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Research Article

Stability of General Newton Functional Equations for Logarithmic Spirals

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We investigate the generalized Hyers-Ulam stability of Newton functional equations for logarithmic spirals.

1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [1] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Later, the result of Hyers was significantly generalized for additive mappings by Aoki [3] and for linear mappings by Rassias [4]. It should be remarked that we can find in the books [5–7] a lot of references concerning the stability of functional equations.

Recently, Jung and Sahoo [8] proved the generalized Hyers-Ulam stability of the functional equation $f(\sqrt{r^2+1}) = f(r) + \arctan(1/r)$ which is closely related to the square root spiral, for the case that $f(1) = 0$ and $f(r)$ is monotone increasing for $r > 0$ (see [9, 10]).

By $\mathcal{F}$ we denote the set of all functions $f : (0,\infty) \to \mathbb{R}$. Let Δ be the difference operator defined by

$$(\Delta f)(r) = f(r+1) - f(r) \quad (r > 0) \tag{1.1}$$

for any $f \in \mathcal{F}$. Throughout this paper, let n be a fixed positive integer, and we define an operator $\Delta^n : \mathcal{F} \to \mathcal{F}$ by

$$(\Delta^n f)(r) = \Delta(\Delta^{n-1} f)(r) \quad (r > 0) \tag{1.2}$$

for all $f \in \mathcal{F}$, where we set $\Delta^0 f = f$. For instance, we see that

$$\begin{aligned}(\Delta^2 f)(r) &= f(r+2) - 2f(r+1) + f(r),\\ (\Delta^3 f)(r) &= f(r+3) - 3f(r+2) + 3f(r+1) - f(r).\end{aligned} \tag{1.3}$$

In this paper, we will investigate the generalized Hyers-Ulam stability of the Newton difference (operator) equations

$$\Delta^n f(r) = A \ln R_n(r) \tag{1.4}$$

for all $r > 0$ and some fixed integer $n > 0$, where $A > 0$ is a constant and

$$R_1(r) = \frac{r+1}{r}, \qquad R_k(r) = \frac{R_{k-1}(r+1)}{R_{k-1}(r)} \tag{1.5}$$

for $k \in \{2, 3, \ldots, n\}$.

We will say that (1.4) has the generalized Hyers-Ulam stability whenever a (given) function $f : (0,\infty) \to \mathbb{R}$ satisfies the inequality

$$\left|\Delta^n f(r) - A \ln R_n(r)\right| \le \varphi_n(r) \tag{1.6}$$

for all $r > 0$, where $\varphi_n : (0,\infty) \to [0,\infty)$ is a given nonnegative function, there exists a solution of (1.4) which is not far from f.

2. Newton n-ary difference equation

The difference equation in (1.4) is called the Newton n-ary difference (operator) equation. In the following theorem, we give a partial solution to the generalized Hyers-Ulam stability problem of (1.4).

Theorem 2.1. *If a function $f : (0,\infty) \to \mathbb{R}$ satisfies the inequality* (1.6) *for all $r > 0$ and some integer $n > 0$, where $\varphi_n : (0,\infty) \to [0,\infty)$ is a function which satisfies*

$$\Phi_n(r) = \sum_{k=0}^{\infty} \varphi_n(r+k) < \infty \tag{2.1}$$

for any $r > 0$, then there exists a unique function $F_n : (0,\infty) \to \mathbb{R}$ such that $\Delta F_n(r) = A \ln R_n(r)$ and

$$\left|F_n(r) - \Delta^{n-1} f(r)\right| \le \Phi_n(r) \tag{2.2}$$

for each $r > 0$.

Proof. It follows from (1.6) that

$$\begin{aligned}
&\left|\Delta^n f(r) - A\ln R_n(r)\right| \le \varphi_n(r), \\
&\left|\Delta^n f(r+1) - A\ln R_n(r+1)\right| \le \varphi_n(r+1), \\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots \\
&\left|\Delta^n f(r+m-1) - A\ln R_n(r+m-1)\right| \le \varphi_n(r+m-1)
\end{aligned} \tag{2.3}$$

for any $r > 0$ and $m \in \mathbb{N}$. In view of triangular inequality, the above inequalities yield

$$\left|\sum_{k=0}^{m-1}\Delta^n f(r+k) - \sum_{k=0}^{m-1} A\ln R_n(r+k)\right| \le \sum_{k=0}^{m-1}\varphi_n(r+k). \tag{2.4}$$

Substitute $r+\ell$ for r in (2.4) and then substitute k for $k+\ell$ in the resulting inequality to obtain

$$\left|\sum_{k=\ell}^{\ell+m-1}\Delta^n f(r+k) - \sum_{k=\ell}^{\ell+m-1} A\ln R_n(r+k)\right| \le \sum_{k=\ell}^{\ell+m-1}\varphi_n(r+k) \tag{2.5}$$

for all $r > 0$ and $\ell, m \in \mathbb{N}$.

