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Research Article

A Functional Inequality in Restricted Domains of Banach Modules

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We investigate the stability problem for the following functional inequality $\|\alpha f((x+y)/2\alpha) + \beta f((y+z)/2\beta) + \gamma f((z+x)/2\gamma)\| \leq \|f(x+y+z)\|$ on restricted domains of Banach modules over a C^*-algebra. As an application we study the asymptotic behavior of a generalized additive mapping.

1. Introduction and Preliminaries

The following question concerning the stability of group homomorphisms was posed by Ulam [1]: *Under what conditions does there exist a group homomorphism near an approximate group homomorphism?*

Hyers [2] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1.1}$$

for all $x, y \in E$.

In 1950, Aoki [3] provided a generalization of the Hyers' theorem for additive mappings and in 1978, Rassias [4] generalized the Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded (see also [5]). The result of Rassias' theorem has been generalized by Forti [6, 7] and Gavruta [8] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers have been published on the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [9–23]). We also refer the readers to the books [24–28].

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$, unit e, and norm $|\cdot|$. Assume that $\mathbb{X}$ is a left A-module and $\mathbb{Y}$ is a left Banach A-module. An additive mapping $T : \mathbb{X} \to \mathbb{Y}$ is called *A-linear* if $T(ax) = aT(x)$ for all $a \in A$ and all $x \in \mathbb{X}$. In this paper, we investigate the stability problem for the following functional inequality:

$$\left\|\alpha f\left(\frac{x+y}{2\alpha}\right) + \beta f\left(\frac{y+z}{2\beta}\right) + \gamma f\left(\frac{z+x}{2\gamma}\right)\right\| \leq \|f(x+y+z)\| \tag{1.2}$$

on restricted domains of Banach modules over a C^*-algebra, where α, β, γ are nonzero positive real numbers. As an application we study the asymptotic behavior of a generalized additive mapping.

2. Solutions of the Functional Inequality (1.2)

Theorem 2.1. *Let $\mathbb{X}$ and $\mathbb{M}$ be left A-modules and let α, β, γ be nonzero real numbers. If a mapping $f : \mathbb{X} \to \mathbb{M}$ with $f(0) = 0$ satisfies the functional inequality*

$$\left\|\alpha f\left(\frac{ax+ay}{2\alpha}\right) + \beta f\left(\frac{ay+az}{2\beta}\right) + \gamma a f\left(\frac{z+x}{2\gamma}\right)\right\| \leq \|f(ax+ay+az)\| \tag{2.1}$$

for all $x, y, z \in \mathbb{X}$ and all $a \in U(A)$, then f is A-linear.

Proof. Letting $z = -x - y$ in (2.1), we get

$$\alpha f\left(\frac{ax+ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma a f\left(-\frac{y}{2\gamma}\right) = 0 \tag{2.2}$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Letting $x = 0$ (resp., $y = 0$) in (2.2), we get

$$\alpha f\left(\frac{ay}{2\alpha}\right) + \gamma a f\left(-\frac{y}{2\gamma}\right) = 0, \qquad \left(\text{resp., } \alpha f\left(\frac{ax}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) = 0\right) \tag{2.3}$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Hence $f(ay) = (-\gamma/\alpha) a f((-\alpha/\gamma)y)$ and it follows from (2.2) and (2.3) that and $f((ax+ay)/2\alpha) - f(ax/2\alpha) - f(ay/2\alpha) = 0$ for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Therefore $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{X}$. Hence $f(rx) = rf(x)$ for all $x \in \mathbb{X}$ and all rational numbers r.

Now let $a \in A$ $(a \neq 0)$ and let m be an integer number with $m > 4|a|$. Then by Theorem 1 of [29], there exist elements $u_1, u_2, u_3 \in U(A)$ such that $(3/m)a = u_1 + u_2 + u_3$. Since f is additive and $f(rbx) = (-\gamma/\alpha)rbf((-\alpha/\gamma)x)$ for all $x \in \mathbb{X}$, all rational numbers r and all $b \in U(A)$, we have

$$\begin{aligned} f(ax) &= \frac{m}{3}f\left(\frac{3}{m}ax\right) = \frac{m}{3}f(u_1x + u_2x + u_3x) = \frac{m}{3}[f(u_1x) + f(u_2x) + f(u_3x)] \\ &= -\frac{m}{3}\frac{\gamma}{\alpha}(u_1 + u_2 + u_3)f\left(-\frac{\alpha}{\gamma}x\right) = -\frac{m}{3}\frac{\gamma}{\alpha}\frac{3}{m}af\left(-\frac{\alpha}{\gamma}x\right) = -\frac{\gamma}{\alpha}af\left(-\frac{\alpha}{\gamma}x\right) \end{aligned} \tag{2.4}$$

