Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 389624, 18 pages
doi:10.1155/2009/389624

Research Article

On the Nonexistence and Existence of Solutions for a Fourth-Order Discrete Boundary Value Problem

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Received 16 July 2009; Accepted 16 October 2009

Recommended by Patricia J. Y. Wong

By using the critical point theory, we establish various sets of sufficient conditions on the nonexistence and existence of solutions for the boundary value problems of a class of fourth-order difference equations.

1. Introduction

In this paper, we denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ the set of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \ldots\}$, $\mathbb{Z}(a,b) = \{a, a+1, \ldots, b\}$ when $a \le b$.

Consider the following boundary value problem (BVP):

$$\begin{gathered}\Delta^2\left[p(n-1)\Delta^2 u(n-2)\right] + \Delta\left[q(n)\Delta u(n-1)\right] = f(n, u(n)), \quad n \in \mathbb{Z}(1,k), \\ u(-1) = u(0) = 0 = u(k+1) = u(k+2).\end{gathered} \tag{1.1}$$

Here, $k \in \mathbb{N}$, $p(n)$ is nonzero and real valued for each $n \in \mathbb{Z}(0, k+1)$, $q(n)$ is real valued for each $n \in \mathbb{Z}(1, k+1)$. $f(n,u)$ is real-valued for each $(n,u) \in \mathbb{Z}(1,k) \times \mathbb{R}$ and continuous in the second variable u. Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, and $\Delta^2 u(n) = \Delta(\Delta u(n))$.

We may think of (1.1) as being a discrete analogue of the following boundary value problem:

$$\begin{gathered}\left[p(t)x''(t)\right]'' + \left[q(t)x'(t)\right]' = f(t, x(t)), \quad t \in [a,b], \\ x(a) = x'(a) = 0, \qquad x(b) = x'(b) = 0,\end{gathered} \tag{1.2}$$

which are used to describe the bending of an elastic beam; see, for example, [1–10] and references therein. Owing to its importance in physics, many methods are applied to study fourth-order boundary value problems by many authors. For example, fixed point theory [1, 3, 5–7], the method of upper and lower solutions [8], and critical point theory [9, 10] are widely used to deal with the existence of solutions for the boundary value problems of fourth-order differential equations.

Because of applications in many areas for difference equations, in recent years, there has been an increased interest in studying of fourth-order difference equation, which include results on periodic solutions [11], results on oscillation [12–14], and results on boundary value problems and other topics [15, 16]. Recently, a few authors have gradually paid attention to applying critical point theory to deal with problems on discrete systems; for example, Yu and Guo in [17] considered the existence of solutions for the following BVP:

$$\begin{gathered}\Delta\left[p(n)\Delta u(n-1)\right]+q(n)u(n)=f(n,u(n)),\\ u(a)+\alpha u(a+1)=A,\qquad u(b+2)+\beta u(b+1)=B.\end{gathered}\tag{1.3}$$

The papers [17–20] show that the critical point theory is an effective approach to the study of the boundary value problems of difference equations. In this paper, we will use critical point theory to establish some sufficient conditions on the nonexistence and existence of solutions for the BVP (1.1).

Let

$$\begin{aligned}a(n)&=q(n+1)-2\left[p(n)+p(n+1)\right],\\ b(n)&=p(n-1)+4p(n)+p(n+1)-q(n)-q(n+1).\end{aligned}\tag{1.4}$$

Then the BVP (1.1) becomes

$$\begin{gathered}Lu(n)=f(n,u(n)),\quad n\in\mathbb{Z}(1,k),\\ u(-1)=u(0)=0=u(k+1)=u(k+2),\end{gathered}\tag{1.5}$$

where

$$\begin{aligned}Lu(n)={}&p(n+1)u(n+2)+a(n)u(n+1)+b(n)u(n)+a(n-1)u(n-1)\\ &+p(n-1)u(n-2).\end{aligned}\tag{1.6}$$

The remaining of this paper is organized as follows. First, in Section 2, we give some preliminaries and establish the variational framework for BVP (1.5). Then, in Section 3, we present a sufficient condition on the nonexistence of nontrivial solutions of BVP (1.5). Finally, in Section 4, we provide various sets of sufficient conditions on the existence of solutions of BVP (1.5) when f is superlinear, sublinear, and Lipschitz. Moreover, in a special case of f we obtain a necessary and sufficient condition for the existence of unique solutions of BVP (1.5).

To conclude the introduction, we refer to [21, 22] for the general background on difference equations.

2. Preliminaries

In order to apply the critical point theory, we are going to establish the corresponding variational framework of BVP (1.5). First we give some notations.

Let $\mathbb{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbb{R}^k$ as follows:

$$(u,v) = \sum_{j=1}^{k} u(j)v(j), \quad \forall u,v \in \mathbb{R}^k, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u^2(j)\right)^{1/2}, \quad \forall u \in \mathbb{R}^k, \tag{2.2}$$

For BVP (1.5), consider the functional J defined on $\mathbb{R}^k$ as follows:

$$J(u) = \frac{1}{2}(Mu,u) - F(u), \qquad \forall u = (u(1),u(2),\ldots,u(k))^T \in \mathbb{R}^k, \tag{2.3}$$

where ^{T} is the transpose of a vector in $\mathbb{R}^k$:

$$M = \begin{pmatrix} b(1) & a(1) & p(2) & 0 & 0 & \cdots & 0 & 0 & 0 \\ a(1) & b(2) & a(2) & p(3) & 0 & \cdots & 0 & 0 & 0 \\ p(2) & a(2) & b(3) & a(3) & p(4) & \cdots & 0 & 0 & 0 \\ 0 & p(3) & a(3) & b(4) & a(4) & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & b(k-2) & a(k-2) & p(k-1) \\ 0 & 0 & 0 & 0 & 0 & \cdots & a(k-2) & b(k-1) & a(k-1) \\ 0 & 0 & 0 & 0 & 0 & \cdots & p(k-1) & a(k-1) & b(k) \end{pmatrix}_{k\times k}, \tag{2.4}$$

$$F(u) = \sum_{j=1}^{k} \int_0^{u(j)} f(j,s)\,\mathrm{d}s. \tag{2.5}$$

After a careful computation, we find that the Fréchet derivative of J is

$$J'(u) = Mu - f(u), \tag{2.6}$$

where $f(u)$ is defined as $f(u) = (f(1,u(1)), f(2,u(2)),\ldots, f(k,u(k)))^T$.

