezis and Nirenberg [9]). *Let X be a Banach space with norm $\|\cdot\|$, $I \in C^1(X,\mathbb{R})$ and suppose that there exist $r>0$, $\alpha>0$ and $e\in X$ such that $\|e\|>r$*

(i) *$I(x) \geq \alpha$ if $\|x\| = r$,*

(ii) *$I(e) < 0$.*

Let $c = \inf_{\gamma\in\Gamma}\{\max_{0\leq t\leq 1} I(\gamma(t))\} \geq \alpha$, where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0)=0,\ \gamma(1)=e\}. \tag{2.9}$$

Then, there exists a $(PS)_c$ sequence for I. Moreover, if I satisfies the $(PS)_c$ condition, then c is a critical value of I, that is, there exists $u_0 \in X$ such that $I(u_0)=c$ and $I'(u_0)=0$.

Note that, by assumption (1.5), the norm $|\cdot|_q$ in ℓ^q is equivalent to

$$\|u\|_q^q = \frac{1}{q}\sum_{k\in\mathbb{Z}}V(k)|u(k)|^q. \tag{2.10}$$

Lemma 2.3. *Suppose that* (F_1)–(F_3) *hold, then there exist* $\rho > 0$, $\alpha > 0$ *and* $e \in \ell^q$ *such that* $\|e\|_q > \rho$ *and*

(1) $J(u) \geq \alpha$ *if* $\|u\|_q = \rho$,

(2) $J(e) < 0$.

Proof. By (F_3), there exists $\delta \in (0,1)$ such that

$$F(k,t) \leq \frac{V_0}{2q\lambda}|t|^q \quad \text{if } |t| \leq \delta. \tag{2.11}$$

Let $\rho = (V_0/q)^{1/q}\delta$ (V_0 defined in (1.6)), then, for u, $\|u\|_q = \rho$,

$$\begin{aligned} \frac{V_0}{q}\delta^q = \rho^q = \|u\|_q^q &= \frac{1}{q}\sum_{k\in\mathbb{Z}} V(k)|u(k)|^q \\ &\geq \frac{V_0}{q}|u(k)|^q \quad \text{for all } k \in \mathbb{Z}, \end{aligned} \tag{2.12}$$

which implies that $|u(k)| \leq \delta$ for all $k \in \mathbb{Z}$.

Hence, by (2.11)

$$\begin{aligned} \sum_{k\in\mathbb{Z}} F(k,u(k)) &\leq \frac{V_0}{2q\lambda}\sum_{k\in\mathbb{Z}}|u(k)|^q \\ &\leq \frac{1}{2q\lambda}\sum_{k\in\mathbb{Z}} V(k)|u(k)|^q = \frac{1}{2\lambda}\|u\|_q^q \\ J(u) &= A(u) - \lambda\sum_{k\in\mathbb{Z}} F(k,u(k)) \\ &\geq \|u\|_q^q - \frac{1}{2}\|u\|_q^q = \frac{1}{2}\|u\|_q^q = \frac{\rho^q}{2} > 0. \end{aligned} \tag{2.13}$$

By (F_2), there exist $c_1, c_2 > 0$ such that $F(k,t) \geq c_1\ t^\mu - c_2$ for all $t > 0$ and $k \in \mathbb{Z}$.

Take $v \in \ell^q$, $v(0) = a > 0$, $v(k) = 0$ if $k \neq 0$. Then, since $\mu > p \geq q$

$$\begin{aligned} J(\kappa v) &= A(\kappa v) - \lambda\sum_{k\in\mathbb{Z}} F(k,\kappa v(k)) \\ &\leq \frac{2}{p}\kappa^p a^p + V(0)\frac{\kappa^q a^q}{q} - \lambda(c_1\kappa^\mu a^\mu - c_2) \\ &< 0, \end{aligned} \tag{2.14}$$

if κ is sufficiently large.