for all $x \in \mathbb{X}$. Replacing $(-\gamma/\alpha)x$ instead of x in the above equation, we have

$$f\left(-\frac{\gamma}{\alpha}ax\right) = -\frac{\gamma}{\alpha}af(x) \tag{2.5}$$

for all $x \in \mathbb{X}$. Since a is an arbitrary nonzero element in A in the previous paragraph, one can replace $(-\alpha/\gamma)a$ instead of a in (2.5). Thus we have $f(ax) = af(x)$ for all $x \in \mathbb{X}$ and all $a \in A$ $(a \neq 0)$. So $f : \mathbb{X} \to \mathbb{Y}$ is A-linear. □

The following theorem is another version of Theorem 2.1 on a restricted domain when $\alpha, \beta, \gamma > 0$.

Theorem 2.2. *Let $\mathbb{X}$ and $\mathbb{M}$ be left A-modules and let d, α, β, γ be nonzero positive real numbers. Assume that a mapping $f : \mathbb{X} \to \mathbb{M}$ satisfies $f(0) = 0$ and the functional inequality* (2.1) *for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \geq d$ and all $a \in U(A)$. Then f is A-linear.*

Proof. Letting $z = -x - y$ with $\|x\| + \|y\| \geq d$ in (2.1), we get

$$\alpha f\left(\frac{ax + ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma af\left(-\frac{y}{2\gamma}\right) = 0 \tag{2.6}$$

for all $a \in U(A)$. Let $\delta = \max\{|\beta|^{-1}d, |\gamma|^{-1}d\}$ and let $\|x\| + \|y\| \geq \delta$. Then

$$\|\beta x\| + \|\gamma y\| \geq \min\{|\beta|, |\gamma|\}(\|x\| + \|y\|) \geq \min\{|\beta|, |\gamma|\}\delta \geq d. \tag{2.7}$$

Therefore replacing x and y by $2\beta x$ and $2\gamma y$ in (2.6), respectively, we get

$$\alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma af(-y) = 0 \tag{2.8}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq \delta$ and all $a \in U(A)$.

Similar to the proof of Theorem 3 of [30] (see also [31]), we prove that f satisfies (2.8) for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Suppose $\|x\| + \|y\| < \delta$. If $\|x\| + \|y\| = 0$, let $z \in \mathbb{X}$ with $\|z\| = \delta$, otherwise

$$z := \begin{cases} (\delta + \|x\|)\dfrac{x}{\|x\|}, & \text{if } \|x\| \geq \|y\|; \\[2ex] (\delta + \|y\|)\dfrac{y}{\|y\|}, & \text{if } \|y\| \geq \|x\|. \end{cases} \tag{2.9}$$

Since $\alpha, \beta, \gamma > 0$, it is easy to verify that

$$\begin{gathered} \left\|\left(2+\beta^{-1}\gamma\right)z+\beta^{-1}\gamma y\right\| + \left\|\beta\gamma^{-1}x - \left(1+2\beta\gamma^{-1}\right)z\right\| \geq \delta, \\ \|x\| + \|z\| \geq \delta, \\ \left\|2\left(1+\beta^{-1}\gamma\right)z\right\| + \|y\| \geq \delta, \\ \left\|2\left(1+\beta^{-1}\gamma\right)z\right\| + \left\|\beta\gamma^{-1}x - \left(1+2\beta\gamma^{-1}\right)z\right\| \geq \delta, \\ \left\|\left(2+\beta^{-1}\gamma\right)z+\beta^{-1}\gamma y\right\| + \|z\| \geq \delta. \end{gathered} \tag{2.10}$$

