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Research Article

Asymptotic Behavior of Impulsive Infinite Delay Difference Equations with Continuous Variables

Zhixia Ma[1] and Liguang Xu[2]

[1] *College of Computer Science & Technology, Southwest University for Nationalities, Chengdu 610064, China*
[2] *Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China*

Correspondence should be addressed to Liguang Xu, xlg132@126.com

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A class of impulsive infinite delay difference equations with continuous variables is considered. By establishing an infinite delay difference inequality with impulsive initial conditions and using the properties of "ϱ-cone," we obtain the attracting and invariant sets of the equations.

1. Introduction

Difference equations with continuous variables are difference equations in which the unknown function is a function of a continuous variable [1]. These equations appear as natural descriptions of observed evolution phenomena in many branches of the natural sciences (see, e.g., [2, 3]). The book mentioned in [3] presents an exposition of some unusual properties of difference equations, specially, of difference equations with continuous variables. In the recent years, the asymptotic behavior and other behavior of delay difference equations with continuous variables have received much attention due to its potential application in various fields such as numerical analysis, control theory, finite mathematics, and computer science. Many results have appeared in the literatures; see, for example, [1, 4–7].

However, besides the delay effect, an impulsive effect likewise exists in a wide variety of evolutionary process, in which states are changed abruptly at certain moments of time. Recently, impulsive difference equations with discrete variable have attracted considerable attention. In particular, delay effect on the asymptotic behavior and other behaviors of impulsive difference equations with discrete variable has been extensively studied by many authors and various results are reported [8–12]. However, to the best of our knowledge, very little has been done with the corresponding problems for impulsive delay difference equations with continuous variables. Motivated by the above discussions, the main aim of this paper is to study the asymptotic behavior of impulsive infinite delay difference equations with continuous variables. By establishing an infinite delay difference inequality with impulsive initial conditions and using the properties of "ϱ-cone," we obtain the attracting and invariant sets of the equations.

2. Preliminaries

Consider the impulsive infinite delay difference equation with continuous variable

$$
\begin{aligned}
x_i(t) &= a_i x_i(t-\tau_1) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau_1)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_2)) \\
&\quad + \int_{-\infty}^{t} p_{ij}(t-s) h_j(x_j(s))\,ds + I_i, \quad t \neq t_k,\ t \geq t_0, \\
x_i(t) &= J_{ik}(x_i(t^-)), \quad t \geq t_0,\ t = t_k,\ k = 1,2,\ldots,
\end{aligned} \tag{2.1}
$$

where a_i, I_i, a_{ij}, and b_{ij} $(i, j \in \mathcal{N})$ are real constants, $p_{ij} \in L^e$ (here, $\mathcal{N}$ and L^e will be defined later), τ_1 and τ_2 are positive real numbers. t_k $(k = 1, 2, \ldots)$ is an impulsive sequence such that $t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$. f_j, g_j, h_j, and J_{ik}: $\mathbb{R} \to \mathbb{R}$ are real-valued functions.

By a solution of (2.1), we mean a piecewise continuous real-valued function $x_i(t)$ defined on the interval $(-\infty, \infty)$ which satisfies (2.1) for all $t \geq t_0$.

In the sequel, by Φ_i we will denote the set of all continuous real-valued functions ϕ_i defined on an interval $(-\infty, 0]$, which satisfies the "compatibility condition"

$$
\phi_i(0) = a_i\phi_i(-\tau_1) + \sum_{j=1}^{n} a_{ij} f_j(\phi_j(-\tau_1)) + \sum_{j=1}^{n} b_{ij} g_j(\phi_j(-\tau_2)) + \int_{-\infty}^{0} p_{ij}(-s) h_j(\phi_j(s))\,ds + I_i. \tag{2.2}
$$

By the method of steps, one can easily see that, for any given initial function $\phi_i \in \Phi_i$, there exists a unique solution $x_i(t), i \in \mathcal{N}$, of (2.1) which satisfies the initial condition

$$
x_i(t + t_0) = \phi_i(t), \quad t \in (-\infty, 0], \tag{2.3}
$$

this function will be called the solution of the initial problem (2.1)–(2.3).

