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Research Article

Uniform Second-Order Difference Method for a Singularly Perturbed Three-Point Boundary Value Problem

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We consider a singularly perturbed one-dimensional convection-diffusion three-point boundary value problem with zeroth-order reduced equation. The monotone operator is combined with the piecewise uniform Shishkin-type meshes. We show that the scheme is second-order convergent, in the discrete maximum norm, independently of the perturbation parameter except for a logarithmic factor. Numerical examples support the theoretical results.

1. Introduction

We consider the following singularly perturbed three-point boundary value problem:

$$Lu := \varepsilon^2 u''(x) + \varepsilon a(x) u'(x) - b(x) u(x) = f(x), \quad 0 < x < \ell, \tag{1.1}$$

$$u(0) = A, \quad L_0 u := u(\ell) - \gamma u(\ell_1) = B, \quad 0 < \ell_1 < \ell, \tag{1.2}$$

where $\varepsilon \in (0,1]$ is the perturbation parameter, and, A, B, and γ are given constants. The functions $a(x) \geq 0$, $b(x) \geq \beta > 0$ and $f(x)$ are sufficiently smooth. For $0 < \varepsilon \ll 1$ the function $u(x)$ has in general boundary layers at $x = 0$ and $x = \ell$.

Equations of this type arise in mathematical problems in many areas of mechanics and physics. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, shear in second-order fluids, control theory, electrical networks, and other physical models [1, 2].

Differential equations with a small parameter $0 < \varepsilon \ll 1$ multiplying the highest order derivatives are called singularly perturbed differential equations. Typically, the solutions of such equations have steep gradients in narrow layer regions of the domain. Classical numerical methods are inappropriate for singularly perturbed problems. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε; that is, methods that are convergence ε-uniformly [1–5]. One of the simplest ways to derive such methods consists of using a class of special piecewise uniform meshes (a Shishkin mesh), (see, e.g., [4–8] for motivation for this type of mesh), which are constructed a priori in function of sizes of parameter ε, the problem data, and the number of corresponding mesh points.

Three-point boundary value problems have been studied extensively in the literature. For a discussion of existence and uniqueness results and for applications of three-point problems, see [9–12] and the references cited in them. Some approaches to approximating this type of problem have also been considered [13, 14]. However, the algorithms developed in the papers cited above are mainly concerned with regular cases (i.e., when boundary layers are absent). Fitted difference scheme on an equidistant mesh for the numerical solution of the linear three-point reaction-diffusion problem have been studied in [15]. A uniform finite difference method, which is first-order convergent, on an S-mesh (Shishkin type mesh) for a singularly perturbed semilinear one-dimensional convection-diffusion three-point boundary value problem have also been studied in [16].

Computational methods for singularly perturbed problems with two small parameters have been studied in different ways [17–21]. In this paper, we propose the hybrid scheme for solving the nonlocal problem (1.1)-(1.2), which comprises three kinds of schemes, such as Samarskii's scheme [22], a finite difference scheme with uniform mesh, and finite difference scheme on piecewise uniform mesh. The considered algorithm is monotone.

We will prove that the method for the numerical solution of the three-point boundary value problem (1.1)-(1.2) is uniformly convergent of order $N^{-2}\ln^2 N$ on special piecewise equidistant mesh, in discrete maximum norm, independently of singular perturbation parameter. In Section 2, we present some analytical results of the three-point boundary value problem (1.1)-(1.2). In Section 3, we describe some monotone finite-difference discretization and introduce the piecewise uniform grid. In Section 4, we analyze the convergence properties of the scheme. Finally, numerical examples are presented in Section 5.

Notation 1. Henceforth, C denote the generic positive constants independent of ε and of the mesh parameter. Such a subscripted constant is also independent of ε and mesh parameter, but whose value is fixed.

Assumption 1. In what follows, we will assume that $\varepsilon \leq CN^{-1}$, which is nonrestrictive in practice.

2. Properties of the Exact Solution

For constructing layer-adapted meshes correctly, we need to know the asymptotic behavior of the exact solution. This behavior will be used later in the analysis of the uniform convergence of the finite difference approximations defined in Section 3. For any continuous function $v(x)$, we use $\|v\|_\infty$ for the continuous maximum norm on the corresponding interval.