Expanding out $J'(u)$, one can easily see that there is an one-to-one correspondence between the critical point of functional J and the solution of BVP (1.5). Furthermore, $u = (u(1), u(2), \ldots, u(k))^T$ is a critical point of J if and only if $\{u(t)\}_{t=-1}^{k+2} = (u(-1), u(0), u(1), \ldots, u(k), u(k+1), u(k+2))^T$ is a solution of BVP (1.5), where $u(-1) = u(0) = 0 = u(k+1) = u(k+2)$.

Therefore, we have reduced the problem of finding a solution of (1.5) to that of seeking a critical point of the functional J defined on $\mathbb{R}^k$.

In order to obtain the existence of critical points of J on $\mathbb{R}^k$, for the convenience of readers, we cite some basic notations and some known results from critical point theory.

Let H be a real Banach space, $J \in C^1(H, \mathbb{R})$, that is, J is a continuously Fréchet differentiable functional defined on H, and J is said to satisfy the Palais-Smale condition (P-S condition), if any sequence $\{x_n\} \subset H$ for which $J(x_n)$ is bounded and $J'(x_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in H.

Let B_r denote the open ball in H about 0 of radius r and let ∂B_r denote its boundary. The following lemmas are taken from [23, 24] and will play an important role in the proofs of our main results.

Lemma 2.1 (Linking theorem). *Let H be a real Banach space, $H = H_1 \bigoplus H_2$, where H_1 is a finite-dimensional subspace of H. Assume that $J \in C^1(H, \mathbb{R})$ satisfies the P-S condition and the following.*

(F_1) *There exist constants $\sigma, \rho > 0$ such that $J|_{\partial B_\rho \cap H_2} \geq \sigma$.*

(F_2) *There is an $e \in \partial B_1 \cap H_2$ and a constant $R_0 > \rho$ such that $J|_{\partial Q} \leq 0$ and $Q = (\overline{B}_{R_0} \cap H_1) \bigoplus \{re \mid 0 < r < R_0\}$.*

Then J possesses a critical value $c \geq \sigma$, where

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{Q}} J(h(u)), \tag{2.7}$$

and $\Gamma = \{h \in C(\overline{Q}, H) | h|_{\partial Q} = \mathrm{id}\}$, *where* id *denotes the identity operator.*

Lemma 2.2 (Saddle point theorem). *Let H be a real Banach space, $H = H_1 \bigoplus H_2$, where $H_1 \neq \{0\}$ and is finite-dimensional. Suppose that $J \in C^1(H, \mathbb{R})$ satisfies the P-S condition and the following.*

(F_3) *There exist constants $\sigma, \rho > 0$ such that $J|_{\partial B_\rho \cap H_1} \leq \sigma$.*

(F_4) *There is $e \in B_\rho \cap H_1$ and a constant $\omega > \sigma$ such that $J|_{e + H_2} \geq \omega$.*

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{B}_\rho \cap H_1} J(h(u)), \tag{2.8}$$

and $\Gamma = \{h \in C(\overline{B}_\rho \cap H_1, H) | h|_{\partial B_\rho \cap H_1} = \mathrm{id}\}$, *where* id *denotes the identity operator.*

Lemma 2.3 (Clark theorem). *Let H be a real Banach space, $J \in C^1(H, \mathbb{R})$, with J being even, bounded from below, and satisfying P-S condition. Suppose $J(\theta) = 0$, there is a set $K \subset H$ such that K is homeomorphic to S^{j-1} ($j-1$ dimension unit sphere) by an odd map, and $\sup_K J < 0$. Then J has at least j distinct pairs of nonzero critical points.*

3. Nonexistence of Nontrivial Solutions

In this section, we give a result of nonexistence of nontrivial solutions to BVP (1.5).

Theorem 3.1. *Suppose that matrix M is negative semidefinite and for $n = 1, 2, \ldots, k$,*

$$zf(n, z) > 0, \quad \text{for } z \neq 0. \tag{3.1}$$

Then BVP (1.5) *has no nontrivial solutions.*

Proof. Assume, for the sake of contradiction, that BVP (1.5) has a nontrivial solution. Then J has a nonzero critical point u^*. Since

$$J'(u) = Mu - f(u), \tag{3.2}$$

we get

$$\left(f(u^*), u^*\right) = (Mu^*, u^*) \leq 0. \tag{3.3}$$

On the other hand, it follows from (3.1) that

$$\left(f(u^*), u^*\right) = \sum_{n=1}^{k} u^*(n) f(n, u^*(n)) > 0. \tag{3.4}$$

This contradicts with (3.3) and hence the proof is complete. □

In the existing literature, results on the nonexistence of solutions of discrete boundary value problems are scarce. Hence Theorem 3.1 complements existing ones.