For convenience, we rewrite (2.1) and (2.3) into the following vector form

$$
\begin{aligned}
x(t) &= A_0 x(t-\tau_1) + Af(x(t-\tau_1)) + Bg(x(t-\tau_2)) \\
&\quad + \int_{-\infty}^{t} P(t-s) h(x(s))\,ds + I, \quad t \neq t_k,\ t \geq t_0, \\
x(t) &= J_k(x(t^-)), \quad t \geq t_0,\ t = t_k,\ k = 1,2,\ldots, \\
x(t_0 + \theta) &= \phi(\theta), \quad \theta \in (-\infty, 0],
\end{aligned} \tag{2.4}
$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$, $A_0 = \operatorname{diag}\{a_1, \dots, a_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $P(t) = (p_{ij}(t))_{n\times n}$, $I = (I_1, \dots, I_n)^T$, $f(x) = (f_1(x_1), \dots, f_n(x_n))^T$, $g(x) = (g_1(x_1), \dots, g_n(x_n))^T$, $h(x) = (h_1(x_1), \dots, h_n(x_n))^T$, $J_k(x) = (J_{1k}(x), \dots, J_{nk}(x))^T$, and $\phi = (\phi_1, \dots, \phi_n)^T \in \Phi$, in which $\Phi = (\Phi_1, \dots, \Phi_n)^T$.

In what follows, we introduce some notations and recall some basic definitions. Let $\mathbb{R}^n(\mathbb{R}^n_+)$ be the space of n-dimensional (nonnegative) real column vectors, $\mathbb{R}^{m\times n}(\mathbb{R}^{m\times n}_+)$ be the set of $m \times n$ (nonnegative) real matrices, E be the n-dimensional unit matrix, and $|\cdot|$ be the Euclidean norm of $\mathbb{R}^n$. For $A, B \in \mathbb{R}^{m\times n}$ or $A, B \in \mathbb{R}^n$, $A \geq B$ $(A \leq B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq$ $(\leq, >, <)$."Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$. $\mathcal{N} \triangleq \{1, 2, \dots, n\}$ and $e_n = (1, 1, \dots, 1)^T \in \mathbb{R}^n$.

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C \triangleq C[(-\infty, 0], \mathbb{R}^n]$

$$PC[\mathbb{J}, \mathbb{R}^n] = \left\{ \psi : \mathbb{J} \longrightarrow \mathbb{R}^n \;\middle|\; \begin{array}{c} \psi(s) \text{ is continuous for all but at most} \\ \text{countable points } s \in \mathbb{J} \text{ and at these points} \\ s \in \mathbb{J},\ \psi(s^+) \text{ and } \psi(s^-) \text{ exist, } \psi(s) = \psi(s^+) \end{array} \right\}, \tag{2.5}$$

where $\mathbb{J} \subset \mathbb{R}$ is an interval, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$, respectively. Especially, let $PC \triangleq PC[(-\infty, 0], \mathbb{R}^n]$

$$L^e = \left\{ \begin{array}{c} \psi(s) : \mathbb{R}_+ \rightarrow \mathbb{R}, \\ \text{where } \mathbb{R}_+ = [0, \infty) \end{array} \;\middle|\; \begin{array}{c} \psi(s) \text{ is piecewise continuous and satisfies} \\ \int_0^\infty e^{\lambda_0 s} \left|\psi(s)\right| ds < \infty, \text{ where } \lambda_0 > 0 \text{ is constant} \end{array} \right\}. \tag{2.6}$$

For $x \in \mathbb{R}^n$, $\phi \in C$ $(\phi \in PC)$, and $A \in \mathbb{R}^{n\times n}$ we define

$$\begin{gathered} [x]^+ = (|x_1|, \dots, |x_n|)^T, \quad [\phi]^+_\infty = \left([\phi_1(t)]^+_\infty, \dots, [\phi_n(t)]^+_\infty\right)^T, \\ [\phi_i(t)]^+_\infty = \sup_{\theta\in(-\infty,0]} \left|\phi_i(t+\theta)\right|, \quad i \in \mathcal{N},\ [A]^+ = \left(\left|a_{ij}\right|\right)_{n\times n}, \end{gathered} \tag{2.7}$$

and $\varrho(A)$ denotes the spectral radius of A.

For any $\phi \in C$ or $\phi \in PC$, we always assume that ϕ is bounded and introduce the following norm:

$$\|\phi\| = \sup_{-\infty<\theta\leq 0} \left|\phi(s)\right|. \tag{2.8}$$

Definition 2.1. The set $S \subset PC$ is called a positive invariant set of (2.4), if for any initial value $\phi \in S$, the solution $x(t, t_0, \phi) \in S$, $t \geq t_0$.

Definition 2.2. The set $S \subset PC$ is called a global attracting set of (2.4), if for any initial value $\phi \in PC$, the solution $x(t, t_0, \phi)$ satisfies

$$\operatorname{dist}(x(t, t_0, \phi), S) \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \tag{2.9}$$

where $\operatorname{dist}(\phi, S) = \inf_{\psi \in S} \operatorname{dist}(\phi, \psi)$, $\operatorname{dist}(\phi, \psi) = \sup_{\theta \in (-\infty, 0]} |\phi(\theta) - \psi(\theta)|$, for $\psi \in PC$.