Lemma 2.1. *If a, b, and $f \in C^2[0,\ell]$, the solution of* (1.1)-(1.2) *satisfies the following estimates:*

$$\|u\|_\infty \le C,$$
$$\left|u^{(k)}(x)\right| \le C\left\{1 + \frac{1}{\varepsilon^k}\left(e^{-\mu_1 x/\varepsilon} + e^{-\mu_2(\ell - x)/\varepsilon}\right)\right\}, \quad 0 \le x \le \ell,\ k = 1,2,3,4, \tag{2.1}$$

provided that $b(x) - \varepsilon a'(x) \ge \beta_* > 0$ and $|\gamma| < 1$, where

$$\mu_1 = \frac{1}{2}\left(\sqrt{a^2(0) + 4\beta_*} + a(0)\right),$$
$$\mu_2 = \frac{1}{2}\left(\sqrt{a^2(\ell) + 4\beta_*} - a(\ell)\right). \tag{2.2}$$

Proof. The proof is almost identical to that of [16, 23]. □

3. Discretization and Piecewise Uniform Mesh

Introduce an arbitrary nonuniform mesh on the segment $[0,\ell]$

$$\omega_N = \{0 < x_1 < \cdots < x_{N-1} < \ell\},$$
$$\overline{\omega}_N = \omega_N \cup \{x_0 = 0,\ x_N = \ell\}. \tag{3.1}$$

Let $h_i = x_i - x_{i-1}$ be a mesh size at the node x_i and $\hbar_i = (h_i + h_{i+1})/2$ be an average mesh size. Before describing our numerical method, we introduce some notation for the mesh functions. Define the following finite differences for any mesh function $v_i = v(x_i)$ given on $\overline{\omega}_N$ by

$$v_{\bar{x},i} = \frac{(v_i - v_{i-1})}{h_i}, \qquad v_{x,i} = \frac{(v_{i+1} - v_i)}{h_{i+1}}, \qquad v_{\overset{0}{x},i} = \frac{(v_{\bar{x},i} + v_{x,i})}{2},$$
$$v_{\hat{x},i} = \frac{(v_{i+1} - v_i)}{\hbar_i}, \qquad \hbar_i = \frac{h_i + h_{i+1}}{2}, \qquad v_{\bar{x}\hat{x},i} = \frac{(v_{x,i} - v_{\bar{x},i})}{\hbar}, \tag{3.2}$$
$$\|w\|_\infty \equiv \|w\|_{\infty,\overline{\omega}_N} := \max_{0 \le i \le N} |w_i|.$$

For equidistant subintervals of the mesh, we use the finite differences in the form

$$v_{\bar{x},i} = \frac{(v_i - v_{i-1})}{h}, \qquad v_{x,i} = \frac{(v_{i+1} - v_i)}{h}, \qquad v_{\bar{x}x,i} = \frac{(v_{x,i} - v_{\bar{x},i})}{h}. \tag{3.3}$$

To approximate the solution of (1.1)-(1.2), we employ a finite difference scheme defined on a piecewise uniform Shishkin mesh. This mesh is defined as follows.

We divide each of the intervals $[0,\sigma_1]$ and $[\ell - \sigma_2, \ell]$ into $N/4$ equidistant subintervals, and we divide $[\sigma_1, \ell - \sigma_2]$ into $N/2$ equidistant subintervals, where N is a positive integer divisible by 4. The transition points σ_1 and σ_2, which separate the fine and coarse portions of the mesh, are obtained by taking

$$\sigma_1 = \min\left\{\frac{\ell}{4}, \mu_1^{-1}\varepsilon \ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{\ell}{4}, \mu_2^{-1}\varepsilon \ln N\right\}, \tag{3.4}$$

where μ_1 and μ_2 are given in Lemma 2.1. In practice, we usually have $\sigma_i \ll \ell$ $(i = 1, 2)$, and so the mesh is fine on $[0, \sigma_1]$, $[\ell - \sigma_2, \ell]$ and coarse on $[\sigma_1, \ell - \sigma_2]$. Hence, if we denote the step sizes in $[0, \sigma_1]$, $[\sigma_1, \ell - \sigma_2]$, and $[\ell - \sigma_2, \ell]$ by $h^{(1)}$, $h^{(2)}$, and $h^{(3)}$, respectively, we have

$$\begin{gathered} h^{(1)} = \frac{4\sigma_1}{N}, \qquad h^{(2)} = \frac{2(\ell - \sigma_2 - \sigma_1)}{N}, \qquad h^{(3)} = \frac{4\sigma_2}{N}, \qquad h^{(2)} + \frac{1}{2}\left(h^{(1)} + h^{(3)}\right) = \frac{2\ell}{N}, \\ h^{(k)} \le \ell N^{-1}, \quad k = 1, 3, \quad \ell N^{-1} \le h^{(2)} < 2\ell N^{-1}, \end{gathered} \tag{3.5}$$

so that

$$\begin{aligned} \overline{\omega}_N = \Bigg\{ & x_i = ih^{(1)}, i = 0, 1, \ldots, \frac{N}{4}; x_i = \sigma_1 + \left(i - \frac{N}{4}\right)h^{(2)}, i = \frac{N}{4} + 1, \ldots, \frac{3N}{4}; \\ & x_i = \ell - \sigma_2 + \left(i - \frac{3N}{4}\right)h^{(3)}, i = \frac{3N}{4} + 1, \ldots, N, h^{(1)} = \frac{4\sigma_1}{N}, h^{(2)} = \frac{2(\ell - \sigma_2 - \sigma_1)}{N}, \\ & h^{(3)} = \frac{4\sigma_2}{N} \Bigg\}. \end{aligned} \tag{3.6}$$