Therefore

$$\begin{aligned} &\alpha f\left(\frac{\beta ax+\gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma a f(-y) \\ &\quad= \left[\alpha f\left(\frac{\beta ax+\gamma ay}{\alpha}\right) + \beta f\left(-\left(2+\beta^{-1}\gamma\right)az - \beta^{-1}\gamma ay\right) + \gamma a f\left(\left(1+2\beta\gamma^{-1}\right)z - \beta\gamma^{-1}x\right)\right] \\ &\qquad+ \left[\alpha f\left(\frac{\beta ax+\gamma az}{\alpha}\right) + \beta f(-ax) + \gamma a f(-z)\right] \\ &\qquad+ \left[\alpha f\left(\frac{2(\beta+\gamma)az+\gamma ay}{\alpha}\right) + \beta f\left(-2\left(1+\beta^{-1}\gamma\right)az\right) + \gamma a f(-y)\right] \\ &\qquad- \left[\alpha f\left(\frac{\beta ax+\gamma az}{\alpha}\right) + \beta f\left(-2\left(1+\beta^{-1}\gamma\right)az\right) + \gamma a f\left(\left(1+2\beta\gamma^{-1}\right)z - \beta\gamma^{-1}x\right)\right] \\ &\qquad- \left[\alpha f\left(\frac{2(\beta+\gamma)az+\gamma ay}{\alpha}\right) + \beta f\left(-\left(2+\beta^{-1}\gamma\right)az - \beta^{-1}\gamma ay\right) + \gamma a f(-z)\right] = 0. \end{aligned} \tag{2.11}$$

Hence f satisfies (2.8) and we infer that f satisfies (2.2) for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. By Theorem 2.1, f is A-linear. □

3. Generalized Hyers-Ulam Stability of (1.2) on a Restricted Domain

In this section, we investigate the stability problem for A-linear mappings associated to the functional inequality (1.2) on a restricted domain. For convenience, we use the following abbreviation for a given function $f:\mathbb{X}\to\mathbb{Y}$ and $a\in U(A)$:

$$D_a f(x,y,z) := \alpha f\left(\frac{ax+ay}{2\alpha}\right)+\beta f\left(\frac{ay+az}{2\beta}\right)+\gamma a f\left(\frac{z+x}{2\gamma}\right) \tag{3.1}$$

for all $x,y,z\in\mathbb{X}$.

Theorem 3.1. *Let $d,\alpha,\beta,\gamma>0$, $p\in(0,1)$, and $\theta,\varepsilon\ge 0$ be given. Assume that a mapping $f:\mathbb{X}\to\mathbb{Y}$ satisfies the functional inequality*

$$f\|D_a f(x,y,z)\| \le \|f(ax+ay+az)\|+\theta+\varepsilon\left(\|x\|^p+\|y\|^p+\|z\|^p\right) \tag{3.2}$$

for all $x,y,z\in\mathbb{X}$ with $\|x\|+\|y\|+\|z\|\ge d$ and all $a\in U(A)$. Then there exist a unique A-linear mapping $T:\mathbb{X}\to\mathbb{Y}$ and a constant $C>0$ such that

$$\|f(x)-T(x)\|\le C+\frac{24\times 2^p\alpha^{p-1}\varepsilon}{(2-2^p)}\|x\|^p \tag{3.3}$$

for all $x\in\mathbb{X}$.

Proof. Let $z=-x-y$ with $\|x\|+\|y\|\ge d$. Then (3.2) implies that

$$\begin{aligned}\left\|\alpha f\left(\frac{ax+ay}{2\alpha}\right)+\beta f\left(-\frac{ax}{2\beta}\right)+\gamma a f\left(-\frac{y}{2\gamma}\right)\right\| &\le \|f(0)\|+\theta+\varepsilon\left(\|x\|^p+\|y\|^p+\|x+y\|^p\right)\\ &\le \|f(0)\|+\theta+2\varepsilon\left(\|x\|^p+\|y\|^p\right).\end{aligned} \tag{3.4}$$

Thus

$$\left\|\alpha f\left(\frac{ax+ay}{\alpha}\right)+\beta f\left(-\frac{ax}{\beta}\right)+\gamma a f\left(-\frac{y}{\gamma}\right)\right\|\le \|f(0)\|+\theta+2^{p+1}\varepsilon\left(\|x\|^p+\|y\|^p\right) \tag{3.5}$$

for all $x,y\in\mathbb{X}$ with $\|x\|+\|y\|\ge d$ and all $a\in U(A)$. Let $\delta=\max\{\beta^{-1}d,\gamma^{-1}d\}$ and let $\|x\|+\|y\|\ge\delta$. Then $\|\beta x\|+\|\gamma y\|\ge d$. Therefore it follows from (3.5) that

