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Research Article

Periodic Solutions for Subquadratic Discrete Hamiltonian Systems

Xiaoqing Deng

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Some existence conditions of periodic solutions are obtained for a class of nonautonomous subquadratic first-order discrete Hamiltonian systems by the minimax methods in the critical point theory.

1. Introduction and statement of main results

We denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ by the set of all natural numbers, integers, real, and complex numbers, respectively. For $a,b\in\mathbb{Z}$, define $\mathbb{Z}(a)=\{a,a+1,\dots\}$, $\mathbb{Z}(a,b)=\{a,a+1,\dots,b\}$ when $a\le b$.

Consider the nonautonomous first-order discrete Hamiltonian systems

$$J\Delta x(n)+\nabla H(n,Lx(n))=0,\quad n\in\mathbb{Z},\tag{1.1}$$

where $J=\begin{pmatrix}0 & -I_N\\ I_N & 0\end{pmatrix}$, $x(n)=\begin{pmatrix}x_1(n)\\ x_2(n)\end{pmatrix}$, $x_i(n)\in\mathbb{R}^N$, $i=1,2,N$ is a given positive integer and I_N denotes the $N\times N$ identity matrix, $\Delta x(n)=x(n+1)-x(n)$, $Lx(n)=\begin{pmatrix}x_1(n+1)\\ x_2(n)\end{pmatrix}$, for all $n\in\mathbb{Z}$, and $H\in C^1(\mathbb{Z}\times\mathbb{R}^{2N},\mathbb{R})$. For a given integer $T>0$, we suppose that $H(n+T,z)=H(n,z)$ for all $n\in\mathbb{Z}$ and $z\in\mathbb{R}^{2N}$, and $\nabla H(n,z)$ denotes the gradient of $H(n,z)$ in $z\in\mathbb{R}^{2N}$.

Our purpose is to establish the existence of T-periodic solutions of (1.1) where H is subquadratic.

Let $H(n,Lx(n)) = H(n,x_1(n+1),x_2(n)) = (1/2)|x_1(n+1)|^2 + V(n,x_2(n))$ with $x_1(n+1) = \Delta x_2(n)$, where $V \in C^1(\mathbb{Z}\times\mathbb{R}^N,\mathbb{R})$ is T-periodic in n, and $\nabla V(n,z)$ denotes the gradient of $V(n,z)$ in $z\in\mathbb{R}^N$. Then, from (1.1) we obtain

$$\Delta^2 x_2(n-1) + \nabla V(n,x_2(n)) = 0, \quad n\in\mathbb{Z},\ x_2(n)\in\mathbb{R}^N. \tag{1.2}$$

As the author knows, in the past two decades, there has been a large number of papers devoted to the existence of periodic and subharmonic solutions for subquadratic first-order (see [1–3]) or second-order (see [4–8]) continuous Hamiltonian systems by using the critical point theory.

On the other hand, in the last five years, by using the critical point theory, the study of existence conditions of periodic and subharmonic solutions for discrete Hamiltonian systems developed rapidly, such as the superquadratic condition for (1.1) (see [9, 10]) or (1.2) (see [11, 12]), the subquadratic condition for (1.1) in [13] or (1.2) in [14, 15], neither superquadratic nor subquadratic condition for (1.2) in [16]. As for the existence of positive solutions of (1.2) with boundary value condition, we can refer to [17, 18].

Recently, in [19] Xue and Tang established the existence of periodic solution for the second-order subquadratic discrete Hamiltonian system (1.2) and generalized the results in [14]. Here, we extend their results to the first-order subquadratic discrete Hamiltonian system (1.1). Our results are more general than those in the literature [13].

Now, we state our main results below.

Theorem 1.1. *Suppose that $H(n,z)$ satisfies the following.*
(H_1) *There exists an integer $T>0$ such that $H(n+T,z) = H(n,z)$ for all $(n,z)\in\mathbb{Z}\times\mathbb{R}^{2N}$,*
(H_2) *there are constants $M_0>0$, $M_1>0$, and $0\le\alpha<1$ such that*

$$|\nabla H(n,z)| \le M_0|z|^\alpha + M_1, \quad \forall (n,z)\in\mathbb{Z}\times\mathbb{R}^{2N}, \tag{1.3}$$

(H_3) $|z|^{-2\alpha}\sum_{n=1}^{T} H(n,z) \to +\infty$ *as* $|z|\to\infty$.
Then problem (1.1) possesses at least one T-periodic solution.