Then, we can take κ large enough, such that for $e = \kappa v$, $\|e\|_q^q = V(0)(\kappa^q\ a^q/q) > \rho^q$ and (2.14) holds. ∎

Lemma 2.4. *Suppose that the assumptions of Lemma 2.3 hold. Then, there exists $c > 0$ and a ℓ^q-bounded $(PS)_c$ sequence for J.*

Proof. By Lemma 2.3 and Theorem 2.2 there exists a sequence $(u_m) \subset \ell^q$ such that

$$J(u_m) \longrightarrow c\ , \qquad J'(u_m) \longrightarrow 0, \tag{2.15}$$

where

$$\begin{gathered} c = \inf_{\gamma\in\Gamma}\left\{\max_{t\in[0,1]} J(\gamma(t))\right\}, \\ \Gamma = \{\gamma \in C([0,1],\ell^q) : \gamma(0) = 0,\ \ \gamma(1) = e\}, \end{gathered} \tag{2.16}$$

and e is defined in the proof of Lemma 2.3.

We will prove that the sequence (u_m) is bounded in ℓ^q. We have for $\mu > p \geq q$

$$\begin{aligned} \langle J'(u_m), u_m\rangle = {} & \sum_{k\in\mathbb{Z}} |\Delta u_m(k-1)|^p \\ & + \sum_{k\in\mathbb{Z}} V(k)|u_m(k)|^q - \lambda \sum_{k\in\mathbb{Z}} f(k,u_m(k))u_m(k), \end{aligned} \tag{2.17}$$

and, by (F_2),

$$\begin{aligned} & \mu J(u_m) - \langle J'(u_m), u_m\rangle \\ & \quad = \left(\frac{\mu}{p} - 1\right)\sum_{k\in\mathbb{Z}} |\Delta u_m(k-1)|^p + \left(\frac{\mu}{q} - 1\right)\sum_{k\in\mathbb{Z}} V(k)|u_m(k)|^q \\ & \qquad - \lambda\sum_{k\in\mathbb{Z}}(\mu F(k,u_m(k)) - f(k,u_m(k))u_m(k)) \\ & \quad \geq \left(\frac{\mu}{q} - 1\right) q\|u_m\|_q^q = (\mu - q)\|u_m\|_q^q, \end{aligned} \tag{2.18}$$

which implies that the sequence u_m is bounded in ℓ^q. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. For any $m \in \mathbb{N}$, the sequence $\{|u_m\ (k)|,\ k \in Z\}$, given in Lemma 2.4, is bounded in ℓ^q and, in consequence, $|u_m(k)| \to 0$ as $|k| \to \infty$. Let $|u_m(k)|$ takes its maximum at $k_m \in \mathbb{Z}$. There exists a unique $j_m \in \mathbb{Z}$, such that $j_m T \leq k_m < (j_m + 1)T$ and let $w_m(k) = u_m(k + j_m T)$. Then $|w_m(k)|$ takes its maximum at $i_m = k_m - j_m T \in [0, T-1]$. By the T-periodicity of V and $f(\cdot, t)$, it follows that

$$\begin{gathered} \|u_m\|_q = \|w_m\|_q, \\ J(u_m) = J(w_m). \end{gathered} \tag{2.19}$$

Since (u_m) is bounded in ℓ^q, there exists $w \in \ell^q$, such that $w_m \rightharpoonup w$ weakly in ℓ^q. The weak convergence in ℓ^q implies that $w_m(k) \to w(k)$ for every $k \in \mathbb{Z}$. Indeed, if we take a test function $v_k \in \ell^q$, $v_k(k) = 1$, $v_k(j) = 0$ if $j \neq k$, then

$$w_m(k) = \langle w_m, v_k \rangle \longrightarrow \langle w, v_k \rangle = w(k). \tag{2.20}$$

Moreover, for any $v \in \ell^q$

$$\begin{aligned} \left|\left\langle J'(w_m), v\right\rangle\right| &= \left|\left\langle J'(u_m), v(\cdot + j_m T)\right\rangle\right| \\ &\le \left\|J'(u_m)\right\|_* \left\|v(\cdot + j_m T)\right\|_q \\ &= \left\|J'(u_m)\right\|_* \|v\|_q \longrightarrow 0, \end{aligned} \tag{2.21}$$

which implies that $J'(w_m) \to 0$, which means that for every $v \in \ell^q$,

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \varphi_p(\Delta w_m(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} V(k)\varphi_q(w_m(k))v(k) \\ &\quad - \lambda \sum_{k\in\mathbb{Z}} f(k, w_m(k))v(k) \longrightarrow 0, \quad \forall k \in \mathbb{Z}, \text{ as } m \longrightarrow \infty. \end{aligned} \tag{2.22}$$