$$\left\|\alpha f\left(\frac{\beta ax+\gamma ay}{\alpha}\right)+\beta f(-ax)+\gamma a f(-y)\right\|\le \|f(0)\|+\theta+2^{p+1}\varepsilon\left(\|\beta x\|^p+\|\gamma y\|^p\right) \tag{3.6}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq \delta$ and all $a \in U(A)$. For the case $\|x\| + \|y\| < \delta$, let z be an element of $\mathbb{X}$ which is defined in the proof of Theorem 2.2. It is clear that $\|z\| \leq 2\delta$. Using (2.11) and (3.6), we get

$$\begin{aligned}
&\left\|\alpha f\left(\frac{\beta a x + \gamma a y}{\alpha}\right) + \beta f(-ax) + \gamma a f(-y)\right\| \\
&\leq \left\| \left[\alpha f\left(\frac{\beta a x + \gamma a y}{\alpha}\right) + \beta f\left(-\left(2 + \beta^{-1}\gamma\right) a z - \beta^{-1}\gamma a y\right) + \gamma a f\left(\left(1 + 2\beta\gamma^{-1}\right) z - \beta\gamma^{-1} x\right)\right] \right\| \\
&\quad + \left\| \left[\alpha f\left(\frac{\beta a x + \gamma a z}{\alpha}\right) + \beta f(-ax) + \gamma a f(-z)\right] \right\| \\
&\quad + \left\| \left[\alpha f\left(\frac{2(\beta + \gamma) a z + \gamma a y}{\alpha}\right) + \beta f\left(-2\left(1 + \beta^{-1}\gamma\right) a z\right) + \gamma a f(-y)\right] \right\| \\
&\quad + \left\| \left[\alpha f\left(\frac{\beta a x + \gamma a z}{\alpha}\right) + \beta f\left(-2\left(1 + \beta^{-1}\gamma\right) a z\right) + \gamma a f\left(\left(1 + 2\beta\gamma^{-1}\right) z - \beta\gamma^{-1} x\right)\right] \right\| \\
&\quad + \left\| \left[\alpha f\left(\frac{2(\beta + \gamma) a z + \gamma a y}{\alpha}\right) + \beta f\left(-\left(2 + \beta^{-1}\gamma\right) a z - \beta^{-1}\gamma a y\right) + \gamma a f(-z)\right] \right\| \\
&\leq 5(\|f(0)\| + \theta) + 4^{p+1}\varepsilon\delta^p \left[2(2\beta + \gamma)^p + 2^p(\beta + \gamma)^p + \gamma^p\right] + 6 \times 2^p \varepsilon\left(\|\beta x\|^p + \|\gamma y\|^p\right)
\end{aligned} \tag{3.7}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| < \delta$ and all $a \in U(A)$. Hence

$$\left\|\alpha f\left(\frac{\beta a x + \gamma a y}{\alpha}\right) + \beta f(-ax) + \gamma a f(-y)\right\| \leq K + 6 \times 2^p \varepsilon\left(\|\beta x\|^p + \|\gamma y\|^p\right) \tag{3.8}$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$, where

$$K := 5(\|f(0)\| + \theta) + 4^{p+1}\varepsilon\delta^p \left[2(2\beta + \gamma)^p + 2^p(\beta + \gamma)^p + \gamma^p\right]. \tag{3.9}$$

Letting $x = 0$ and $y = 0$ in (3.8), respectively, we get

$$\begin{aligned}
&\left\|\alpha f\left(\frac{\gamma a y}{\alpha}\right) + \beta f(0) + \gamma a f(-y)\right\| \leq K + 6 \times 2^p \varepsilon \|\gamma y\|^p, \\
&\left\|\alpha f\left(\frac{\beta a x}{\alpha}\right) + \beta f(-ax) + \gamma a f(0)\right\| \leq K + 6 \times 2^p \varepsilon \|\beta x\|^p
\end{aligned} \tag{3.10}$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. It follows from (3.8) and (3.10) that

$$\|f(x + y) - f(x) - f(y)\| \leq \alpha^{-1}\left[(\beta + \gamma)\|f(0)\| + 3K + 12 \times 2^p \varepsilon\left(\|\alpha x\|^p + \|\alpha y\|^p\right)\right] \tag{3.11}$$

for all $x, y \in \mathbb{X}$. By the results of Hyers [2] and Rassias [4], there exists a unique additive mapping $T : \mathbb{X} \to \mathbb{Y}$ given by $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ such that

$$\|f(x) - T(x)\| \le \alpha^{-1}\left[(\beta+\gamma)\|f(0)\| + 3K\right] + \frac{24 \times 2^p \alpha^{p-1}\varepsilon}{(2-2^p)}\|x\|^p \tag{3.12}$$

for all $x \in \mathbb{X}$. It follows from the definition of T and (3.2) that $T(0) = 0$ and $\|D_a T(x,y,z)\| \le \|T(ax+ay+az)\|$ for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \ge d$ and all $a \in U(A)$. Hence T is A-linear by Theorem 2.2. □