Definition 2.3. System (2.4) is said to be globally exponentially stable if for any solution $x(t, t_0, \phi)$, there exist constants $\xi > 0$ and $\kappa_0 > 0$ such that

$$|x(t, t_0, \phi)| \leq \kappa_0 \|\phi\| e^{-\xi(t-t_0)}, \quad t \geq t_0. \tag{2.10}$$

Lemma 2.4 (See [13, 14]). *If $M \in \mathbb{R}_+^{n\times n}$ and $\varrho(M) < 1$, then $(E - M)^{-1} \geq 0$.*

Lemma 2.5 (La Salle [14]). *Suppose that $M \in \mathbb{R}_+^{n\times n}$ and $\varrho(M) < 1$, then there exists a positive vector z such that $(E - M)z > 0$.*

For $M \in \mathbb{R}_+^{n\times n}$ and $\varrho(M) < 1$, we denote

$$\Omega_\varrho(M) = \{z \in \mathbb{R}^n \mid (E - M)z > 0, z > 0\}, \tag{2.11}$$

which is a nonempty set by Lemma 2.5, satisfying that $k_1 z_1 + k_2 z_2 \in \Omega_\varrho(M)$ for any scalars $k_1 > 0$, $k_2 > 0$, and vectors $z_1, z_2 \in \Omega_\varrho(M)$. So $\Omega_\varrho(M)$ is a cone without vertex in $\mathbb{R}^n$, we call it a "ϱ-cone" [12].

3. Main Results

In this section, we will first establish an infinite delay difference inequality with impulsive initial conditions and then give the attracting and invariant sets of (2.4).

Theorem 3.1. *Let $P = (p_{ij})_{n\times n}, W = (w_{ij})_{n\times n} \in \mathbb{R}_+^{n\times n}$, $I = (I_1, \ldots, I_n)^T \in \mathbb{R}_+^n$, and $Q(t) = (q_{ij}(t))_{n\times n}$, where $0 \leq q_{ij}(t) \in L^e$. Denote $\widetilde{Q} = (\widetilde{q}_{ij})_{n\times n} \triangleq (\int_0^\infty q_{ij}(t)dt)_{n\times n}$ and let $\varrho(P + W + \widetilde{Q}) < 1$ and $u(t) \in \mathbb{R}^n$ be a solution of the following infinite delay difference inequality with the initial condition $u(\theta) \in PC[(-\infty, t_0], \mathbb{R}^n]$:*

$$u(t) \leq Pu(t - \tau_1) + Wu(t - \tau_2) + \int_0^\infty Q(s)u(t - s)ds + I, \quad t \geq t_0. \tag{3.1}$$

(a) Then

$$u(t) \leq ze^{-\lambda(t-t_0)} + (E - P - W - \widetilde{Q})^{-1} I, \quad t \geq t_0, \tag{3.2}$$

provided the initial conditions

$$u(\theta) \leq ze^{-\lambda(\theta-t_0)} + (E - P - W - \widetilde{Q})^{-1} I, \quad \theta \in (-\infty, t_0], \tag{3.3}$$

where $z = (z_1, z_2, \ldots, z_n)^T \in \Omega_\varrho(P + W + \widetilde{Q})$ *and the positive number* $\lambda \le \lambda_0$ *is determined by the following inequality:*

$$\left(e^{\lambda}\left(Pe^{\lambda\tau_1} + We^{\lambda\tau_2} + \int_0^{\infty} Q(s)e^{\lambda s}ds\right) - E\right)z \le 0. \tag{3.4}$$

(b) Then

$$u(t) \le d(E - P - W - \widetilde{Q})^{-1}I, \quad t \ge t_0, \tag{3.5}$$

provided the initial conditions

$$u(\theta) \le d(E - P - W - \widetilde{Q})^{-1}I, \quad d \ge 1,\ \theta \in (-\infty, t_0]. \tag{3.6}$$

Proof. (a): Since $\varrho(P + W + \widetilde{Q}) < 1$ and $P + W + \widetilde{Q} \in \mathbb{R}_+^{n\times n}$, then, by Lemma 2.5, there exists a positive vector $z \in \Omega_\varrho(P + W + \widetilde{Q})$ such that $(E - (P + W + \widetilde{Q}))z > 0$. Using continuity and noting $q_{ij}(t) \in L^e$, we know that (3.4) has at least one positive solution $\lambda \le \lambda_0$, that is,

$$\sum_{j=1}^{n}\left[p_{ij}e^{\lambda\tau_1} + w_{ij}e^{\lambda\tau_2} + \int_0^{\infty} q_{ij}(s)e^{\lambda s}ds\right]z_j \le z_i, \quad i \in \mathcal{N}. \tag{3.7}$$