Let us take $v \in \ell^q$ with compact support, that is, there exist $a, b \in \mathbb{Z}$, $a < b$ such that $v(k) = 0$ if $k \in \mathbb{Z} \setminus [a, b]$ and $v(k) \neq 0$ if $k \in \{a+1, b-1\}$. The set of such elements ℓ_0^q is dense in ℓ^q because if $v \in \ell^q$ and $v_k \in \ell_0^q$ is such that $v_k(j) = 0$ if $|j| \ge k+1$, $v_k(j) = v(j)$ if $|j| \le k$, then $\|v - v_k\|_q \to 0$ as $k \to \infty$. Taking $v \in \ell_0^q$ in (2.22), due to the finite sums and the continuity of functions $f(k, \cdot)$, we obtain, passing to a limit, that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \varphi_p(\Delta w(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} V(k)\varphi_q(w(k))v(k) \\ &\quad - \lambda \sum_{k\in\mathbb{Z}} f(k, w(k))v(k) = 0, \quad \forall v \in l_0^q. \end{aligned} \tag{2.23}$$

From the density of l_0^q in ℓ^q, we deduce that the previous equality is fulfilled for all $v \in \ell^q$ and, in consequence, w is a critical point of the functional J, that is, w is a solution of (1.10).

It remains to show that $w \neq 0$.

Assuming, on the contrary, that $w = 0$, we conclude that

$$|u_m|_\infty = |w_m|_\infty = \max\{|w_m(k)| : k \in \mathbb{Z}\} \longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty. \tag{2.24}$$

By (F_3), for a given $\varepsilon > 0$, there exists $\delta > 0$, such that if $|x| < \delta$ then, for every $k \in [0, T-1]$, the following inequalities holds:

$$\begin{aligned} |F(k, x)| &\le \varepsilon |x|^q, \\ \left|f(k, x)x\right| &\le \varepsilon |x|^q. \end{aligned} \tag{2.25}$$

By (2.24), for every $k \in [0, T-1]$, there exists a positive integer M_k such that for all $m > M_k$ it follows that $|w_m(k)| < \delta$. Since the maximum value of $|w_m|$ is attained at $i_m \in [0, T-1]$, it follows that for $m > M = \max\{M_k : k \in [0, T-1]\}$ and every $k \in \mathbb{Z}$

$$|w_m(k)| \le |w_m(i_m)| \le \delta. \tag{2.26}$$

Then, by (2.25), for $m > M$ and every $k \in \mathbb{Z}$:

$$\begin{gathered} |F(k, w_m(k))| \le \varepsilon |w_m(k)|^q, \\ \left| f(k, w_m(k)) w_m(k) \right| \le \varepsilon |w_m(k)|^q, \end{gathered} \tag{2.27}$$

which implies that

$$\begin{aligned} 0 \le qJ(w_m) &= \frac{q}{p} \sum_{k \in \mathbb{Z}} |\Delta w_m(k-1)|^p + \sum_{k \in \mathbb{Z}} V(k) |w_m(k)|^q - \lambda \sum_{k \in \mathbb{Z}} qF(k, w_m(k)) \\ &\le \sum_{k \in \mathbb{Z}} |\Delta w_m(k-1)|^p + \sum_{k \in \mathbb{Z}} V(k) |w_m(k)|^q - \lambda \sum_{k \in \mathbb{Z}} f(k, w_m(k)) w_m(k) \\ &\quad - \lambda \sum_{k \in \mathbb{Z}} \left(qF(k, w_m(k)) - f(k, w_m(k)) w_m(k) \right) \\ &\le \left\langle J'(w_m), w_m \right\rangle + \lambda \left(q\varepsilon |w_m|_q^q + \varepsilon |w_m|_q^q \right) \\ &\le \left\| J'(w_m) \right\|_* \|w_m\|_q + \lambda \varepsilon \frac{q+1}{V_0} \, \|w_m\|_q^q. \end{aligned} \tag{2.28}$$

Since $\|w_m\|$ is bounded in ℓ^q, $J'(w_m) \to 0$ and ε is arbitrary, by (2.28) we obtain a contradiction with $J(w_m) = J(u_m) \to c > 0$. The proof of the first part is complete.