We apply the result of Theorem 3.1 to study the asymptotic behavior of a generalized additive mapping. An asymptotic property of additive mappings has been proved by Skof [32] (see also [30, 33]).

Corollary 3.2. *Let α, β, γ be nonzero positive real numbers. Assume that a mapping $f : \mathbb{X} \to \mathbb{Y}$ with $f(0) = 0$ satisfies*

$$\|D_a f(x,y,z) - f(ax+ay+az)\| \longrightarrow 0 \quad \text{as } \|x\| + \|y\| + \|z\| \longrightarrow \infty \tag{3.13}$$

for all $a \in U(A)$, then f is A-linear.

Proof. It follows from (3.13) that there exists a sequence $\{\delta_n\}$, monotonically decreasing to zero, such that

$$\|D_a f(x,y,z) - f(ax+ay+az)\| \le \delta_n \tag{3.14}$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \ge n$ and all $a \in U(A)$. Therefore

$$\|D_a f(x,y,z)\| \le \|f(ax+ay+az)\| + \delta_n \tag{3.15}$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \ge n$ and all $a \in U(A)$. Applying (3.15) and Theorem 3.1, we obtain a sequence $\{T_n : \mathbb{X} \to \mathbb{Y}\}$ of unique A-linear mappings satisfying

$$\|f(x) - T_n(x)\| \le 15\alpha^{-1}\delta_n \tag{3.16}$$

for all $x \in \mathbb{X}$. Since the sequence $\{\delta_n\}$ is monotonically decreasing, we conclude

$$\|f(x) - T_m(x)\| \le 15\alpha^{-1}\delta_m \le 15\alpha^{-1}\delta_n \tag{3.17}$$

for all $x \in \mathbb{X}$ and all $m \ge n$. The uniqueness of T_n implies $T_m = T_n$ for all $m \ge n$. Hence letting $n \to \infty$ in (3.16), we obtain that f is A-linear. □

The following theorem is another version of Theorem 3.1 for the case $p > 1$.

Theorem 3.3. *Let $p > 1, d > 0, \varepsilon \geq 0$ be given and let α, β, γ be nonzero real numbers. Assume that a mapping $f : \mathbb{X} \to \mathbb{Y}$ with $f(0) = 0$ satisfies the functional inequality*

$$\|D_a f(x, y, z)\| \leq \|f(ax + ay + az)\| + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.18}$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \leq d$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $\phi : \mathbb{X} \to \mathbb{Y}$ such that

$$\|\phi(x) - f(x)\| \leq \frac{(6 + 2^p) \times 2^p |\alpha|^{p-1} \varepsilon}{2^p - 2} \|x\|^p \tag{3.19}$$

for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$ and $\phi(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$.

Proof. Letting $z = -x - y$ in (3.18), we get

$$\left\|\alpha f\left(\frac{ax + ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma a f\left(-\frac{y}{2\gamma}\right)\right\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|x + y\|^p) \tag{3.20}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \leq d/2$ and all $a \in U(A)$. Hence

$$\left\|\alpha f\left(\frac{ax + ay}{\alpha}\right) + \beta f\left(-\frac{ax}{\beta}\right) + \gamma a f\left(-\frac{y}{\gamma}\right)\right\| \leq 2^p \varepsilon(\|x\|^p + \|y\|^p + \|x + y\|^p) \tag{3.21}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \leq d/4$ and all $a \in U(A)$. It follows from (3.21) that

$$\begin{aligned} \left\|\alpha f\left(\frac{ax}{\alpha}\right) + \beta f\left(-\frac{ax}{\beta}\right)\right\| &\leq 2^{p+1} \varepsilon \|x\|^p, \\ \left\|\alpha f\left(\frac{ay}{\alpha}\right) + \gamma a f\left(-\frac{y}{\gamma}\right)\right\| &\leq 2^{p+1} \varepsilon \|y\|^p \end{aligned} \tag{3.22}$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\| \leq d/4$ and all $a \in U(A)$. Adding (3.21) to (3.22), we get