4. Existence of Solutions

Theorem 3.1 gives a set of sufficient conditions on the nonexistence of solutions of BVP (1.5). In this section, with part of the conditions being violated, we establish the existence of solutions of BVP (1.5) by distinguishing three cases: f is superlinear, f is sublinear, and f is Lipschitzian.

4.1. The Superlinear Case

In this subsection, we need the following conditions.

(P_1) For any $(n, z) \in \mathbb{Z}(1, k) \times \mathbb{R}$, $\int_0^z f(n, s)ds \geq 0$, and $\int_0^z f(n, s)ds = o(|z|^2)$, as $z \to 0$.

(P_2) There exist constants $a_1 > 0$, $a_2 > 0$ and $\beta > 2$ such that

$$\int_0^z f(n, s)ds \geq a_1 |z|^{\beta} - a_2, \quad \forall (n, z) \in \mathbb{Z}(1, k) \times \mathbb{R}. \tag{4.1}$$

(P_3) Matrix M exists at least one positive eigenvalue.

(P_4) $f(n,z)$ is odd for the second variable z, namely,

$$f(n,-z)=-f(n,z),\quad \forall (n,z)\in\mathbb{Z}(1,k)\times\mathbb{R}. \tag{4.2}$$

Theorem 4.1. *Suppose that $f(n,z)$ satisfies (P_2). Then BVP (1.5) possesses at least one solution.*

Proof. For any $u=(u(1),u(2),\dots,u(k))^T\in\mathbb{R}^k$, we have

$$\begin{aligned}F(u)&=\sum_{j=1}^{k}\int_0^{u(j)}f(j,s)\mathrm{d}s\ge a_1\left(\sum_{j=1}^{k}|u(j)|^{\beta}\right)-a_2k\\&\ge a_1\left(k^{(2-\beta)/\beta}\sum_{j=1}^{k}|u(j)|^2\right)^{\beta/2}-a_2k=a_1k^{(2-\beta)/2}\|u\|^{\beta}-a_2k.\end{aligned} \tag{4.3}$$

Let $A_1=a_1k^{(2-\beta)/2}$, $A_2=a_2k$. We have, for any $u=(u(1),u(2),\dots,u(k))^T\in\mathbb{R}^k$,

$$F(u)\ge A_1\|u\|^{\beta}-A_2. \tag{4.4}$$

Since matrix M is symmetric, its all eigenvalues are real. We denote by $\lambda_1,\lambda_2,\dots,\lambda_k$ its eigenvalues. Set $\lambda_{\max}=\max\{|\lambda_1|,|\lambda_2|,\dots,|\lambda_k|\}$. Thus for any $u=(u(1),u(2),\dots,u(k))^T\in\mathbb{R}^k$,

$$\begin{aligned}J(u)&=\frac{1}{2}(Mu,u)-F(u)\\&\le\frac{1}{2}\lambda_{\max}\|u\|^2-A_1\|u\|^{\beta}+A_2\\&\longrightarrow-\infty\quad(\text{as }\|u\|\longrightarrow\infty).\end{aligned} \tag{4.5}$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, J attains its maximum at some point, and we denote it by $\tilde{u}$, that is, $J(\tilde{u})=c_{\max}$, where $c_{\max}=\sup_{u\in\mathbb{R}^k}(J(u))$. Clearly, $\tilde{u}$ is a critical point of J. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Suppose that $f(n,z)$ satisfies the assumptions (P_1), (P_2), and (P_3). Then BVP (1.5) possesses at least two nontrivial solutions.*

To prove Theorem 4.2, we need the following lemma.

Lemma 4.3. *Assume that (P_2) holds, then the functional J satisfies the P-S condition.*

Proof. Assume that $\{u^{(n)}\} \subset \mathbb{R}^k$ is a P-S sequence. Then there exists a constant c_1 such that for any $n \in \mathbb{Z}(1)$, $|J(u^{(n)})| \le c_1$ and $J'(u^{(n)}) \to 0$ as $n \to \infty$. By (4.5) we have

$$\begin{aligned} -c_1 \le J\left(u^{(n)}\right) &= \frac{1}{2}\left(Mu^{(n)}, u^{(n)}\right) - F\left(u^{(n)}\right) \\ &\le \frac{1}{2}\lambda_{\max}\left\|u^{(n)}\right\|^2 - A_1\left\|u^{(n)}\right\|^\beta + A_2, \end{aligned} \tag{4.6}$$

and so

$$A_1\left\|u^{(n)}\right\|^\beta - \frac{1}{2}\lambda_{\max}\left\|u^{(n)}\right\|^2 \le c_1 + A_2. \tag{4.7}$$

Due to $\beta > 2$, the above inequality means $\{u^{(n)}\}$ is bounded. Since $\mathbb{R}^k$ is a finite-dimensional Hilbert space, there must exist a subsequence of $\{u^{(n)}\}$ which is convergent in $\mathbb{R}^k$. Therefore, P-S condition is satisfied. □

Proof of Theorem 4.2. Let λ_i, $1 \le i \le l$, $-\mu_j$, $1 \le j \le m$ be the positive eigenvalues and the negative eigenvalues, where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_l$, $0 > -\mu_1 \ge -\mu_2 \ge \cdots \ge -\mu_m$. Let ξ_i be an eigenvector of M corresponding to the eigenvalue λ_i, $1 \le i \le l$, and let η_j be an eigenvector of M corresponding to the eigenvalue $-\mu_j$, $1 \le j \le m$, such that

$$\begin{gathered} \left(\xi_i, \xi_j\right) = \begin{cases} 0, & \text{as } i \ne j, \\ 1, & \text{as } i = j, \end{cases} \qquad \left(\eta_i, \eta_j\right) = \begin{cases} 0, & \text{as } i \ne j, \\ 1, & \text{as } i = j, \end{cases} \\ \left(\xi_i, \eta_j\right) = 0, \quad \text{for any } 1 \le i \le l,\ 1 \le j \le m. \end{gathered} \tag{4.8}$$