On this mesh, we define the following finite difference schemes:

$$\begin{gathered} L_1^h u_i \equiv \varepsilon^2 k_i u_{\overline{x}x,i} + \varepsilon a_i u_{x,i} - b_i u_i = f_i - R_i^{(1)}, \quad \text{for } i = 1, 2, \ldots, \frac{N}{4} - 1;\ i = \frac{3N}{4} + 1, \ldots, N, \\ L_2^h u_i \equiv \varepsilon^2 u_{\overline{x}x,i} + \varepsilon a_i u_{x,i} - b_i u_i = f_i - R_i^{(2)}, \quad \text{for } i = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1, \\ L_3^h u_i \equiv \varepsilon^2 u_{\overline{x}\hat{x},i} + \varepsilon a_i u_{x,i} - b_i u_i = f_i - R_i^{(3)}, \quad \text{for } i = \frac{N}{4}, \frac{3N}{4}, \end{gathered} \tag{3.7}$$

where

$$k_i = \frac{1}{1 + a_i h/2\varepsilon}, \tag{3.8}$$

$$R_i^{(1)} = -\frac{\varepsilon^2 h}{6}\int_{x_{i-1}}^{x_{i+1}} \varphi_i^{(1)}(x)u^{(4)}(x)dx - \frac{\varepsilon a_i h}{4}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'''(x)dx - \frac{a_i^2 h^2}{4(1 + a_i h/2\varepsilon)} u_{\overline{x}x,i}, \tag{3.9}$$

$$R_i^{(2)} = -\frac{\varepsilon^2}{2}\int_{x_{i-1}}^{x_{i+1}} \varphi_i^{(2)}(x)u'''(x)dx - \varepsilon a_i h^{-1}\int_{x_i}^{x_{i+1}} (x_{i+1} - x)u''(x)dx, \tag{3.10}$$

$$R_i^{(3)} = -\frac{\varepsilon^2}{2}\int_{x_{i-1}}^{x_{i+1}} \varphi_i^{(3)}(x)u'''(x)dx - \varepsilon a_i h_{i+1}^{-1}\int_{x_i}^{x_{i+1}} (x_{i+1} - x)u''(x)dx, \tag{3.11}$$

with the usual piecewise linear basis functions

$$\varphi_i(x) = \begin{cases} \left(\dfrac{x - x_{i-1}}{h}\right)^2, & x_{i-1} < x < x_i, \\ \left(\dfrac{x_{i+1} - x}{h}\right)^2, & x_i < x < x_{i+1}, \end{cases}$$

$$\varphi_i^{(1)}(x) = \left(1 - h^{-1}|x - x_i|\right)^3 = \begin{cases} \left(\dfrac{x - x_{i-1}}{h}\right)^3, & x_{i-1} < x < x_i, \\ \left(\dfrac{x_{i+1} - x}{h}\right)^3, & x_i < x < x_{i+1}, \end{cases}$$

$$\varphi_i^{(2)}(x) = \begin{cases} -\left(\dfrac{x - x_{i-1}}{h}\right)^2, & x_{i-1} < x < x_i, \\ \left(\dfrac{x_{i+1} - x}{h}\right)^2, & x_i < x < x_{i+1}, \end{cases} \tag{3.12}$$

$$\varphi_i^{(3)}(x) = \begin{cases} \dfrac{(x - x_{i-1})^2}{h_i \hbar_i}, & x_{i-1} < x < x_i, \\ \dfrac{(x_{i+1} - x)^2}{h_i \hbar_{i+1}}, & x_i < x < x_{i+1}. \end{cases}$$

It is now necessary to define an approximation for the second boundary condition of (1.2). Let x_{N_0} be the mesh point nearest to ℓ_1. Then, using interpolating quadrature formula with respect to x_{N_0} and x_{N_0+1}, we can write

$$u(x) = \frac{x - x_{N_0+1}}{x_{N_0} - x_{N_0+1}} u(x_{N_0}) + \frac{x - x_{N_0}}{x_{N_0+1} - x_{N_0}} u(x_{N_0+1}) + r(x), \tag{3.13}$$

where

$$r(x) = \frac{1}{2} f''(\xi)(x - x_{N_0})(x - x_{N_0+1}), \quad \xi \in (x_{N_0}, \ell_1). \tag{3.14}$$