Let $N \triangleq (E - P - W - \widetilde{Q})^{-1}I$, $N = (N_1, \ldots, N_n)^T$, one can get that $(E - P - W - \widetilde{Q})N = I$, or

$$\sum_{j=1}^{n}(p_{ij} + w_{ij} + \tilde{q}_{ij})N_j + I_i = N_i, \quad i \in \mathcal{N}. \tag{3.8}$$

To prove (3.2), we first prove, for any given $\varepsilon > 0$, when $u(\theta) \le ze^{-\lambda(\theta - t_0)} + N, \theta \in (-\infty, t_0]$,

$$u_i(t) \le (1 + \varepsilon)\left[z_i e^{-\lambda(t - t_0)} + N_i\right] \triangleq y_i(t), \quad t \ge t_0,\ i \in \mathcal{N}. \tag{3.9}$$

If (3.9) is not true, then there must be a $t^* > t_0$ and some integer r such that

$$u_r(t^*) > y_r(t^*), \quad u_i(t) \le y_i(t), \quad t \in (-\infty, t^*),\ i \in \mathcal{N}. \tag{3.10}$$

By using (3.1), (3.7)–(3.10), and $q_{ij}(t) \geq 0$, we have

$$
\begin{aligned}
u_r(t^*) &\leq \sum_{j=1}^{n} p_{rj}(1+\varepsilon)\left[z_j e^{-\lambda(t^*-\tau_1-t_0)} + N_j\right] \\
&\quad + \sum_{j=1}^{n} w_{rj}(1+\varepsilon)\left[z_j e^{-\lambda(t^*-\tau_2-t_0)} + N_j\right] \\
&\quad + \sum_{j=1}^{n}\int_0^\infty q_{rj}(s)(1+\varepsilon)\left[z_j e^{-\lambda(t^*-s-t_0)} + N_j\right]ds + I_r \\
&= \sum_{j=1}^{n}\left(p_{rj}e^{\lambda\tau_1} + w_{rj}e^{\lambda\tau_2} + \int_0^\infty q_{rj}(s)e^{\lambda s}ds\right)z_j(1+\varepsilon)e^{-\lambda(t^*-t_0)} \\
&\quad + \sum_{j=1}^{n}\left(p_{rj} + w_{rj} + \widetilde{q}_{rj}\right)N_j(1+\varepsilon) + (1+\varepsilon)I_r - \varepsilon I_r \\
&\leq (1+\varepsilon)\left[z_r e^{-\lambda(t^*-t_0)} + N_r\right] \\
&= y_r(t^*),
\end{aligned}
\tag{3.11}
$$

which contradicts the first equality of (3.10), and so (3.9) holds for all $t \geq t_0$. Letting $\varepsilon \to 0$, then (3.2) holds, and the proof of part (a) is completed.

(b) For any given initial function: $u(t_0+\theta) = \phi(\theta)$, $\theta \in (-\infty, 0]$, where $\phi \in PC$, there is a constant $d \geq 1$ such that $[\phi]^+_\infty \leq dN$. To prove (3.5), we first prove that

$$
u(t) \leq dN + \Lambda \triangleq (\overline{x}_1, \ldots, \overline{x}_n)^T = \overline{x}, \quad t \geq t_0, \tag{3.12}
$$

where $\Lambda = (E - P - W - \widetilde{Q})^{-1} e_n \varepsilon$ ($\varepsilon > 0$ small enough), provided that the initial conditions satisfies $[\phi]^+_\infty \leq \overline{x}$.

If (3.12) is not true, then there must be a $t^* > t_0$ and some integer r such that

$$
u_r(t^*) > \overline{x}_r, \quad u(t) \leq \overline{x}, \quad t \in (-\infty, t^*). \tag{3.13}
$$

By using (3.1), (3.8), (3.13) $q_{ij}(t) \geq 0$, and $\varrho(P + W + \widetilde{Q}) < 1$, we obtain that

$$
\begin{aligned}
u(t^*) &\leq \left(P + W + \widetilde{Q}\right)\overline{x} + I \\
&= \left(P + W + \widetilde{Q}\right)(dN + \Lambda) + I \\
&\leq d\left[\left(P + W + \widetilde{Q}\right)N + I\right] + \left(P + W + \widetilde{Q}\right)\Lambda \\
&\leq dN + \Lambda \\
&= \overline{x},
\end{aligned}
\tag{3.14}
$$

which contradicts the first equality of (3.13), and so (3.12) holds for all $t \geq t_0$. Letting $\varepsilon \to 0$, then (3.5) holds, and the proof of part (b) is completed. □

Remark 3.2. Suppose that $Q(t) = 0$ in part (a) of Theorem 3.1, then we get [15, Lemma 3].