Let E^+, E^0, and E^- be subspaces of $\mathbb{R}^k$ defined as follows:

$$\begin{gathered} E^+ = \operatorname{span}\{\xi_i, 1 \le i \le l\}, \qquad E^- = \operatorname{span}\{\eta_j, 1 \le i \le m\}, \\ E^0 = \left(E^+ \bigoplus E^-\right)^\perp. \end{gathered} \tag{4.9}$$

For any $u \in \mathbb{R}^k, u = u^+ + u^0 + u^-$, where $u^+ \in E^+, u^0 \in E^0, u^- \in E^-$. Then

$$\lambda_1\|u^+\|^2 \le \left(Mu^+, u^+\right) \le \lambda_l\|u^+\|^2, \qquad -\mu_m\left\|u^-\right\|^2 \le \left(Mu^-, u^-\right) \le -\mu_1\left\|u^-\right\|^2. \tag{4.10}$$

Let $X_1 = E^- \bigoplus E^0$, $X_2 = E^+$, then $\mathbb{R}^k$ has the following decomposition of direct sum:

$$\mathbb{R}^k = X_1 \bigoplus X_2. \tag{4.11}$$

By assumption (P_1), there exists a constant $\rho > 0$, such that for any $n \in \mathbb{Z}(1,k)$, $z \in B_\rho$, $\int_0^z f(n,s)\mathrm{d}s \le (1/4)\lambda_1 z^2$. So for any $u \in \partial B_\rho \cap X_2$, $n \in \mathbb{Z}(1,k)$,

$$\begin{aligned} J(u) &= \frac{1}{2}(Mu,u) - F(u) \\ &\ge \frac{1}{2}\lambda_1\|u\|^2 - \frac{1}{4}\lambda_1\|u\|^2 = \frac{1}{4}\lambda_1\rho^2. \end{aligned} \tag{4.12}$$

Denote $\sigma = (1/4)\lambda_1\rho^2$. Then

$$J(u) \ge \sigma, \quad \forall u \in \partial B_\rho \cap X_2. \tag{4.13}$$

That is to say, J satisfies assumption (F_1) of Linking theorem.

Take $e \in \partial B_1 \cap X_2$. For any $\omega \in X_1, r \in R$, let $u = re + \omega$, because $\omega = \omega^0 + \omega^-$, where $\omega^0 \in E^0, \omega^- \in E^-$. Then

$$\begin{aligned} J(u) &= \frac{1}{2}(M(re+\omega), re+\omega) - F(re+\omega) \\ &= \frac{1}{2}(Mre, re) + \frac{1}{2}(M\omega^-, \omega^-) - \sum_{j=1}^{k}\int_0^{re(j)+\omega(j)} f(j,s)\mathrm{d}s \\ &\le \frac{1}{2}\lambda_l r^2 - \frac{1}{2}\mu_1\|\omega^-\|^2 - a_1\left(\sum_{j=1}^{k}|re(j)+\omega(j)|^\beta\right) + a_2 k \\ &\le \frac{1}{2}\lambda_l r^2 - \frac{1}{2}\mu_1\|\omega^-\|^2 - a_1\left(k^{(2-\beta)/\beta}\sum_{j=1}^{k}|re(j)+\omega(j)|^2\right)^{\beta/2} + a_2 k \\ &= \frac{1}{2}\lambda_l r^2 - \frac{1}{2}\mu_1\|\omega^-\|^2 - a_1 k^{(2-\beta)/2}\left(\sum_{j=1}^{k}\left(r^2e^2(j)+\omega^2(j)\right)\right)^{\beta/2} + a_2 k \\ &\le \frac{1}{2}\lambda_l r^2 - a_1 k^{(2-\beta)/2} r^\beta - a_1 k^{(2-\beta)/2}\|\omega\|^\beta + a_2 k. \end{aligned} \tag{4.14}$$

Set $g_1(r) = (1/2)\lambda_l r^2 - a_1 k^{(2-\beta)/2} r^\beta$ and $g_2(\tau) = -a_1 k^{(2-\beta)/2}\tau^\beta + a_2 k$. Then $\lim_{r\to+\infty} g_1(r) = -\infty, \lim_{\tau\to+\infty} g_2(\tau) = -\infty$. Furthermore, $g_1(r)$ and $g_2(\tau)$ are bounded from above. Accordingly, there is some $R_0 > \rho$, such that for any $u \in \partial Q, J(u) \le 0$, where $Q = (\overline{B}_{R_0} \cap X_1) \bigoplus \{re \mid 0 < r < R_0\}$. By Linking theorem, J possesses a critical value $c \ge \sigma > 0$, where

$$c = \inf_{h\in\Gamma}\max_{u\in\overline{Q}} J(h(u)), \qquad \Gamma = \left\{h \in C\left(\overline{Q}, \mathbb{R}^k\right) |h|_{\partial\overline{Q}} = \mathrm{id}\right\}. \tag{4.15}$$

Let $\overline{u} \in \mathbb{R}^k$ be a critical point corresponding to the critical value c of J, that is, $J(\overline{u}) = c$. Clearly, $\overline{u} \neq 0$ since $c > 0$. On the other hand, by Theorem 4.1, J has a critical point $\tilde{u}$ satisfying $J(\tilde{u}) = \sup_{u\in\mathbb{R}^k}(J(u)) \geq c$. If $\overline{u} \neq \tilde{u}$, then Theorem 4.2 holds. Otherwise, $\overline{u} = \tilde{u}$. Then $c_{\max} = J(\tilde{u}) = J(\overline{u}) = c$, which is the same as $\sup_{u\in\mathbb{R}^k} J(u) = \inf_{h\in\Gamma}\sup_{u\in\overline{Q}} J(h(u))$.