Substituting $x = \ell_1$ into (3.13), for the second boundary condition of (1.2), we obtain

$$u_N - \gamma\left[\frac{\ell_1 - x_{N_0+1}}{x_{N_0} - x_{N_0+1}}u(x_{N_0}) + \frac{\ell_1 - x_{N_0}}{x_{N_0+1} - x_{N_0}}u(x_{N_0+1})\right] + r(x) = B. \tag{3.15}$$

Based on (3.7) and (3.15), we propose the following difference scheme for approximating (1.1)-(1.2):

$$\ell_1^h y_i \equiv \varepsilon^2 k_i y_{\bar{x}x,i} + \varepsilon a_i y_{x,i} - b_i y_i = f_i \quad i = 1,2,\dots,\frac{N}{4} - 1;\ i = \frac{3N}{4} + 1,\dots,N, \tag{3.16}$$

$$\ell_2^h y_i \equiv \varepsilon^2 y_{\bar{x}x,i} + \varepsilon a_i y_{x,i} - b_i y_i = f_i \quad i = \frac{N}{4} + 1,\dots,\frac{3N}{4} - 1, \tag{3.17}$$

$$\ell_3^h y_i \equiv \varepsilon^2 y_{\bar{x}\hat{x},i} + \varepsilon a_i y_{x,i} - b_i y_i = f_i \quad i = \frac{N}{4}, \frac{3N}{4}, \tag{3.18}$$

$$y_0 = A, \qquad \ell_0 y \equiv y_N - \gamma\left[\frac{\ell_1 - x_{N_0+1}}{x_{N_0} - x_{N_0+1}}y(x_{N_0}) + \frac{\ell_1 - x_{N_0}}{x_{N_0+1} - x_{N_0}}y(x_{N_0+1})\right] = B. \tag{3.19}$$

4. Uniform Error Estimates

Let $z = y - u$, $x \in \overline{\omega}_N$. Then, the error in the numerical solution satisfies

$$\begin{gathered} \ell^h z \equiv R_i, \quad i = 1,2,\dots,N-1, \\ z_0 = 0, \qquad z_N - \gamma\left[\frac{\ell_1 - x_{N_0+1}}{x_{N_0} - x_{N_0+1}}z_{N_0} + \frac{\ell_1 - x_{N_0}}{x_{N_0+1} - x_{N_0}}z_{N_0+1}\right] = r, \end{gathered} \tag{4.1}$$

where

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}, \tag{4.2}$$

and r is defined by (3.14).

Lemma 4.1. *Let z_i be the solution to* (4.1)*. Then, the estimate*

$$\|z\|_{\infty,\overline{\omega}_N} \le C\{\|R\|_{\infty,\omega_N} + |r|\} \tag{4.3}$$

holds.

Proof. The proof is almost identical to that of [16, 23]. □

Lemma 4.2. *Under the above assumptions of Section 1 and Lemma 2.1, the following estimates hold for the error functions R_i and r:*

$$\|R\|_{\infty,\omega_N} \leq C\left(N^{-1}\ln N\right)^2,$$
$$|r| \leq C\left(N^{-1}\ln N\right)^2. \tag{4.4}$$

Proof. The argument now depends on whether $\sigma_1 = \sigma_2 = \ell/4$ or $\sigma_1 = \mu_1^{-1}\varepsilon\ln N$ and $\sigma_2 = \mu_2^{-1}\varepsilon\ln N$. In the first case

$$\mu_1^{-1}\varepsilon\ln N \geq \frac{\ell}{4}, \qquad \mu_2^{-1}\varepsilon\ln N \geq \frac{\ell}{4}, \tag{4.5}$$

and the mesh is uniform with $h^{(1)} = h^{(2)} = h^{(3)} = \ell N^{-1}$ for all $i, 1 \leq i \leq N$. Therefore, from (3.9), we have

$$\left|R_i^{(1)}\right| \leq C\left\{\varepsilon^2 h\int_{x_{i-1}}^{x_{i+1}}\left|u^{(4)}(x)\right|dx + \varepsilon h\int_{x_{i-1}}^{x_{i+1}}\left|u'''(x)\right|dx + h\int_{x_{i-1}}^{x_{i+1}}\left|u''(x)\right|dx\right\}$$
$$\leq C\left\{\frac{h^2}{\varepsilon^2}\right\} \leq C\left\{\frac{16\mu_1^{-2}\ln^2 N}{\ell^2}\frac{4\ell^2}{N^2}\right\} \leq C\left(N^{-1}\ln N\right)^2. \tag{4.6}$$

The same estimate is obtained for $R_i^{(2)}$ and $R_i^{(3)}$ in a similar manner.