By some manipulation, we further have

$$\begin{aligned}
&\Bigg|\sum_{k=0}^{\ell+m-1}\Delta^n f(r+k) - \sum_{k=0}^{\ell+m-1} A\ln R_n(r+k) + \Delta^{n-1} f(r) \\
&\qquad - \sum_{k=0}^{\ell-1}\Delta^n f(r+k) + \sum_{k=0}^{\ell-1} A\ln R_n(r+k) - \Delta^{n-1} f(r)\Bigg| \le \sum_{k=\ell}^{\ell+m-1}\varphi_n(r+k)
\end{aligned} \tag{2.6}$$

for every $r > 0$ and $\ell, m \in \mathbb{N}$. Thus, considering (2.1), we see that the sequence

$$\left\{\sum_{k=0}^{m-1}\left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] + \Delta^{n-1} f(r)\right\}_{n=1,2,3,\ldots} \tag{2.7}$$

is a Cauchy sequence for any $r > 0$. Hence, we can define a function $F_n : (0,\infty)\to\mathbb{R}$ by

$$F_n(r) = \sum_{k=0}^{\infty}\left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] + \Delta^{n-1} f(r). \tag{2.8}$$

By (2.8), we get

$$\begin{aligned}
\Delta F_n(r) &= F_n(r+1) - F_n(r) \\
&= \sum_{k=1}^{\infty}\left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] + \Delta^{n-1} f(r+1) \\
&\quad - \sum_{k=0}^{\infty}\left[\Delta^n f(r+k) - A\ln R_n(r+k)\right] - \Delta^{n-1} f(r) \\
&= A\ln R_n(r)
\end{aligned} \tag{2.9}$$

for all $r > 0$. In view of (2.1) and (2.8), if we let m go to infinity in (2.4), then we obtain (2.2).

It only remains to prove the uniqueness of the function F_n. If a function $H : (0,\infty)\to\mathbb{R}$ satisfies $\Delta H(r) = A\ln R_n(r)$ for each $r > 0$, then we can easily show that

$$H(r+m) - H(r) = \sum_{k=0}^{m-1} A\ln R_n(r+k) \tag{2.10}$$

for all $r > 0$ and $m \in \mathbb{N}$. Now, assume that $G_n : (0,\infty)\to\mathbb{R}$ satisfies $\Delta G_n(r) = A\ln R_n(r)$ and the inequality (2.2) in place of F_n. By (2.1), (2.2), and (2.10), we obtain

$$\left|F_n(r) - G_n(r)\right| = \left|F_n(r+m) - G_n(r+m)\right| \le 2\Phi_n(r+m) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{2.11}$$

for any $r > 0$, which proves the uniqueness of F_n. □

3. Application to logarithmic spirals

For given $\alpha > 1$ and $c > 0$, the equation

$$r = ce^{\theta/\sqrt{\alpha^2-1}} \tag{3.1}$$

represents a logarithmic spiral in the polar coordinates (r,θ). We know that this formula is equivalent to

$$\theta = \sqrt{\alpha^2-1}(\ln r - \ln c). \tag{3.2}$$

Let us define $f(r) = \sqrt{\alpha^2-1}(\ln r - \ln c)$ so that we can write the above expression in a simpler form, $\theta = f(r)$. Then f is a solution of (1.4) for $n = 1$ and $A = \sqrt{\alpha^2-1}$, that is, f is a solution of the equation

$$\Delta f(r) = \sqrt{\alpha^2-1}\ln\frac{r+1}{r}, \tag{3.3}$$

which may be called the equation for logarithmic spirals.

We will now solve (3.3) by using [9, Theorem 1].

Theorem 3.1. *If a function $f : (0,\infty)\to\mathbb{R}$ satisfies* (3.3)*, then there exists a periodic function $\sigma : \mathbb{R}\to\mathbb{R}$ of period 1 such that*

$$f(r) = \sigma(r) + \sqrt{\alpha^2-1}\ln r \tag{3.4}$$

for all $r > 0$.

Proof. If we set

$$\psi(r) = \sqrt{\alpha^2-1}\ln\frac{r+1}{r} \tag{3.5}$$

for all $r > 0$, then we have

$$\psi(r+s) - \psi(r) = \sqrt{\alpha^2-1}\ln\frac{r^2+(s+1)r}{r^2+(s+1)r+s} < 0 \tag{3.6}$$

for any $r, s > 0$, which implies that ψ is monotonically decreasing. Moreover, we also see that

$$\lim_{r\to\infty}\psi(r) = \sqrt{\alpha^2-1}\lim_{r\to\infty}\ln\frac{r+1}{r} = 0. \tag{3.7}$$

According to [9, Theorem 1], the general solution of (3.3) is given by

$$f(r) = \sigma(r) + \sum_{k=0}^{\infty}[\psi(k+1) - \psi(r+k)] = \sigma(r) + \sqrt{\alpha^2-1}\ln r, \tag{3.8}$$

where σ is an arbitrary periodic function of period 1. □

If we set $n = 1$ in Theorem 2.1 and apply Theorem 3.1, then we get the following corollary concerning the generalized Hyers-Ulam stability of (3.3).

Corollary 3.2. *If a given function $f : (0,\infty)\to\mathbb{R}$ satisfies the inequality*

$$\left|\Delta f(r) - \sqrt{\alpha^2-1}\ln\frac{r+1}{r}\right| \leq \varphi(r) \tag{3.9}$$

for all $r > 0$ and some $\alpha > 1$, where $\varphi : (0,\infty)\to[0,\infty)$ is a function which satisfies the condition

$$\Phi(r) = \sum_{k=0}^{\infty}\varphi(r+k) < \infty \tag{3.10}$$

for any $r > 0$, then there exists a unique periodic function $\sigma : \mathbb{R}\to\mathbb{R}$ of period 1 such that

$$\left|f(r) - \sigma(r) - \sqrt{\alpha^2-1}\ln r\right| \leq \Phi(r) \tag{3.11}$$

for all $r > 0$.

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