$$\left\|\alpha f\left(\frac{ax + ay}{\alpha}\right) - \alpha f\left(\frac{ax}{\alpha}\right) - \alpha f\left(\frac{ay}{\alpha}\right)\right\| \leq 2^p \varepsilon(3\|x\|^p + 3\|y\|^p + \|x + y\|^p) \tag{3.23}$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\| \leq d/8$ and all $a \in U(A)$. Therefore

$$\|f(x + y) - f(x) - f(y)\| \leq 2^p |\alpha|^{p-1} \varepsilon(3\|x\|^p + 3\|y\|^p + \|x + y\|^p) \tag{3.24}$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\| \leq d/8|\alpha|$. Let $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$. We may put $y = x$ in (3.24) to obtain

$$\|f(2x) - 2f(x)\| \leq (6 + 2^p) \times 2^p |\alpha|^{p-1} \varepsilon \|x\|^p. \tag{3.25}$$

We can replace x by $x/2^{n+1}$ in (3.25) for all nonnegative integers n. Then using a similar argument given in [4], we have

$$\left\|2^{n+1} f(2^{-n-1}x) - 2^n f(2^{-n}x)\right\| \le (6+2^p) \times \left(\frac{2}{2^p}\right)^n |\alpha|^{p-1} \varepsilon \|x\|^p. \tag{3.26}$$

Hence we have the following inequality:

$$\begin{aligned}\left\|2^{n+1} f\left(2^{-n-1}x\right) - 2^m f(2^{-m}x)\right\| &\le \sum_{k=m}^{n} \left\|2^{k+1} f\left(2^{-k-1}x\right) - 2^k f\left(2^{-k}x\right)\right\| \\ &\le (6+2^p)|\alpha|^{p-1} \varepsilon \sum_{k=m}^{n} \left(\frac{2}{2^p}\right)^k \|x\|^p\end{aligned} \tag{3.27}$$

for all $x \in \mathbb{X}$ with $\|x\| \le d/8|\alpha|$ and all integers $n \ge m \ge 0$. Since Y is complete, (3.27) shows that the limit $T(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ exists for all $x \in \mathbb{X}$ with $\|x\| \le d/8|\alpha|$. Letting $m = 0$ and $n \to \infty$ in (3.27), we obtain that T satisfies inequality (3.19) for all $x \in \mathbb{X}$ with $\|x\| \le d/8|\alpha|$. It follows from the definition of T and (3.24) that

$$T(x+y) = T(x) + T(y) \tag{3.28}$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\|, \|x+y\| \le d/8|\alpha|$. Hence

$$T\left(\frac{x}{2}\right) = \frac{1}{2} T(x) \tag{3.29}$$

for all $x \in \mathbb{X}$ with $\|x\| \le d/8|\alpha|$. We extend the additivity of T to the whole space $\mathbb{X}$ by using an extension method of Skof [34]. Let $\delta := d/8|\alpha|$ and $x \in \mathbb{X}$ be given with $\|x\| > \delta$. Let $k = k(x)$ be the smallest integer such that $2^{k-1}\delta < \|x\| \le 2^k \delta$. We define the mapping $\phi : \mathbb{X} \to \mathbb{Y}$ by

$$\phi(x) := \begin{cases} T(x), & \text{if } \|x\| \le \delta, \\ \\ 2^k T(2^{-k}x), & \text{if } \|x\| > \delta. \end{cases} \tag{3.30}$$

Let $x \in \mathbb{X}$ be given with $\|x\| > \delta$ and let $k = k(x)$ be the smallest integer such that $2^{k-1}\delta < \|x\| \le 2^k \delta$. Then $k-1$ is the smallest integer satisfying $2^{k-2}\delta < \|x/2\| \le 2^{k-1}\delta$. If $k = 1$, we have $\phi(x/2) = T(x/2)$ and $\phi(x) = 2T(x/2)$. Therefore $\phi(x/2) = (1/2)\phi(x)$. For the case $k > 1$, it follows from the definition of ϕ that

$$\phi\left(\frac{x}{2}\right) = 2^{k-1} T\left(2^{-(k-1)} \frac{x}{2}\right) = \frac{1}{2} \cdot 2^k T\left(2^{-k}x\right) = \frac{1}{2}\phi(x). \tag{3.31}$$