In the second case

$$\mu_1^{-1}\varepsilon\ln N < \frac{\ell}{4}, \qquad \mu_2^{-1}\varepsilon\ln N < \frac{\ell}{4}, \tag{4.7}$$

and the mesh is piecewise uniform with the mesh spacing $4\sigma_1/N$ and $4\sigma_2/N$ in the subintervals $[0,\sigma_1]$ and $[\ell-\sigma_2,\ell]$, respectively, and $2(\ell-\sigma_2-\sigma_1)/N$ in the subinterval $[\sigma_1,\ell-\sigma_2]$. We have the estimate $R_i^{(1)}$ in $[0,\sigma_1]$ and $[\ell-\sigma_2,\ell]$ and the estimate $R_i^{(2)}$ in $[\sigma_1,\ell-\sigma_2]$. In the layer region $[0,\sigma_1]$, the estimate $R_i^{(1)}$ reduces to

$$\left|R_i^{(1)}\right| \leq C\left(\frac{h^{(1)}}{\varepsilon}\right)^2 \leq C\left(\frac{16\mu_1^{-2}\varepsilon^2\ln^2 N}{\varepsilon^2 N^2}\right), \quad 1 \leq i \leq \frac{N}{4}-1. \tag{4.8}$$

Hence,

$$\left|R_i^{(1)}\right| \leq CN^{-2}\ln^2 N, \quad 1 \leq i \leq \frac{N}{4}-1. \tag{4.9}$$

The same estimate is obtained in the layer region $[\ell - \sigma_2, \ell]$ in a similar manner. We now have to estimate $R_i^{(2)}$ for $N/4 + 1 \leq i \leq 3N/4 - 1$. In this case, we are able to rewrite $R_i^{(2)}$ as follows:

$$\begin{aligned} \left|R_i^{(2)}\right| &\leq C\left\{\varepsilon^2 \int_{x_{i-1}}^{x_{i+1}} \left|u'''(x)\right| dx + \varepsilon \int_{x_{i-1}}^{x_{i+1}} \left|u''(x)\right| dx\right\} \\ &\leq C\Big\{\varepsilon^2 h^{(2)} + \varepsilon h^{(2)} + \mu_1^{-1}\left(e^{-\mu_1 x_{i-1}/\varepsilon} - e^{-\mu_1 x_{i+1}/\varepsilon}\right) \\ &\quad + \mu_2^{-1}\left(e^{-\mu_2(\ell - x_{i+1})/\varepsilon} - e^{-\mu_2(\ell - x_{i-1})/\varepsilon}\right)\Big\}, \quad \frac{N}{4} + 1 \leq i \leq \frac{3N}{4} - 1. \end{aligned} \tag{4.10}$$

Since

$$x_i = 2\mu_1^{-1}\varepsilon \ln N + \left(i - \frac{N}{4}\right)h^{(2)}, \tag{4.11}$$

it follows that

$$e^{-\mu_1 x_{i-1}/\varepsilon} - e^{-\mu_1 x_{i+1}/\varepsilon} = \frac{1}{N^2} e^{-\mu_1 (i-1-N/4)h^{(2)}/\varepsilon}\left(1 - e^{-2\mu_1 h^{(2)}/\varepsilon}\right) < N^{-2}. \tag{4.12}$$

Also, if we rewrite the mesh points in the form $x_i = \ell - \sigma_2 - (3N/4 - i)h^{(2)}$, evidently

$$e^{-\mu_2(\ell - x_{i+1})/\varepsilon} - e^{-\mu_2(\ell - x_{i-1})/\varepsilon} = \frac{1}{N^2} e^{-\mu_2(3N/4 - i - 1)h^{(2)}/\varepsilon}\left(1 - e^{-2\mu_2 h^{(2)}/\varepsilon}\right) < N^{-2}. \tag{4.13}$$

The last two inequalities together, (4.10), give the bound

$$\left|R_i^{(2)}\right| \leq CN^{-2}, \qquad \frac{N}{4} + 1 \leq i \leq \frac{3N}{4}. \tag{4.14}$$