In the following, we will obtain attracting and invariant sets of (2.4) by employing Theorem 3.1. Here, we firstly introduce the following assumptions.

(A_1) For any $x \in \mathbb{R}^n$, there exist nonnegative diagonal matrices $\overline{F}, \overline{G}, \overline{H}$ such that

$$[f(x)]^+ \leq \overline{F}[x]^+, \quad [g(x)]^+ \leq \overline{G}[x]^+, \quad [h(x)]^+ \leq \overline{H}[x]^+. \tag{3.15}$$

(A_2) For any $x \in \mathbb{R}^n$, there exist nonnegative matrices R_k such that

$$[J_k(x)]^+ \leq R_k[x]^+, \quad k = 1, 2, \dots. \tag{3.16}$$

(A_3) Let $\varrho(\widehat{P} + \widehat{W} + \widehat{Q}) < 1$, where

$$\widehat{P} = [A_0]^+ + [A]^+\overline{F}, \qquad \widehat{W} = [B]^+\overline{G}, \qquad \widehat{Q} = \int_0^\infty \overline{Q}(s)ds, \qquad \overline{Q}(s) = [P(s)]^+\overline{H}. \tag{3.17}$$

(A_4) There exists a constant γ such that

$$\frac{\ln \gamma_k}{t_k - t_{k-1}} \leq \gamma < \lambda, \quad k = 1, 2, \dots, \tag{3.18}$$

where the scalar λ satisfies $0 < \lambda \leq \lambda_0$ and is determined by the following inequality

$$\left(e^{\lambda}\left(\widehat{P}e^{\lambda\tau_1} + \widehat{W}e^{\lambda\tau_2} + \int_0^\infty \overline{Q}(s)e^{\lambda s}ds\right) - E\right)z \leq 0, \tag{3.19}$$

where $z = (z_1, \dots, z_n)^T \in \Omega_\varrho(\widehat{P} + \widehat{W} + \widehat{Q})$, and

$$\gamma_k \geq 1, \quad \gamma_k z \geq R_k z, \quad k = 1, 2, \dots. \tag{3.20}$$

(A_5) Let

$$\sigma = \sum_{k=1}^{\infty} \ln \sigma_k < \infty, \quad k = 1, 2, \dots, \tag{3.21}$$

where $\sigma_k \geq 1$ satisfy

$$R_k(E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+ \leq \sigma_k(E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+. \tag{3.22}$$

Theorem 3.3. *If* (A_1)*–*(A_5) *hold, then* $S = \{\phi \in PC \mid [\phi]_\infty^+ \le e^{\sigma}(E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+\}$ *is a global attracting set of* (2.4).

Proof. Since $\varrho(\widehat{P} + \widehat{W} + \widehat{Q}) < 1$ and $\widehat{P}, \widehat{W}, \widehat{Q} \in \mathbb{R}_+^{n\times n}$, then, by Lemma 2.5, there exists a positive vector $z \in \Omega_\varrho(\widehat{P} + \widehat{W} + \widehat{Q})$ such that $(E - (\widehat{P} + \widehat{W} + \widehat{Q}))z > 0$. Using continuity and noting $p_{ij}(t) \in L^e$, we obtain that inequality (3.19) has at least one positive solution $\lambda \le \lambda_0$.

From (2.4) and condition (A_1), we have

$$\begin{aligned}
[x(t)]^+ &\le [A_0 x(t-\tau_1)]^+ + \left[A f(x(t-\tau_1))\right]^+ + \left[B g(x(t-\tau_2))\right]^+ \\
&\quad + \left[\int_{-\infty}^{t} P(t-s) h(x(s)) ds\right]^+ + [I]^+ \\
&\le [A_0]^+[x(t-\tau_1)]^+ + [A]^+\overline{F}[(x(t-\tau_1))]^+ + [B]^+\overline{G}[(x(t-\tau_2)]^+ \\
&\quad + \int_0^\infty [P(s)]^+\overline{H}[(x(t-s))]^+ ds + [I]^+ \\
&= \widehat{P}[x(t-\tau_1)]^+ + \widehat{W}[(x(t-\tau_2))]^+ + \int_0^\infty \overline{Q}(s)[(x(t-s))]^+ ds + [I]^+,
\end{aligned} \tag{3.23}$$

where $t_{k-1} \le t < t_k, k = 1, 2, \ldots$.

Since $\varrho(\widehat{P} + \widehat{W} + \widehat{Q}) < 1$ and $\widehat{P}, \widehat{W}, \widehat{Q} \in \mathbb{R}_+^{n\times n}$, then, by Lemma 2.4, we can get $(E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1} \ge 0$, and so $\widehat{N} \triangleq (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+ \ge 0$.