Now, let u be a nonzero homoclinic solution of problem (1.10). Assume that it attains positive local maximums and/or negative local minimums at infinitely many points k_n. In particular we can assume that $\{|k_n|\} \to \infty$. In consequence $\Delta_p^2 u(k_n - 1) u(k_n) \le 0$ and $u(k_n) \to 0$.

From this, multiplying in (1.10) by $u(k_n)/|u(k_n)|^q$, we have

$$\lambda \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^q} \ge \frac{\Delta_p^2 u(k_n - 1) u(k_n)}{|u(k_n)|^q} + \lambda \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^q} = V(k_n) \ge V_0 > 0. \tag{2.29}$$

By means of condition (F_3) we arrive at the following contradiction:

$$0 = \lambda \lim_{n \to \infty} \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^q} \ge V_0 > 0. \tag{2.30}$$

Suppose now that function u vanishes at infinitely many points l_n. From condition (F_3) we conclude that $\Delta_p^2 u(l_n - 1) = 0$ and, in consequence, $u(l_n - 1) u(l_n + 1) < 0$. Therefore it has an unbounded sequence of positive local maximums and negative local minimums, in contradiction with the previous assertion.

As a direct consequence of the two previous properties, we deduce that, for $|k|$ large enough, function u has constant sign and it is strictly monotone. □

To illustrate the optimality of the obtained results, we present in the sequel an example in which it is pointed out that condition (F_2) cannot be removed to deduce the existence result proved in Theorem 1.1.

Example 2.5. Let $W(k) > 0$ be a T-periodic sequence, $W_1 = \max\{W(k) : k \in [0, T-1]\}$, $p \geq q > 1$ and $r > q$ be fixed. Consider problem (1.10) with

$$f(k,t) = \begin{cases} W(k)\varphi_r(t) & \text{if } |t| \leq 1, \\ W(k)\varphi_q(t) & \text{if } |t| \geq 1. \end{cases} \tag{2.31}$$

It is obvious that condition (F_1) holds. Since $r > q$ we have that condition (F_3) is trivially fulfilled. Concerning to condition (F_2), we have that

$$F(k,t) = \begin{cases} W(k)\dfrac{|t|^r}{r} & \text{if } |t| \leq 1, \\ W(k)\left(\dfrac{|t|^q}{q} + \dfrac{q-r}{qr}\right) & \text{if } |t| \geq 1. \end{cases} \tag{2.32}$$

It is clear that $F(k,t) > 0$ for all $t \neq 0$ and that $\mu F(k,t) \leq t\ f(k,t)$ for all $t \in [-1,1]$ if and only if $0 < \mu \leq r$.

When $|t| \geq 1$, the inequality $\mu F(k,t) \leq t f(k,t)$ holds if and only if either $\mu = q$ or $\mu > q$ and

$$|t|^q \leq \frac{\mu(r-q)}{r(\mu - q)} < \infty. \tag{2.33}$$

As consequence, the inequality $\mu F(k,t) \leq t f(k,t)$ for all $t \neq 0$ is satisfied if and only if $\mu = q$, that is, condition (F_2) does not hold.

Let us see that this problem has only the trivial solution for small values of the parameter λ.

Since $r > q$, it is not difficult to verify that, for $0 < \lambda < (q-1)V_0/(r-1)W_1$, the function $\lambda f(k,t) - V(k)\varphi_q(t)$ is strictly decreasing for every integer k. So, for λ in that situation, we have that

$$(\lambda f(k,t) - V(k)\varphi_q(t))t < 0 \quad \text{for all } t \neq 0 \text{ and all } \ k \in \mathbb{Z}. \tag{2.34}$$

Suppose that there is a nontrivial solution u of the considered problem, and moreover it takes some positive values. Let k_0 be such that $u(k_0) = \max\{u(k); k \in \mathbb{Z}\} > 0$. In such a case we deduce the following contradiction:

$$0 = \Delta_p^2 u(k_0 - 1) - V(k_0)\varphi_q(u(k_0)) + \lambda f(k_0, u(k_0)) < \Delta_p^2 u(k_0 - 1) \leq 0. \tag{2.35}$$

Analogously it can be verified that the solution u has no negative values on $\mathbb{Z}$.