From the definition of ϕ and (3.29), we get that $\phi(x/2) = (1/2)\phi(x)$ holds true for all $x \in \mathbb{X}$. Let $x \in \mathbb{X}$ and let k be an integer such that $\|x\| \le 2^k\delta$. Then

$$\phi(x) = 2^k\phi\left(2^{-k}x\right) = 2^kT\left(2^{-k}x\right) = \lim_{n\to\infty} 2^{n+k} f\left(2^{-(n+k)}x\right) = \lim_{n\to\infty} 2^n f\left(2^{-n}x\right). \tag{3.32}$$

It remains to prove that ϕ is A-linear. Let $x, y \in \mathbb{X}$ and let n be a positive integer such that $\|x\|, \|y\|, \|x+y\| \le 2^n\delta$. Since $\phi(x/2) = (1/2)\phi(x)$ for all $x \in \mathbb{X}$ and T satisfies (3.28), we have

$$\begin{aligned}\phi(x+y) = 2^n\phi\left(\frac{x+y}{2^n}\right) = 2^nT\left(\frac{x+y}{2^n}\right) &= 2^n\left[T\left(\frac{x}{2^n}\right) + T\left(\frac{y}{2^n}\right)\right] \\ &= 2^n\left[\phi\left(\frac{x}{2^n}\right) + \phi\left(\frac{y}{2^n}\right)\right] = \phi(x) + \phi(y).\end{aligned} \tag{3.33}$$

Hence ϕ is additive. Since $\phi(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ for all $x \in \mathbb{X}$, we have from (3.22) that $\alpha\phi(ay/\alpha) = \gamma a\phi(y/\gamma))$ for all $y \in \mathbb{X}$ and all $a \in U(A)$. Letting $a = e$, we get $\alpha\phi(y/\alpha) = \gamma\phi(y/\gamma))$. Therefore $\phi(ay) = a\phi(y)$ for all $y \in \mathbb{X}$ and all $a \in U(A)$. This proves that ϕ is A-linear. Also, ϕ satisfies inequality (3.19) for all $x \in \mathbb{X}$ with $\|x\| \le d/8|\alpha|$, by the definition of ϕ. □

For the case $p = 1$ we use the Gajda's example [35] to give the following counterexample.

Example 3.4. Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) := \begin{cases} x, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \ge 1. \end{cases} \tag{3.34}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\phi(2^n x). \tag{3.35}$$

It is clear that f is continuous, bounded by 2 on $\mathbb{C}$ and

$$\left|f(x+y) - f(x) - f(y)\right| \le 6\left(|x| + \left|y\right|\right) \tag{3.36}$$

for all $x, y \in \mathbb{C}$ (see [35]). It follows from (3.36) that the following inequality:

$$\left|f(x+y+z) - f(x) - f(y) - f(z)\right| \le 12\left(|x| + \left|y\right| + |z|\right) \tag{3.37}$$

holds for all $x, y, z \in \mathbb{C}$. First we show that

$$\left|f(\lambda x) - \lambda f(x)\right| \le 2(1 + |\lambda|)^2|x| \tag{3.38}$$

for all $x, \lambda \in \mathbb{C}$. If f satisfies (3.38) for all $|\lambda| \geq 1$, then f satisfies (3.38) for all $\lambda \in \mathbb{C}$. To see this, let $0 < |\lambda| < 1$ (the result is obvious when $\lambda = 0$). Then $|f(\lambda^{-1}x) - \lambda^{-1}f(x)| \leq 2(1 + |\lambda|^{-1})^2|x|$ for all $x \in \mathbb{C}$. Replacing x by λx, we get that $|f(\lambda x) - \lambda f(x)| \leq 2|\lambda|^2(1 + |\lambda|^{-1})^2|x| = 2(1+|\lambda|)^2|x|$ for all $x \in \mathbb{C}$. Hence we may assume that $|\lambda| \geq 1$. If $\lambda x = 0$ or $|\lambda x| \geq 1$, then

$$\left|f(\lambda x) - \lambda f(x)\right| \leq 2(1+|\lambda|) \leq 2|\lambda|(1+|\lambda|)|x| \leq 2(1+|\lambda|)^2|x|. \tag{3.39}$$