Choosing $h = \mathrm{id}$, we have $\sup_{u\in\overline{Q}} J(u) = c_{\max}$. Because the choice of $e \in \partial B_1 \cap X_2 \in Q = (\overline{B}_{R_0} \cap X_1) \bigoplus \{re \mid 0 < r < R_0\}$ is arbitrary, we can take $-e \in \partial B_1 \cap X_2$. Similarly, there exists a positive number $R_1 > \rho$, for any $u \in \partial Q_1$, $J(u) \leq 0$, where $Q_1 = (\overline{B}_{R_1} \cap X_1) \bigoplus \{-re \mid 0 < r < R_1\}$. Again, by the Linking theorem, J possesses a critical value $c_0 \geq \sigma > 0$, where

$$c_0 = \inf_{h\in\Gamma_1} \max_{u\in\overline{Q}_1} J(h(u)), \qquad \Gamma_1 = \left\{ h \in C\left(\overline{Q}_1, \mathbb{R}^k\right) | h|_{\partial\overline{Q}_1} = \mathrm{id} \right\}. \tag{4.16}$$

If $c_0 \neq c_{\max}$, then the proof is complete. Otherwise $c_0 = c_{\max}, \sup_{u\in\overline{Q}_1} J(u) = c_{\max}$. Because $J|_{\partial Q} \leq 0$ and $J|_{\partial Q_1} \leq 0$, then J attains its maximum at some point in the interior of sets Q and Q_1. But $Q \cap Q_1 \subset X_1$, and $J(u) \leq 0$ for $u \in X_1$. Thus there is a critical point $\hat{u} \in \mathbb{R}^k$ satisfying $\hat{u} \neq \tilde{u}$, $J(\hat{u}) = c_0 = c_{\max}$.

The proof of Theorem 4.2 is now complete. □

Theorem 4.4. *Suppose that $f(n, z)$ satisfies the assumptions (P_1), (P_2), (P_3), and (P_4). Then BVP* (1.5) *possesses at least l distinct pairs of nontrivial solutions, where l is the dimension of the space spanned by the eigenvectors corresponding to the positive eigenvalues of M.*

Proof. From the proof of Theorem 4.2, it is easy to know that J is bounded from above and satisfies the P-S condition. It is clear that J is even and $J(0) = 0$, and we should find a set K and an odd map such that K is homeomorphic to S^{l-1} by an odd map.

We take $K = \partial B_\rho \cap X_2$, where ρ and X_2 are defined as in the proof of Theorem 4.2. It is clear that K is homeomorphic to S^{l-1}($l-1$ dimension unit sphere) by an odd map. With (4.13), we get $\sup_K(-J) < 0$. Thus all the conditions of Lemma 2.3 are satisfied, and J has at least l distinct pairs of nonzero critical points. Consequently, BVP (1.5) possesses at least l distinct pairs nontrivial solutions. The proof of Theorem 4.4 is complete. □

4.2. The Sublinear Case

In this subsection, we will consider the case where f is sublinear. First, we assume the following.

(P_5) There exist constants $a_1 > 0, a_2 > 0, R > 0$ and $1 < \alpha < 2$ such that

$$F(u) \leq a_1 \|u\|^\alpha + a_2, \qquad \forall u = (u(1), u(2), \ldots, u(k))^T \in \mathbb{R}^k, \ \|u\| \geq R. \tag{4.17}$$

The first result is as follows.

Theorem 4.5. *Suppose that (P_5) is satisfied and that matrix M is positive definite. Then BVP* (1.5) *possesses at least one solution.*

Proof. The proof will be finished when the existence of one critical point of functional J defined as in (2.3) is proved.

Assume that matrix M is positive definite. We denote by $\lambda_1, \lambda_2, \ldots, \lambda_k$ its eigenvalues, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then for any $u = (u(1), u(2), \ldots, u(k))^T \in \mathbb{R}^k$, $\|u\| \geq R$, followed by (P_5) we have

$$\begin{aligned} J(u) &= \frac{1}{2}(Mu, u) - F(u) \\ &\leq \lambda_1 \|u\|^2 - a_1 \|u\|^\alpha - a_2 \\ &\longrightarrow +\infty \quad (\text{as } \|u\| \longrightarrow \infty). \end{aligned} \tag{4.18}$$

By the continuity of J on $\mathbb{R}^k$, the above inequality means that there exists a lower bound of values of functional J. Classical calculus shows that J attains its minimal value at some point, and then there exist u' such that $J(u') = \min\{J(u) \mid u \in \mathbb{R}^k\}$. Clearly, u' is a critical point of the functional J. □

Corollary 4.6. *Suppose that matrix M is positive definite, and $f(n, z)$ satisfies that there exist constants $a_1 > 0$, $a_2 > 0$ and $1 < \alpha < 2$ such that*

$$\int_0^z f(n, s) ds \leq a_1 |z|^\alpha + a_2, \quad \forall (n, z) \in \mathbb{Z}(1, k) \times \mathbb{R}. \tag{4.19}$$

Then BVP (1.5) *possesses at least one solution.*

Corollary 4.7. *Suppose that matrix M is positive definite, and $f(n, z)$ satisfies the following.*

(P_6) *There exists a constant $t_0 > 0$ such that for any $(n, z) \in \mathbb{Z}(1, k) \times \mathbb{R}$, $|f(n, z)| \leq t_0$.*

Then BVP (1.5) *possesses at least one solution.*

Proof. Assume that matrix M is positive definite. In this case, for any $u = (u(1), u(2), \ldots, u(k))^T \in \mathbb{R}^k$,

$$|F(u)| \leq \sum_{j=1}^k \left| \int_0^{u(j)} f(j, s) \mathrm{d}s \right| \leq \sum_{j=1}^k t_0 |u(j)| \leq t_0 \sqrt{k} \|u\|. \tag{4.20}$$

Since the rest of the proof is similar to Theorem 4.5, we do not repeat them here. □

When M is neither positive definite nor negative definite, we now assume that M is nonsingular, and we have the following result.