For the initial conditions: $x(t_0 + \theta) = \phi(\theta), \theta \in (-\infty, 0]$, where $\phi \in PC$, we have

$$[x(t)]^+ \le \kappa_0 z e^{-\lambda(t-t_0)} \le \kappa_0 z e^{-\lambda(t-t_0)} + \widehat{N}, \quad t \in (-\infty, t_0], \tag{3.24}$$

where

$$\kappa_0 = \frac{\|\phi\|}{\min_{1\le i\le n}\{z_i\}}, \quad z \in \Omega_\varrho\left(\widehat{P} + \widehat{W} + \widehat{Q}\right). \tag{3.25}$$

By the property of ϱ-cone and $z \in \Omega_\varrho(\widehat{P} + \widehat{W} + \widehat{Q})$, we have $\kappa_0 z \in \Omega_\varrho(\widehat{P} + \widehat{W} + \widehat{Q})$. Then, all the conditions of part (a) of Theorem 3.1 are satisfied by (3.23), (3.24), and condition (A_3), we derive that

$$[x(t)]^+ \le \kappa_0 z e^{-\lambda(t-t_0)} + \widehat{N}, \quad t \in [t_0, t_1). \tag{3.26}$$

Suppose for all $\iota = 1, \ldots, k$, the inequalities

$$[x(t)]^+ \le \gamma_0 \cdots \gamma_{\iota-1}\kappa_0 z e^{-\lambda(t-t_0)} + \sigma_0 \cdots \sigma_{\iota-1}\widehat{N}, \quad t \in [t_{\iota-1}, t_\iota), \tag{3.27}$$

hold, where $\gamma_0 = \sigma_0 = 1$. Then, from (3.20), (3.22), (3.27), and (A_2), the impulsive part of (2.4) satisfies that

$$\begin{aligned}[x(t_k)]^+ &= \left[J_k(x(t_k^-))\right]^+ \leq R_k[x(t_k^-)]^+ \\ &\leq R_k\left[\gamma_0\cdots\gamma_{k-1}\kappa_0 z e^{-\lambda(t_k-t_0)} + \sigma_0\cdots\sigma_{k-1}\widehat{N}\right] \\ &\leq \gamma_0\cdots\gamma_{k-1}\gamma_k\kappa_0 z e^{-\lambda(t_k-t_0)} + \sigma_0\cdots\sigma_{k-1}\sigma_k\widehat{N}.\end{aligned} \tag{3.28}$$

This, together with (3.27), leads to

$$[x(t)]^+ \leq \gamma_0\cdots\gamma_{k-1}\gamma_k\kappa_0 z e^{-\lambda(t-t_0)} + \sigma_0\cdots\sigma_{k-1}\sigma_k\widehat{N}, \quad t\in(-\infty, t_k]. \tag{3.29}$$

By the property of ϱ-cone again, the vector

$$\gamma_0\cdots\gamma_{k-1}\gamma_k\kappa_0 z \in \Omega_\varrho\left(\widehat{P} + \widehat{W} + \widehat{Q}\right). \tag{3.30}$$

On the other hand,

$$[x(t)]^+ \leq \widehat{P}[x(t-\tau_1)]^+ + \widehat{W}[(x(t-\tau_2))]^+ + \int_0^\infty \overline{Q}(t)[(x(t-s))]^+ ds + \sigma_0,\ldots,\sigma_k[I]^+, \quad t\neq t_k. \tag{3.31}$$

It follows from (3.29)–(3.31) and part (a) of Theorem 3.1 that

$$[x(t)]^+ \leq \gamma_0\cdots\gamma_{k-1}\gamma_k\kappa_0 z e^{-\lambda(t-t_0)} + \sigma_0\cdots\sigma_{k-1}\sigma_k\widehat{N}, \quad t\in[t_k, t_{k+1}). \tag{3.32}$$

By the mathematical induction, we can conclude that

$$[x(t)]^+ \leq \gamma_0\cdots\gamma_{k-1}\kappa_0 z e^{-\lambda(t-t_0)} + \sigma_0\cdots\sigma_{k-1}\widehat{N}, \quad t\in[t_{k-1}, t_k),\ k=1,2,\ldots. \tag{3.33}$$

From (3.18) and (3.21),

$$\gamma_k \leq e^{\gamma(t_k-t_{k-1})}, \quad \sigma_0\cdots\sigma_{k-1}\leq e^{\sigma}, \tag{3.34}$$

we can use (3.33) to conclude that

$$\begin{aligned}[x(t)]^+ &\leq e^{\gamma(t_1-t_0)}\cdots e^{\gamma(t_{k-1}-t_{k-2})}\kappa_0 z e^{-\lambda(t-t_0)} + \sigma_0\cdots\sigma_{k-1}\widehat{N} \\ &\leq \kappa_0 z e^{\gamma(t-t_0)} e^{-\lambda(t-t_0)} + e^{\sigma}\widehat{N} \\ &= \kappa_0 z e^{-(\lambda-\gamma)(t-t_0)} + e^{\sigma}\widehat{N}, \quad t\in[t_{k-1}, t_k),\ k=1,2,\ldots.\end{aligned} \tag{3.35}$$