Now suppose that $0 < |\lambda x| < 1$. Then there exists an integer $k \geq 0$ such that

$$\frac{1}{2^{k+1}} \leq |\lambda x| < \frac{1}{2^k}. \tag{3.40}$$

Therefore

$$2^k|x|,\ 2^k|\lambda x| \in (-1,1). \tag{3.41}$$

Hence

$$2^m|x|,\ 2^m|\lambda x| \in (-1,1) \tag{3.42}$$

for all $m = 0, 1, \ldots, k$. From the definition of f and (3.40), we have

$$\begin{aligned} \left|f(\lambda x) - \lambda f(x)\right| &= \left|\sum_{n=k+1}^{\infty} \frac{1}{2^n}\left[\phi(2^n\lambda x) - \lambda\phi(2^n x)\right]\right| \\ &\leq (1+|\lambda|)\sum_{n=k+1}^{\infty}\frac{1}{2^n} = \frac{1+|\lambda|}{2^k} \leq 2|\lambda|(1+|\lambda|)|x| \leq 2(1+|\lambda|)^2|x|. \end{aligned} \tag{3.43}$$

Therefore f satisfies (3.38). Now we prove that

$$\begin{aligned} &\left|D_\mu f(x,y,z) - f(\mu x + \mu y + \mu z)\right| \\ &\quad \leq \left(16 + |\alpha|^{-1}(1+|\alpha|)^2 + |\beta|^{-1}(1+|\beta|)^2 + |\gamma|^{-1}(1+|\gamma|)^2\right)\ (|x| + |y| + |z|) \end{aligned} \tag{3.44}$$

for all $x, y, z \in \mathbb{C}$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, where

$$D_\mu f(x,y,z) := \alpha f\left(\frac{\mu x + \mu y}{2\alpha}\right) + \beta f\left(\frac{\mu y + \mu z}{2\beta}\right) + \gamma\mu f\left(\frac{z+x}{2\gamma}\right). \tag{3.45}$$

It follows from (3.37) and (3.38) that

$$
\begin{aligned}
&\left|D_\mu f(x,y,z) - f(\mu x+\mu y+\mu z)\right| \\
&\quad\le \left|\alpha f\left(\frac{\mu x+\mu y}{2\alpha}\right) - f\left(\frac{\mu x+\mu y}{2}\right)\right| + \left|\beta f\left(\frac{\mu y+\mu z}{2\beta}\right) - f\left(\frac{\mu y+\mu z}{2}\right)\right| \\
&\qquad+ \left|\gamma\mu f\left(\frac{z+x}{2\gamma}\right) - \mu f\left(\frac{z+x}{2}\right)\right| + \left|\mu f\left(\frac{z+x}{2}\right) - f\left(\frac{\mu z+\mu x}{2}\right)\right| \\
&\qquad+ \left|f\left(\frac{\mu x+\mu y}{2}\right) + f\left(\frac{\mu y+\mu z}{2}\right) + f\left(\frac{\mu z+\mu x}{2}\right) - f(\mu x+\mu y+\mu z)\right| \\
&\quad\le \left(6+|\alpha|^{-1}(1+|\alpha|)^2\right)|x+y| + \left(6+|\beta|^{-1}(1+|\beta|)^2\right)|y+z| + \left(10+|\gamma|^{-1}(1+|\gamma|)^2\right)|x+z| \\
&\quad\le \left(16+|\alpha|^{-1}(1+|\alpha|)^2+|\beta|^{-1}(1+|\beta|)^2+|\gamma|^{-1}(1+|\gamma|)^2\right)(|x|+|y|+|z|)
\end{aligned}
\tag{3.46}
$$

for all $x, y, z \in \mathbb{C}$ and all $\mu \in \mathbb{T}^1$. Thus f satisfies inequality (3.18) for $p = 1$. Let $T : \mathbb{C} \to \mathbb{C}$ be a linear functional such that

$$
\left|f(x) - T(x)\right| \le M|x| \tag{3.47}
$$

for all $x \in \mathbb{C}$, where M is a positive constant. Then there exists a constant $c \in \mathbb{C}$ such that $T(x) = cx$ for all rational numbers x. So we have

$$
\left|f(x)\right| \le (M+|c|)|x| \tag{3.48}
$$

for all rational numbers x. Let $m \in \mathbb{N}$ with $m > M+|c|$. If $x_0 \in (0, 2^{-m+1}) \cap \mathbb{Q}$, then $2^n x_0 \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. So

$$
f(x_0) \ge \sum_{n=0}^{m-1} \frac{1}{2^n}\phi(2^n x_0) = m x_0 > (M+|c|)x_0, \tag{3.49}
$$

which contradicts (3.48).

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