Theorem 4.8. *Suppose that M is nonsingular, $f(n, z)$ satisfies (P_6). Then BVP* (1.5) *possesses at least one solution.*

Proof. We may assume that M is neither positive definite nor negative definite. Let $\lambda_{-l}, \lambda_{-l+1}, \ldots, \lambda_{-1}, \lambda_1, \lambda_2, \ldots, \lambda_m$ denote all eigenvalues of M, where $\lambda_{-l} \leq \lambda_{-l+1} \leq \cdots \leq \lambda_{-1} <$

$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ and $l + m = k$. For any $j \in \mathbb{Z}(-l, -1) \cup \mathbb{Z}(1, m)$, let ξ_j be an eigenvector of M corresponding to the eigenvalue λ_j, $j = -l, \ -l+1, \ldots, -1, 1, 2, \ldots, m$, such that

$$(\xi_i, \xi_j) = \begin{cases} 0, & \text{as } i \neq j, \\ 1, & \text{as } i = j, \end{cases} \tag{4.21}$$

Let X_1 and X_2 be subspaces of $\mathbb{R}^k$ defined as follows:

$$X_1 = \operatorname{span}\{\xi_i, i \in \mathbb{Z}(1, m)\}, \qquad X_2 = \operatorname{span}\{\xi_j, j \in \mathbb{Z}(-l, -1)\}. \tag{4.22}$$

Then $\mathbb{R}^k$ has the following decomposition of direct sum:

$$\mathbb{R}^k = X_1 \bigoplus X_2. \tag{4.23}$$

Let $J(u)$ be defined as in (2.3). Then $J \in C^1(\mathbb{R}^k, \mathbb{R})$. By (4.20),

$$|F(u)| \leq t_0 \sqrt{k} \|u\|, \quad \forall u \in \mathbb{R}^k. \tag{4.24}$$

Suppose that $\{u^{(n)}\} \subset \mathbb{R}^k$ is a P-S sequence. Then there exists a constant t_1 such that for any $n \in \mathbb{Z}(1)$, $|(J(u^{(n)}))| \leq t_1$ and $J'(u^{(n)}) \to 0$ as $n \to \infty$. Thus, for sufficiently large n and for any $u \in \mathbb{R}^k$, we have $|(J'(u^{(n)}), u)| \leq \|u\|$.

Let $u^{(n)} = x^{(n)} + y^{(n)} \in X_1 \bigoplus X_2$. We have, by (2.6), for any $u = (u(1), u(2), \ldots, u(k))^T \in \mathbb{R}^k$,

$$\left(J'\left(u^{(n)}\right), u\right) = \left(Mu^{(n)}, u\right) - \sum_{j=1}^{k} f\left(j, u^{(n)}(j)\right) \cdot u(j). \tag{4.25}$$

Thus for sufficiently large n, we get

$$\begin{aligned} \left(Mu^{(n)}, x^{(n)}\right) &\leq \sum_{j=1}^{k} f\left(j, u^{(n)}(j)\right) \cdot x^{(n)}(j) + \left\|x^{(n)}\right\| \\ &\leq t_0 \sum_{j=1}^{k} \left|x^{(n)}(j)\right| + \left\|x^{(n)}\right\| \\ &\leq \left(t_0 \sqrt{k} + 1\right) \left\|x^{(n)}\right\|, \\ \left(Mu^{(n)}, x^{(n)}\right) &= \left(Mx^{(n)}, x^{(n)}\right) \geq \lambda_1 \left\|x^{(n)}\right\|^2. \end{aligned} \tag{4.26}$$

Thus,

$$\lambda_1 \left\| x^{(n)} \right\|^2 \leq \left(t_0 \sqrt{k} + 1 \right) \left\| x^{(n)} \right\|, \tag{4.27}$$

which implies that $\{x^{(n)}\}$ is bounded.

Now we are going to prove that $\{y^{(n)}\}$ is also bounded. By (4.25),

$$\begin{aligned}
\left(M u^{(n)}, y^{(n)} \right) &\geq \sum_{j=1}^{k} f\left(j, u^{(n)}(j) \right) \cdot y^{(n)}(j) - \left\| y^{(n)} \right\| \\
&\geq -\left(t_0 \sqrt{k} + 1 \right) \left\| y^{(n)} \right\|, \\
\left(M u^{(n)}, y^{(n)} \right) &= \left(M y^{(n)}, y^{(n)} \right) \leq \lambda_{-1} \left\| y^{(n)} \right\|^2.
\end{aligned} \tag{4.28}$$

Thus we have

$$\lambda_{-1} \left\| y^{(n)} \right\|^2 \geq -\left(t_0 \sqrt{k} + 1 \right) \left\| y^{(n)} \right\|. \tag{4.29}$$

And so

$$-\lambda_{-1} \left\| y^{(n)} \right\|^2 - \left(t_0 \sqrt{k} + 1 \right) \left\| y^{(n)} \right\| \leq 0. \tag{4.30}$$

Due to $\lambda_{-1} < 0, \{y^{(n)}\}$ is bounded. Then $\{u^{(n)}\}$ is bounded. Since $\mathbb{R}^k$ is a finite-dimensional Hilbert space, there must exist a subsequence of $\{u^{(n)}\}$ which is convergent in $\mathbb{R}^k$. Therefore, P-S condition is satisfied.