3. Remarks and Examples

In this section we will consider some examples and remarks on applications and extensions of Theorem 1.1 to the existence of homoclinic solutions of difference equations of following types:

(A) Second-order discrete p-Laplacian equations of the form

$$\Delta_p^2 u(k-1) - V(k)u(k)|u(k)|^{q-2} + \lambda b(k)u(k)|u(k)|^{r-2} = 0, \tag{3.1}$$

with $r > p \geq q > 1$.

(B) Higher even-order difference equations. A model equation is the fourth-order extended Fisher-Kolmogorov equation

$$\Delta^4 u(k-2) - a\Delta^2 u(k-1) + V(k)u(k)|u(k)|^{q-2} - \lambda b(k)u(k)|u(k)|^{r-2} = 0, \tag{3.2}$$

with $r > q > 1$.

(C) Second-order difference equations with cubic and quintic nonlinearities of the forms

$$\Delta^2 u(k-1) - V(k)u(k) + \lambda\Big(b(k)u^3(k) + c(k)u^5(k)\Big) = 0, \tag{3.3}$$

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda\Big(b(k)u^2(k) + c(k)u^3(k)\Big) = 0, \tag{3.4}$$

arising in mathematical physics and biology.

(A) Second-Order Discrete p-Laplacian Equations.

The spectrum of the Dirichlet problem (D_N) for (3.1), subject to Dirichlet boundary conditions

$$u(0) = u(N+1) = 0, \tag{3.5}$$

is studied in [17]. It is proved that if $2 < r < q$, $N \geq 2$ and $b : [1, N] \to (0, \infty)$ is a given function, then there exist two positive constants $\lambda_0(N)$ and $\lambda_1(N)$ with $\lambda_0(N) \leq \lambda_1(N)$ such that no $\lambda \in (0, \lambda_0(N))$ is an eigenvalue of problem (D_N) while any $\lambda \in [\lambda_1(N), \infty)$ is an eigenvalue of problem (D_N). Moreover, we have

$$\lambda_1(N) \leq \frac{r}{2}\lambda_0(N), \qquad \frac{4}{(N+1)^2|b|_\infty} \leq \lambda_0(N) \leq \lambda_1(N) \leq B(r, q, b, N), \tag{3.6}$$

where $B(r,q,b,N) = r(q-2)/((q-r)\sum_{k=1}^{N} b(k))(N(q-r)/(q(r-2)))^{(r-2)/(q-2)}$ and $|b|_\infty = \max_{k\in[1,N]} b(k)$. Note that if $b(k)$ is positive and $b(k) \geq b > 0$ then

$$B(r,q,b,N) \leq KN^{(r-q)/(q-2)}, \tag{3.7}$$

where K is a constant depending on p,q,b, which implies that $\lambda_0(N) \to 0$ and $\lambda_1(N) \to 0$ as $N \to \infty$. It implies that for a given $\varepsilon > 0$, there exists N_0 such that for any $N > N_0$, the problem (D_N) has a solution for every $\lambda > \varepsilon > 0$.

We extend this phenomenon, looking for homoclinic solutions of (3.1). Applying Theorem 1.1 with $f(k,t) = b(k)\varphi_r(t)$, $F(k,t) = b(k)\Phi_r(t)$ and $\mu = r > p \geq q > 1$, we obtain the following.

Corollary 3.1. *Suppose that the function $V : \mathbb{Z} \to \mathbb{R}$ is positive and T-periodic and $r > p \geq q > 1$. Then, for each $\lambda > 0$, (3.1) has a nonzero homoclinic solution.*

Moreover, given a nontrivial solution u of problem (3.1), there exist $k_\pm$ two integer numbers such that for all $k > k_+$ and $k < k_-$, the sequence $u(k)$ is strictly monotone.

(B) Higher Even-Order Difference Equations.