Finally, we estimate $R_i^{(3)}$ for the mesh points $x_{N/4}$ and $x_{3N/4}$. For the mesh point $x_{N/4}$, $R_i^{(3)}$ reduces to

$$\begin{aligned} \left|R_i^{(3)}\right| &\leq C\Bigg\{\varepsilon^2 \int_{x_{N/4-1}}^{x_{N/4}} \frac{(x_{N/4-1} - x)^2}{h^{(1)}\left(h^{(1)} + h^{(2)}\right)} \left|u'''(x)\right| dx + \varepsilon^2 \int_{x_{N/4}}^{x_{N/4+1}} \frac{(x_{N/4+1} - x)^2}{h^{(2)}\left(h^{(1)} + h^{(2)}\right)} \left|u'''(x)\right| dx \\ &\quad + \varepsilon\left(h^{(2)}\right)^{-1} \int_{x_{N/4}}^{x_{N/4+1}} (x_{N/4} - x)\left|u''(x)\right| dx\Bigg\} \\ &\leq C\Bigg\{\varepsilon^2 h^{(1)} + \varepsilon^2 h^{(2)} + \varepsilon h^{(2)} + \frac{1}{\varepsilon}\int_{x_{N/4-1}}^{x_{N/4}} \left(e^{-\mu_1 x/\varepsilon} + e^{-\mu_2(\ell - x)/\varepsilon}\right) dx \\ &\quad + \frac{1}{\varepsilon}\int_{x_{N/4}}^{x_{N/4+1}} \left(e^{-\mu_1 x/\varepsilon} + e^{-\mu_2(\ell - x)/\varepsilon}\right) dx\Bigg\}. \end{aligned} \tag{4.15}$$

Since

$$\begin{aligned}
e^{-\mu_1 x_{N/4-1}/\varepsilon} - e^{-\mu_1 x_{N/4}/\varepsilon} &= e^{-\mu_1 (N/4-1)h^{(1)}/\varepsilon}\left(1 - e^{-\mu_1 h^{(1)}/\varepsilon}\right) \\
&= \frac{1}{N^2}\left(1 - e^{-\mu_1 h^{(1)}/\varepsilon}\right) < N^{-2}, \\
e^{-\mu_2(\ell - x_{N/4})/\varepsilon} - e^{-\mu_2(\ell - x_{N/4-1})/\varepsilon} &= e^{-\mu_2(\ell - x_{N/4})/\varepsilon}\left(1 - e^{-\mu_2 h^{(1)}/\varepsilon}\right) \\
&= \frac{1}{N^2} e^{-\mu_2 N/2h^{(2)}/\varepsilon}\left(1 - e^{-\mu_2 h^{(1)}/\varepsilon}\right) < N^{-2}, \\
e^{-\mu_1 x_{N/4}/\varepsilon} - e^{-\mu_1 x_{N/4+1}/\varepsilon} &= \frac{1}{N^2}\left(1 - e^{-\mu_1 h^{(2)}/\varepsilon}\right) < N^{-2}, \\
e^{-\mu_2(\ell - x_{N/4+1})/\varepsilon} - e^{-\mu_2(\ell - x_{N/4})/\varepsilon} &= \frac{1}{N^2} e^{-\mu_2 (N/2-1)h^{(2)}/\varepsilon}\left(1 - e^{-\mu_2 h^{(2)}/\varepsilon}\right) < N^{-2},
\end{aligned} \tag{4.16}$$

it then follows that

$$\left|R_i^{(3)}\right| \le CN^{-2}. \tag{4.17}$$

The same estimate is obtained for $i = 3N/4$ in a similar manner. This estimate is valid when only one of the values of σ_1 or σ_2 is equal to $\ell/4$. Next, we estimate the remainder term r. Suppose that $\ell_1 \in [2\alpha^{-1}\varepsilon|\ln\varepsilon|, \ell - 2\alpha^{-1}\varepsilon|\ln\varepsilon|]$, and the second derivative of f on this interval is bounded. From (3.14), we obtain

$$\begin{aligned}
|r| &\le C\left|f''(\xi)(x - x_{N_0})(x - x_{N_0+1})\right| \\
&\le C|(x - x_{N_0})(x - x_{N_0+1})| \\
&\le C\left(\left(h^{(2)}\right)^2\right) \\
&\le C\left(N^{-1}\ln N\right)^2.
\end{aligned} \tag{4.18}$$

□

Combining Lemmas 2 and 3 gives us the following convergence result.

Theorem 4.3. *Let $u(x)$ be the solution of (1) and y be the solution of (29). Then,*

$$\|y - u\|_{\infty,\varpi_N} \le CN^{-2}\ln^2 N. \tag{4.19}$$

5. Algorithm and Numerical Results

In this section, we present some numerical results which illustrate the present method.