This implies that the conclusion of the theorem holds and the proof is complete. □

Theorem 3.4. *If* (A_1)*–*(A_3) *with* $R_k \le E$ *hold, then* $S = \{\phi \in PC \mid [\phi]_\infty^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+\}$ *is a positive invariant set and also a global attracting set of* (2.4)*.*

Proof. For the initial conditions: $x(t_0 + s) = \phi(s)$, $s \in (-\infty, 0]$, where $\phi \in S$, we have

$$[x(t)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+, \quad t \in (-\infty, t_0]. \tag{3.36}$$

By (3.36) and the part (b) of Theorem 3.1 with $d = 1$, we have

$$[x(t)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+, \quad t \in [t_0, t_1). \tag{3.37}$$

Suppose for all $\iota = 1, \ldots, k$, the inequalities

$$[x(t)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+, \quad t \in [t_{\iota-1}, t_\iota), \tag{3.38}$$

hold. Then, from (A_2) and $R_k \le E$, the impulsive part of (2.4) satisfies that

$$[x(t_k)]^+ \le \left[J_k\left(x(t_k^-)\right)\right]^+ \le R_k[x(t_k^-)]^+ \le E[x(t_k^-)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+. \tag{3.39}$$

This, together with (3.36) and (3.38), leads to

$$[x(t)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+, \quad t \in (-\infty, t_k]. \tag{3.40}$$

It follows from (3.40) and the part (b) of Theorem 3.1 that

$$[x(t)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+, \quad t \in [t_k, t_{k+1}). \tag{3.41}$$

By the mathematical induction, we can conclude that

$$[x(t)]^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots. \tag{3.42}$$

Therefore, $S = \{\phi \in PC \mid [\phi]_\infty^+ \le (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^+\}$ is a positive invariant set. Since $R_k \le E$, a direct calculation shows that $\gamma_k = \sigma_k = 1$ and $\sigma = 0$ in Theorem 3.3. It follows from Theorem 3.3 that the set S is also a global attracting set of (2.4). The proof is complete. □

For the case $I = 0$, we easily observe that $x(t) \equiv 0$ is a solution of (2.4) from (A_1) and (A_2). In the following, we give the attractivity of the zero solution and the proof is similar to that of Theorem 3.3.

Corollary 3.5. *If* $(A_1) - (A_4)$ *hold with* $I = 0$*, then the zero solution of* (2.4) *is globally exponentially stable.*

Remark 3.6. If $J_k(x) = x$, that is, they have no impulses in (2.4), then by Theorem 3.4, we can obtain the following result.

Corollary 3.7. *If* (A_1) *and* (A_3) *hold, then* $S = \{\phi \in PC \mid [\phi]_{\infty}^{+} \leq (E - \widehat{P} - \widehat{W} - \widehat{Q})^{-1}[I]^{+}\}$ *is a positive invariant set and also a global attracting set of* (2.4).

4. Illustrative Example

The following illustrative example will demonstrate the effectiveness of our results.

Example 4.1. Consider the following impulsive infinite delay difference equations:

$$\left.\begin{aligned} x_1(t) &= \frac{1}{4}x_1(t-1) + \frac{1}{12}\sin(x_1(t-1)) + \frac{1}{15}x_2(t-1) \\ &\quad + \frac{4}{15}|x_2(t-2)| - \int_{-\infty}^{t} e^{-6(t-s)}|x_1(s)|ds + 2 \\ x_2(t) &= -\frac{1}{4}x_2(t-1) + \frac{1}{5}\sin(x_1(t-1)) + \frac{1}{6}x_2(t-1) \\ &\quad + \frac{2}{15}|x_1(t-2)| + \int_{-\infty}^{t} e^{-12(t-s)}|x_2(s)|ds + 3 \end{aligned}\right\}, m \neq m_k, \tag{4.1}$$

with

$$\begin{aligned} x_1(t_k) &= \alpha_{1k}x_1(t_k^-) - \beta_{1k}x_2(t_k^-) \\ x_2(t_k) &= \beta_{2k}x_1(t_k^-) + \alpha_{2k}x_2(t_k^-), \end{aligned} \tag{4.2}$$

where α_{ik} and β_{ik} are nonnegative constants, and the impulsive sequence t_k $(k = 1, 2, \ldots)$ satisfies: $t_1 < t_2 < \cdots, \lim_{k\to\infty} t_k = \infty$. For System (4.1), we have $p_{11}(s) = -e^{-6s}$, $p_{22}(s) = e^{-12s}, p_{12}(s) = p_{21}(s) = 0$. So, it is easy to check that $p_{ij}(s) \in L^e$, $i, j = 1, 2$, provided that $0 < \lambda_0 < 1$. In this example, we may let $\lambda_0 = 0.1$.