The statement of Theorem 1.1 can be extended to higher even-order difference equations. For simplicity we consider the fourth-order difference equations of the form

$$\Delta^2\left(\varphi_{p_2}\left(\Delta^2 u(k-2)\right)\right) - a\Delta\left(\varphi_{p_1}(\Delta u(k-1))\right) + V(k)\varphi_q(u(k)) - \lambda f(k,u(k)) = 0, \tag{3.8}$$

where $f(k,\cdot) \in C(\mathbb{R},\mathbb{R})$ for each $k \in \mathbb{Z}$, satisfy the assumptions (F_1)–(F_3).

We consider the functional $J_1 : \ell^q \to \mathbb{R}$,

$$J_1(u) = A_1(u) - \lambda \sum_{k\in\mathbb{Z}} F(k,u(k)), \tag{3.9}$$

where

$$A_1(u) = \sum_{k\in\mathbb{Z}} \Phi_{p_2}\left(\Delta^2 u(k-2)\right) + a\Phi_{p_1}(\Delta u(k-1)) + V(k)\Phi_q(u(k)), \tag{3.10}$$

which is well defined for $\mu > p_j \geq q > 1$, $j = 1,2$.

Note that the series $\sum_{k\in\mathbb{Z}} \Phi_{p_2}(\Delta^2 u(k-2))$ is convergent because

$$\begin{aligned}\Phi_{p_2}\left(\Delta^2 u(k-2)\right) &= \Phi_{p_2}(u(k) - 2u(k-1) + u(k-2))\\ &\leq \frac{2.3^{p_2-1}}{p_2}\left(|u(k)|^{p_2} + |u(k-1)|^{p_2} + |u(k-2)|^{p_2}\right),\\ \sum_{k\in\mathbb{Z}} \Phi_{p_2}\left(\Delta^2 u(k-2)\right) &\leq \frac{2.3^{p_2}}{p_2}\sum_{k\in\mathbb{Z}}|u(k)|^{p_2},\end{aligned} \tag{3.11}$$

while $\sum_{k\in\mathbb{Z}} |u(k)|^{p_2}$ is convergent since $q \leq p_2$.

Now following the steps of the proof of Theorem 1.1 one can prove the following.

Theorem 3.2. *Suppose that $a > 0$, the function $V : \mathbb{Z} \to \mathbb{R}$ is positive and T-periodic and the functions $f(k,\cdot) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfy assumptions (F_1)–(F_3) and $\mu > p_j \geq q > 1,\ j = 1,2$. Then, for each $\lambda > 0$,* (3.8) *has a nonzero homoclinic solution $u \in \ell^q$, which is a critical point of the functional $J_1 : \ell^q \to \mathbb{R}$.*

A typical example of (3.8) is (3.2), which is a discretization of a fourth-order extended Fisher-Kolmogorov equation. Homoclinic solutions for fourth-order ODEs are studied in [7] using variational approach and concentration-compactness arguments. As a consequence of Theorem 3.2 we obtain the following corollary.

Corollary 3.3. *Suppose that $a > 0$, the function $V : \mathbb{Z} \to \mathbb{R}$ is positive and T-periodic and $r > q > 1$. Then, for each $\lambda > 0$,* (3.2) *has a nonzero homoclinic solution $u \in \ell^q$.*

(C) Second-Order Difference Equations with Cubic and Quintic Nonlinearities.

Our next example is (3.3), known as stationary Ginzubrg-Landau equation with cubic-quintic nonlinearity. We refer to [18, 19] and references therein. From physical point of view it is interesting the case $b(k) = b,\ c(k) = c,\ \ bc < 0$. Theorem 1.1 can be applied for $f(k,t) = b(k)t^3 + c(k)t^5$ with $b(k), c(k)$, T-periodic, and $c(k)$ positive. Then $f(k,t)$ satisfies assumptions (F_1)–(F_3) with $\mu = 4$ and as a consequence we have the following corollary.