(a) The difference scheme (3.16)–(3.19) can be rewritten as

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, \ldots, N-1, \tag{5.1}$$

where

$$A_i = \frac{2\varepsilon^3}{\left(h^{(1)}\right)^2\left(2\varepsilon + a_i h^{(1)}\right)}, \qquad B_i = \frac{2\varepsilon^3}{\left(h^{(1)}\right)^2\left(2\varepsilon + a_i h^{(1)}\right)} + \frac{\varepsilon a_i}{h^{(1)}},$$
$$C_i = \frac{4\varepsilon^3}{\left(h^{(1)}\right)^2\left(2\varepsilon + a_i h^{(1)}\right)} + \frac{\varepsilon a_i}{h^{(1)}} + b_i, \quad i = 1,2,\ldots,\frac{N}{4} - 1;\ \frac{3N}{4} + 1,\ldots,N,$$
$$A_i = \frac{\varepsilon^2}{\left(h^{(2)}\right)^2}, \quad B_i = \frac{\varepsilon^2}{\left(h^{(2)}\right)^2} + \frac{\varepsilon a_i}{h^{(2)}}, \quad C_i = \frac{\varepsilon^2}{\left(h^{(2)}\right)^2} + \frac{\varepsilon a_i}{h^{(2)}} + b_i, \quad i = \frac{N}{4} + 1,\ldots,\frac{3N}{4} - 1,$$
$$A_i = \frac{\varepsilon^2}{\hbar h_i}, \quad B_i = \frac{\varepsilon^2}{\hbar h_{i+1}} + \frac{\varepsilon a_i}{h_{i+1}}, \quad C_i = \frac{\varepsilon^2}{\hbar h_{i+1}} + \frac{\varepsilon^2}{\hbar h_i} + \frac{\varepsilon a_i}{h_{i+1}} + b_i, \quad \hbar = \frac{h_i + h_{i+1}}{2}, \quad i = \frac{N}{4}, \frac{3N}{4},$$
$$F_i = -f_i, \quad i = 1,2,\ldots,N-1. \tag{5.2}$$

System (5.1) and (3.19) is solved by the following factorization procedure:

$$\alpha_1 = 0, \qquad \beta_1 = 0,$$
$$\alpha_{i+1} = \frac{B_i}{C_i - A_i\alpha_i}, \quad \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, \quad i = 1,2,\ldots,N-1,$$
$$\sigma_1 = \min\left\{\frac{\ell}{4}, \mu_1^{-1}\varepsilon \ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{\ell}{4}, \mu_2^{-1}\varepsilon \ln N\right\}, \qquad h^{(2)} = \frac{2(\ell - \sigma_2 - \sigma_1)}{N},$$
$$N_0^* = \left[\frac{\ell_1 - \sigma_1 + N h^{(2)}/4}{h^{(2)}}\right], \qquad N_0 = \begin{cases} N_0^*, & \text{if } \ell_1 - x_{N_0^*} \le x_{N_0^*} - \ell_1, \\ N_0^* + 1, & \text{if } \ell_1 - x_{N_0^*} > x_{N_0^*} - \ell_1, \end{cases}$$
$$Q_{i,N_0} = \begin{cases} 1, & i = N_0 + 1, \\ \displaystyle\prod_{j=N_0+1}^{i-1} \alpha_j, & N_0 + 2 \le i \le N, \end{cases} \tag{5.3}$$
$$y_N = \frac{B\alpha_{N_0+1} - \gamma\mu\beta_{N_0+1} + \gamma\left(\delta\alpha_{N_0+1} - \mu\right)\sum_{i=N_0+1}^{N} Q_{i,N_0}\beta_i}{\alpha_{N_0+1} - \gamma\left(\delta\alpha_{N_0+1} - \mu\right)\prod_{i=N_0+1}^{N}\alpha_i},$$
$$\delta = \frac{\ell_1 - x_{N_0+1}}{x_{N_0} - x_{N_0+1}}, \qquad \mu = \frac{\ell_1 - x_{N_0}}{x_{N_0+1} - x_{N_0}},$$
$$y_i = \alpha_{i+1}y_{i+1} + \beta_{i+1}, \quad i = N-1,\ldots,2,1.$$