In order to apply the saddle point theorem to prove the conclusion, we consider the functional $-J$ and verify the conditions of Lemma 2.2.

For any $y \in X_2$, $y = (y(1), y(2), \ldots, y(k))^T$, we have

$$\begin{aligned}
-J(y) &= -\frac{1}{2}(My, y) + F(y) \\
&\geq -\frac{1}{2}\lambda_{-1}\|y\|^2 - t_0\sqrt{k}\|y\| \\
&\geq \frac{1}{2\lambda_{-1}}\left(t_0 \sqrt{k} \right)^2.
\end{aligned} \tag{4.31}$$

This implies that (F_4) is true.

For any $x \in X_1$, $x = (x(1), x(2), \dots, x(k))^T$,

$$\begin{aligned} -J(x) &= -\frac{1}{2}(Mx, x) + F(x) \\ &\leq -\frac{1}{2}\lambda_1 \|x\|^2 + t_0\sqrt{k}\|x\| \\ &\longrightarrow -\infty \quad (\text{as } \|x\| \longrightarrow +\infty). \end{aligned} \tag{4.32}$$

This implies that (F_3) is true.

So far we have verified all the assumptions of Lemma 2.2 and hence $-J$ has at least a critical point in $\mathbb{R}^k$. This completes the proof. □

Consider the following special case

$$\begin{gathered} \Delta^4 u(n-2) = f(n, u(n)), \quad n \in \mathbb{Z}(1, k), \\ u(-1) = u(0) = 0 = u(k+1) = u(k+2). \end{gathered} \tag{4.33}$$

Here,

$$M = \begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & -4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix}_{k \times k}. \tag{4.34}$$

It can be verified that M is positive definite, then we have the following corollaries.

Corollary 4.9. *Suppose that there exist constants $a_1 > 0, a_2 > 0$ and $1 < \alpha < 2$ such that*

$$\int_0^z f(n, s)\mathrm{d}s \leq a_1|z|^\alpha + a_2, \quad \forall (n, z) \in \mathbb{Z}(1, k) \times \mathbb{R}. \tag{4.35}$$

Then BVP (4.33) *possesses at least one solution.*

Corollary 4.10. *Suppose that $f(n, z)$ satisfies (P_6). Then BVP* (4.33) *possesses at least one solution.*

4.3. The Lipschitz Case

In this subsection, we suppose the following.

(P_7) Assume that there exist positive constants L, K such that for any $(n,z) \in \mathbb{Z}(1,k) \times \mathbb{R}$,

$$\left|f(n,z)\right| \le L|z| + K. \tag{4.36}$$

When $f(n,z)$ is Lipschitzian in the second variable z, namely, there exists a constant $L > 0$ such that for any $n \in \mathbb{Z}(1,k)$, z_1, $z_2 \in \mathbb{R}$,

$$\left|f(n,z_1) - f(n,z_2)\right| \le L|z_1 - z_2|, \tag{4.37}$$

then condition (4.36) is satisfied.

Theorem 4.11. *Suppose that* (P_7) *is satisfied and M is nonsingular. If $L < \lambda_{\min} = \min\{\lambda_1, -\lambda_{-1}\}$, where λ_1 and λ_{-1} are the minimal positive eigenvalue and maximal negative eigenvalue of M, respectively, then BVP* (1.5) *possesses at least one solution.*

Proof. Assume that $\{u^{(n)}\} \subset \mathbb{R}^k$ is a P-S sequence. Then $J'(u^{(n)}) \to 0$ as $n \to +\infty$. Thus for sufficiently large n, we get $\|J'(u^{(n)})\| \le 1$. Since $J'(u^{(n)}) = Mu^{(n)} - f(u^{(n)})$, then for sufficiently large n,

$$\left\|Mu^{(n)}\right\| \le \left\|f\left(u^{(n)}\right)\right\| + 1. \tag{4.38}$$

In view of (4.36), we have

$$\begin{aligned}\left\|f\left(u^{(n)}\right)\right\|^2 &= \sum_{j=1}^{k} f^2\left(j, u^{(n)}(j)\right) \le \sum_{j=1}^{k}\left(L\left|u^{(n)}(j)\right| + K\right)^2 \\ &= L^2\sum_{j=1}^{k}\left|u^{(n)}(j)\right|^2 + 2LK\sum_{j=1}^{k}\left|u^{(n)}(j)\right| + K^2 k.\end{aligned} \tag{4.39}$$

It follows, by using the inequality $\sqrt{a+b+c} \le \sqrt{a} + \sqrt{b} + \sqrt{c}$ for $a \ge 0, b \ge 0, c \ge 0$ and Hölder's inequality, that

$$\left\|f\left(u^{(n)}\right)\right\| \le L\left\|u^{(n)}\right\| + (2LK)^{1/2}k^{1/4}\left\|u^{(n)}\right\|^{1/2} + Kk^{1/2}. \tag{4.40}$$