Corollary 3.4. *Suppose that the functions $V : \mathbb{Z} \to \mathbb{R}$, $b : \mathbb{Z} \to \mathbb{R}$ and $c : \mathbb{Z} \to \mathbb{R}$ are T-periodic and V and c are positive. Then, for each $\lambda > 0$,* (3.3) *has a nonzero homoclinic solution $u \in \ell^2$.*

Moreover, given a nontrivial solution u of problem (1.10)*, there exist $k_{\pm}$ two integer numbers such that for all $k > k_+$ and $k < k_-$, the sequence $u(k)$ is strictly monotone.*

Moreover, we can prove that if in addition to conditions (F_1)–(F_3) the following condition holds:

(F_4) $f(k,t) > 0$ for all $t < 0$ and all $k \in \mathbb{Z}$,

the homoclinic solution of (1.10) is positive.

Indeed, let u be a homoclinic solution of (1.10) and assume that (F_4) holds. Suppose that there exists k_0 such that $u(k_0) < 0$ and let k_1 be such that $u(k_1) = \min\{u(k),\ k \in \mathbb{Z}\} < 0$. In consequence $\Delta_p^2 u(k_1 - 1) \geq 0$, which implies that

$$\lambda f(k_1, u(k_1)) = -\Delta_p^2 u(k_1 - 1) + V(k_1)\varphi_q(u(k_1)) < 0, \tag{3.12}$$

in contradiction with (F_4). Then $u(k) \geq 0$ for every $k \in \mathbb{Z}$.

If $u(k_2) = 0$ for some $k_2 \in \mathbb{Z}$, we know that $\Delta_p u(k_2 - 1) = 0$ and, in consequence, $u(k_2 - 1)u(k_2 + 1) < 0$, and we arrive at a contradiction as in the previous case.

We summarize above observations in the following.

Theorem 3.5. *Suppose that the function $V : \mathbb{Z} \to \mathbb{R}$ is positive and T-periodic and the functions $f(k,\cdot) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfy assumptions (F_1), (F_2), and (F_3). Then, for each $\lambda > 0$,* (1.10) *has a nonzero homoclinic solution $u \in \ell^q$. If moreover (F_4) holds, u is a positive solution on $\mathbb{Z}$ that is strictly monotone for $|k|$ large enough.*

In the case $q = 2$ we can estimate the maximum of the solution u, provided the additional assumption

(F_5) Assume that for all $t > 0$ and $k \in \mathbb{Z}$ function $f(k,\cdot)$ has the form $f(k,t) = tg(k,t)$, where $g(k,t)$ is T-periodic in k, $g(k,0) = 0$ and for each k, $g(k,t)$ is increasing in t for $t > 0$.

Let $g^{-1}(k,t)$ be the inverse function of $g(k,t)$ for $t > 0$. We have that $g^{-1}(k,t)$ is increasing in t for $t > 0$. Let u be a positive homoclinic solution of (1.10) in view of last theorem and $u(k_0) > 0$ is its maximum. Note that, in view of the periodicity of coefficients, if $u(\cdot)$ is a solution of (1.10), then $u(\cdot + jT), j \in \mathbb{Z}$ is also a solution of (1.10). Hence, we may assume that $k_0 \in [0, T-1]$. Then $\Delta_p^2 u(k_0 - 1) \leq 0$ and

$$\lambda u(k_1) g(k_1, u(k_1)) - V(k_1) u(k_1) \geq 0, \tag{3.13}$$

and hence by properties of g and V

$$u(k_1) \geq g^{-1}\left(k_1, \frac{V_0}{\lambda}\right). \tag{3.14}$$

Thus

$$\max\{u(k) : k \in [0, T-1]\} \geq \min\left\{g^{-1}\left(k, \frac{V_0}{\lambda}\right) : k \in [0, T-1]\right\}. \tag{3.15}$$

We summarize above observation in the following.

Corollary 3.6. *Let $q = 2$ and suppose that the functions $V : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{Z} \to \mathbb{R}$ satisfy assumptions of Theorem 3.5. Then, if in addition, f satisfies condition (F_5), the positive homoclinic solution of the equation*

$$\Delta_p^2 u(k-1) - V(k) u(k) + \lambda u(k) g(k, u(k)) = 0, \tag{3.16}$$

satisfies the estimate (3.15).