The parameters of (A_1)–(A_3) are as follows:

$$\begin{gathered} A_0 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{12} & \frac{1}{15} \\ \frac{1}{5} & \frac{1}{6} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{4}{15} \\ \frac{2}{15} & 0 \end{pmatrix}, \\ \overline{F} = \overline{G} = \overline{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{15} \\ \frac{1}{5} & \frac{5}{12} \end{pmatrix}, \quad \widehat{W} = \begin{pmatrix} 0 & \frac{4}{15} \\ \frac{2}{15} & 0 \end{pmatrix}, \\ \widehat{Q} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}, \quad R_k = \begin{pmatrix} \alpha_{1k} & \beta_{1k} \\ \beta_{2k} & \alpha_{2k} \end{pmatrix}, \quad \widehat{P} + \widehat{W} + \widehat{Q} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \end{gathered} \tag{4.3}$$

It is easy to prove that $\varrho(\widehat{P}+\widehat{W}+\widehat{Q}) = 5/6 < 1$ and

$$\Omega_\rho\left(\widehat{P}+\widehat{W}+\widehat{Q}\right) = \left\{ (z_1, z_2)^T > 0 \,\middle|\, \frac{2}{3}z_1 < z_2 < \frac{3}{2}z_1 \right\}. \tag{4.4}$$

Let $z = (1,1)^T \in \Omega_\varrho(\widehat{P}+\widehat{W}+\widehat{Q})$ and $\lambda = 0.01 < \lambda_0$ which satisfies the inequality

$$\left(e^{\lambda}\left(\widehat{P}e^{\lambda} + \widehat{W}e^{2\lambda} + \int_0^{\infty}\overline{Q}(s)e^{\lambda s}ds\right) - E\right)z < 0. \tag{4.5}$$

Let $\gamma_k = \max\{\alpha_{1k}+\beta_{1k}, \alpha_{2k}+\beta_{2k}\}$, then γ_k satisfy $\gamma_k z \geq R_k z$, $k = 1,2,\ldots$.

Case 1. Let $\alpha_{1k} = \alpha_{2k} = (1/3)e^{1/25^k}$, $\beta_{1k} = \beta_{2k} = (2/3)e^{1/25^k}$, and $t_k - t_{k-1} = 5k$, then

$$\gamma_k = e^{1/25^k} \geq 1, \quad \frac{\ln \gamma_k}{t_k - t_{k-1}} = \frac{\ln e^{1/25^k}}{5k} = \frac{1}{25^k \times 5k} \leq 0.008 = \gamma < \lambda. \tag{4.6}$$

Moreover, $\sigma_k = e^{1/25^k} \geq 1$, $\sigma = \sum_{k=1}^{\infty} \ln \sigma_k = \sum_{k=1}^{\infty} \ln e^{1/25^k} = 1/24$. Clearly, all conditions of Theorem 3.3 are satisfied. So $S = \{\phi \in PC \mid [\phi]_\infty^+ \leq e^{1/24}(E-\widehat{P}-\widehat{W}-\widehat{Q})^{-1}I\} = (6e^{1/24}, 6e^{1/24})^T$ is a global attracting set of (4.1).

Case 2. Let $\alpha_{1k} = \alpha_{2k} = (1/3)e^{1/2^k}$ and $\beta_{1k} = \beta_{2k} = 0$, then $R_k = (1/3)e^{1/2^k}E \leq E$. Therefore, by Theorem 3.4, $S = \{\phi \in PC \mid [\phi]_\infty^+ \leq \widehat{N} = (E-\widehat{P}-\widehat{W}-\widehat{Q})^{-1}I\} = (6,6)^T$ is a positive invariant set and also a global attracting set of (4.1).

Case 3. If $I = 0$ and let $\alpha_{1k} = \alpha_{2k} = (1/3)e^{0.04k}$ and $\beta_{1k} = \beta_{2k} = (2/3)e^{0.04k}$, then

$$\gamma_k = e^{0.04k} \geq 1, \quad \frac{\ln \gamma_k}{t_k - t_{k-1}} = \frac{\ln e^{0.04k}}{5k} = 0.008 = \gamma < \lambda. \tag{4.7}$$

Clearly, all conditions of Corollary 3.5 are satisfied. Therefore, by Corollary 3.5, the zero solution of (4.1) is globally exponentially stable.

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