By a similar argument to the proof of Theorem 4.8, we can decompose $\mathbb{R}^k$ into the following form of direct sum:

$$\mathbb{R}^k = X_1 \bigoplus X_2, \tag{4.41}$$

where X_1 and X_2 can be referred to (4.22). Thus $u^{(n)}$ can be expressed by

$$u^{(n)} = x^{(n)} + y^{(n)}, \tag{4.42}$$

and $\|Mu^{(n)}\|^2 = \|Mx^{(n)}\|^2 + \|My^{(n)}\|^2$, where $x^{(n)} \in X_1, y^{(n)} \in X_2$. Therefore,

$$\sqrt{\lambda_1^2\|x^{(n)}\|^2 + \lambda_{-1}^2\|y^{(n)}\|^2} \leq \left\|Mu^{(n)}\right\| \leq 1 + L\left\|u^{(n)}\right\| + (2LK)^{1/2}k^{1/4}\left\|u^{(n)}\right\|^{1/2} + Kk^{1/2}. \tag{4.43}$$

Hence,

$$(\lambda_{\min} - L)\left\|u^{(n)}\right\| \leq 1 + (2LK)^{1/2}k^{1/4}\left\|u^{(n)}\right\|^{1/2} + Kk^{1/2}. \tag{4.44}$$

By the fact that $L < \lambda_{\min}$, we know that the sequence $\{u^{(n)}\}$ is bounded and therefore the P-S condition is verified.

Now we are going to check conditions (F_3) and (F_4) for functional $-J$. In fact, by (4.36), we have for any $u \in \mathbb{R}^k$,

$$|F(u)| \leq \sum_{j=1}^{k}\left|\int_0^{u(j)}|f(j,s)|\mathrm{d}s\right| \leq \frac{1}{2}L\|u\|^2 + K\sqrt{k}\|u\|. \tag{4.45}$$

Thus, for any $y \in X_2$, $y = (y(1), y(2), \ldots, y(k))^T$,

$$\begin{aligned} -J(y) &= -\frac{1}{2}(My, y) + F(y) \\ &\geq -\frac{1}{2}\lambda_{-1}\|y\|^2 - \frac{1}{2}L\|y\|^2 - K\sqrt{k}\|y\| \\ &= \frac{1}{2}(-\lambda_{-1} - L)\|y\|^2 - K\sqrt{k}\|y\| \geq \omega, \end{aligned} \tag{4.46}$$

for some positive constant ω.

For any $x \in X_1$, $x = (x(1), x(2), \ldots, x(k))^T$, we have

$$\begin{aligned} -J(x) &= -\frac{1}{2}(Mx, x) + F(x) \\ &\leq -\frac{1}{2}\lambda_1\|x\|^2 + \frac{1}{2}L\|x\|^2 + K\sqrt{k}\|x\| \\ &= -\frac{1}{2}(\lambda_1 - L)\|x\|^2 + K\sqrt{k}\|x\| \longrightarrow -\infty \quad (\text{as } \|x\| \longrightarrow +\infty). \end{aligned} \tag{4.47}$$

Until now, we have verified all the assumptions of Lemma 2.2 and hence $-J$ has at least a critical point in $\mathbb{R}^k$. This completes the proof. □

Finally, we consider the special case that $f(n,z)$ is independent of the second variable z; that is, $f(n,z) \equiv g(n)$ for any $(n,z) \in \mathbb{Z}(1,k) \times \mathbb{R}$, the BVP (1.1) becomes

$$\begin{gathered} \Delta^2\left[p(n-1)\Delta^2 u(n-2)\right] + \Delta\left[q(n)\Delta u(n-1)\right] = g(n), \quad n \in \mathbb{Z}(1,k), \\ u(-1) = u(0) = 0 = u(k+1) = u(k+2). \end{gathered} \tag{4.48}$$

As in Section 2, we reduce the existence of solutions of BVP (4.48) to the existence of critical points of a functional J_1 defined on $\mathbb{R}^k$ as follows:

$$J_1(u) = \frac{1}{2}(Mu,u) - (G,u), \quad \forall u = (u(1), u(2), \ldots, u(k))^T \in \mathbb{R}^k, \tag{4.49}$$

where M is defined as in (2.4), and $G = (g(1), g(2), \ldots, g(k))^T$. Then we can see that the critical point of J_1 is just the solution to the following system of linear algebraic equations:

$$Mu - G = 0. \tag{4.50}$$

By using the theory of linear algebra, we have the next necessary and sufficient conditions.

Theorem 4.12. (i) *BVP* (4.48) *has at least one solution if and only if* $r(M) = r((M,G))$, *where* $r(M)$ *denotes the rank of matrix* M *and* (M,G) *is the augmented matrix defined as follows:*

$$(M,G) = \begin{pmatrix} b(1) & a(1) & p(2) & 0 & \cdots & 0 & 0 & 0 & \vdots & g(1) \\ a(1) & b(2) & a(2) & p(3) & \cdots & 0 & 0 & 0 & \vdots & g(2) \\ p(2) & a(2) & b(3) & a(3) & \cdots & 0 & 0 & 0 & \vdots & g(3) \\ 0 & p(3) & a(3) & b(4) & \cdots & 0 & 0 & 0 & \vdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a(k-2) & p(k-1) & \vdots & g(k-2) \\ 0 & 0 & 0 & 0 & \cdots & a(k-2) & b(k-1) & a(k-1) & \vdots & g(k-1) \\ 0 & 0 & 0 & 0 & \cdots & p(k-1) & a(k-1) & b(k) & \vdots & g(k) \end{pmatrix}_{k\times(k+1)}. \tag{4.51}$$

(ii) *BVP* (4.48) *has a unique solution if and only if* $r(M) = k$.

Acknowledgment

This work is supported by the Specialized Fund for the Doctoral Program of Higher Eduction (no. 20071078001).

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