Table 1: Approximate errors e_ε^N and e^N and the computed orders of convergence p_ε^N on the piecewise uniform mesh ω_N for various values of ε and N.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-2}	0.0094302	0.0048322	0.0027402	0.0016792	0.0005534
	1.78	1.87	1.95	1.98	2.02
2^{-4}	0.0095503	0.0056215	0.0033157	0.0017325	0.0005988
	1.73	1.85	1.92	1.96	1.99
2^{-6}	0.0096054	0.0056215	0.0033157	0.0017325	0.0005988
	1.76	1.85	1.92	1.96	1.99
2^{-8}	0.0095502	0.0056215	0.0033157	0.0017325	0.0005988
	1.73	1.85	1.92	1.96	1.99
2^{-10}	0.0095502	0.0056215	0.0033157	0.0017325	0.0005988
	1.73	1.85	1.92	1.96	1.99
2^{-12}	0.0095502	0.0056215	0.0033157	0.0017325	0.0005988
	1.73	1.85	1.92	1.96	1.99
2^{-14}	0.0095502	0.0056215	0.0033157	0.0017325	0.0005988
	1.73	1.85	1.92	1.96	1.99
2^{-16}	0.0095502	0.0056215	0.0033157	0.0017325	0.0005988
	1.73	1.85	1.92	1.96	1.99
$\vdots$					
e^N	0.0096054	0.0056215	0.0033157	0.0017325	0.0005988
p^N	1.73	1.85	1.92	1.96	1.99

It is easy to verify that

$$A_i > 0, \quad B_i > 0, \quad C_i > A_i + B_i, \quad i = 1, 2, \ldots, N. \tag{5.4}$$

Therefore, the described factorization algorithm is stable.

(b) We apply the numerical method (3.16)–(3.19) to the following problem:

$$\begin{gathered} \varepsilon^2 u''(x) + \varepsilon(1 + \cos(\pi x))u'(x) - \left(1 + \sin\left(\frac{\pi x}{2}\right)\right)u(x) = f(x), \quad 0 < x < 1, \\ u(0) = 0, \qquad u(1) - \frac{1}{2}u\left(\frac{1}{2}\right) = 1, \end{gathered} \tag{5.5}$$

with

$$f(x) = 2(\varepsilon\pi)^2 \cos(2\pi x) + \varepsilon\pi(1 + \cos(\pi x)) \sin(2\pi x) - \left(1 + \sin\left(\frac{\pi x}{2}\right)\right)\sin^2(\pi x). \tag{5.6}$$

The exact solution of the problem is

$$u(x) = \frac{2\exp((1 - x)(1 + \cos(\pi x) + d)/2\varepsilon)\left[1 - \exp(xd/\varepsilon)\right]}{(-1 + \exp(d/2\varepsilon))(-2 - 2\exp(d/2\varepsilon) + \exp(1 + \cos(\pi x) + d)/4\varepsilon)} + \sin^2(\pi x), \tag{5.7}$$

where

$$d = \sqrt{5 + 2\cos(\pi x) + \cos^2(\pi x) + 4\sin\left(\frac{\pi x}{2}\right)}. \tag{5.8}$$

This $u(x)$ has the typical boundary layers at $x = 0$ and $x = 1$. In the computations in this section, we take

$$\begin{gathered}
A = 0, \qquad B = 1, \qquad \gamma = \frac{1}{2}, \qquad \ell_1 = \frac{1}{2}, \qquad \mu_1 = 2.414213562, \qquad \mu_2 = 1, \\
\sigma_1 = \min\left\{\frac{1}{4}, 2.414213562\,\varepsilon \ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{1}{4}, \varepsilon \ln N\right\}, \\
h^{(2)} = \frac{2(1 - \sigma_2 - \sigma_1)}{N}, \qquad N_0^* = \left[\frac{2 - 4\sigma_1 + Nh^{(2)}}{4h^{(2)}}\right], \\
N_0 = \begin{cases} N_0^*, & \text{if } \dfrac{1}{2} - x_{N_0^*} \le x_{N_0^*} - \dfrac{1}{2}, \\ N_0^* + 1, & \text{if } \dfrac{1}{2} - x_{N_0^*} > x_{N_0^*} - \dfrac{1}{2}. \end{cases}
\end{gathered} \tag{5.9}$$

The error of the scheme is measured in the discrete maximum norm. For any values of ε and N, the maximum pointwise errors e_ε^N and the ε-uniform e^N are calculated using

$$e_\varepsilon^N = \max_i \left| u(x_i) - y_i^N \right|, \qquad e^N = \max_\varepsilon e_\varepsilon^N, \tag{5.10}$$

where u is the exact solution of (5.5) and y is the numerical solution of the finite difference scheme (3.16)–(3.19).The convergence rates are

$$P_\varepsilon^N = \frac{\ln\left(e_\varepsilon^N / e_\varepsilon^{2N}\right)}{\ln 2}. \tag{5.11}$$

The corresponding ε-uniform convergence rates are computed using the formula

$$P^N = \frac{\ln\left(e^N / e^{2N}\right)}{\ln 2}. \tag{5.12}$$

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