Our next example, concerning Theorem 3.5, are (3.4) and

$$\Delta_p^2 u(k-1) - a(k) u(k) + \lambda\left(b(k) u^2(k) + c(k) u_+^3(k)\right) = 0, \tag{3.17}$$

where $u_+ = \max\{u, 0\}$.

Positive homoclinic solutions of corresponding differential equation are studied in [3] and periodic solutions in [20]. We suppose that the coefficients $a(k), b(k)$, and $c(k)$ are T-periodic and there are constants a, b, B, c, and C such that

$$0 < a \leq a(k), \quad 0 \leq b \leq b(k) \leq B, \quad 0 < c \leq c(k) \leq C, \quad \forall k \in [0, T-1]. \tag{3.18}$$

By Theorem 3.5, (3.17) has a positive solution u, which is a critical point of the functional $I : l^2 \to \mathbb{R}$,

$$I(u) = \sum_{k\in\mathbb{Z}} \Phi_p(\Delta u(k-1)) + \frac{1}{2}\sum_{k\in\mathbb{Z}} a(k)u^2(k) - \lambda \sum_{k\in\mathbb{Z}} \left(\frac{1}{3}b(k)u^3(k) + \frac{1}{4}c(k)u_+^4(k)\right). \tag{3.19}$$

Clearly, the positive solution of (3.17) is a positive solution of (3.4) too.

Further, let u take its positive maximum at $k_1 \in [0, T-1]$, then $\Delta_p^2 u(k_1 - 1) \le 0$ and, since $u(k_1) > 0$, we have from (3.4) that

$$-a(k_1) + \lambda\Big(b(k_1)u(k_1) + c(k_1)u^2(k_1)\Big) \ge 0. \tag{3.20}$$

In view of (3.18), the last inequality implies

$$u(k_1) \ge \frac{-\lambda b(k_1) + \sqrt{\lambda^2 b(k_1)^2 + 4\lambda a(k_1)c(k_1)}}{2\lambda c(k_1)}, \tag{3.21}$$

or

$$\max\{u(k) : k \in [0, T-1]\} \ge \frac{-\lambda B + \sqrt{\lambda^2 b^2 + 4\lambda ac}}{2\lambda C} = \frac{-B + \sqrt{b^2 + 4ac/\lambda}}{2C}. \tag{3.22}$$

We obtain a positive lower bound for $\max\{u(k) : k \in [0, T-1]\}$ in the case

$$0 < \lambda < \frac{4ac}{B^2 - b^2} \tag{3.23}$$

and (3.22) shows that $\max\{u(k) : k \in [0, T-1]\}$ blows up, that is, tends to $+\infty$ as $\lambda \to 0$.

We summarize above facts in the following.

Corollary 3.7. *Let $p > 1, \lambda > 0$ and $a(k), b(k)$ and $c(k)$ be T-periodic sequences.*

Assume that there are constants a, b, B, c, and C such that (3.18) *holds. Then,*

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda\Big(b(k)u^2(k) + c(k)u^3(k)\Big) = 0, \tag{3.24}$$

has a positive homoclinic solution and for $0 < \lambda < 4ac/(B^2 - b^2)$,

$$\max\{u(k) : k \in [0, T-1]\} \ge \frac{-B + \sqrt{b^2 + 4ac/\lambda}}{2C} > 0. \tag{3.25}$$

Let $\lambda_m \to 0$. By the last statement, if u_m is the solution of the equation

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda_m\Big(b(k)u^2(k) + c(k)u^3(k)\Big) = 0, \tag{3.26}$$

then

$$\lim_{m\to\infty} \max\{u_m(k) : k \in [0, T-1]\} = +\infty. \tag{3.27}$$

Let $k_m \in [0, T-1]$ be such that $u_m(k_m) = \max\{u_m(k) : k \in [0, T-1]\}$. Since k_m is an infinite sequence of integers, by Dirichlet principle, there exists a fixed $k_* \in [0, T-1]$ and a subsequence of u_m, still denoted by u_m, such that $u_m(k_m) = u_m(k_*)$ and $\lim_{m\to\infty} u_m(k_*) = +\infty$. Note that if $T = 2$, then $k_* = 0$ or $k_* = 1$.

Dedication

This work is dedicated to Professor Gheorghe Moroşanu on the occasion of his 60-th birthday.

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