Remark 1.2. Theorem 1.1 extends [13, Theorem 1.1] which is the special case of this theorem by letting $\alpha = 0$.

Corollary 1.3. *If $H(n,z)$ satisfies (H_1)-(H_2) and*
(H_3') $|z|^{-2\alpha}\sum_{n=1}^{T} H(n,z) \to -\infty$ *as* $|z|\to\infty$,
then the conclusion of Theorem 1.1 holds.

Remark 1.4. Corollary 1.3 extends [13, Corollary 1.1] which is the special case of this corollary by letting $\alpha = 0$.

Theorem 1.5. *Suppose that $H(n,z)$ satisfies (H_1) and*
(H_4) $\lim_{|z|\to\infty}(H(n,z)/|z|^2) = 0$ *for all* $n\in\mathbb{Z}(1,T)$,
(H_5) $\lim_{|z|\to\infty}[(\nabla H(n,z),z) - 2H(n,z)] = -\infty$ *for all* $n\in\mathbb{Z}(1,T)$.
Then problem (1.1) has at least one T-periodic solution.

Corollary 1.6. *If $H(n,z)$ satisfies (H_1), (H_4), and*
(H_5') $\lim_{|z|\to\infty}[(\nabla H(n,z),z) - 2H(n,z)] = +\infty$ *for all* $n\in\mathbb{Z}(1,T)$,
then the conclusion of Theorem 1.5 holds.

Corollary 1.7. *If $H(n,z)$ satisfies (H_1), (H_5), or (H_5'), and*
(H_4') $\lim_{|z|\to\infty}(|\nabla H(n,z)|/|z|) = 0$ *for all* $n \in \mathbb{Z}(1,T)$,
then the conclusion of Theorem 1.5 holds.

Corollary 1.8. *If $H(n,z)$ satisfies (H_1) and*
(H_6) *there exist constants* $0 < \beta < 2$ *and* $R_1 > 0$ *such that for all* $(n,z) \in \mathbb{Z} \times \mathbb{R}^{2N}$,

$$(\nabla H(n,z),z) \le \beta H(n,z), \quad \forall\, |z| \ge R_1, \tag{1.4}$$

(H_7) $H(n,z) \to +\infty$ *as* $|z| \to \infty$ *for all* $n \in \mathbb{Z}(1,T)$,
then the conclusion of Theorem 1.5 holds.

Remark 1.9. Comparing [13, Theorem 1.3] with Corollary 1.8, we extend the interval in which β is and delete the constraint of $(\nabla H(n,z),z) > 0$. Furthermore, condition (H_7) is more general than condition (H_6) of [13, Theorem 1.3].

2. Variational structure and some lemmas

In order to apply critical point theory, we need to state the corresponding Hilbert space and to construct a variational functional. Then we reduce the problem of finding the T-periodic solutions of (1.1) to the one of seeking the critical points of the functional.

First we give some notations. Let N be a given positive integer, and

$$S = \left\{ x = \{x(n)\} : x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \in \mathbb{R}^{2N},\ x_i(n) \in \mathbb{R}^N,\ i = 1,2,\ n \in \mathbb{Z} \right\}. \tag{2.1}$$

For any $x,y \in S$, $a,b \in \mathbb{R}$, $ax+by$ is defined by

$$ax + by \triangleq \{ax(n) + by(n)\}. \tag{2.2}$$

Then S is a vector space.

For any given positive integer $T > 0$, E_T is defined as a subspace of S by

$$E_T = \{x = \{x(n)\} \in S : x(n+T) = x(n),\ n \in \mathbb{Z}\} \tag{2.3}$$

with the inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$ as follows:

$$\langle x,y \rangle = \sum_{n=1}^{T} (x(n),y(n)), \quad \|x\| = \left(\sum_{n=1}^{T} |x(n)|^2 \right)^{1/2}, \quad \forall x,y \in E_T, \tag{2.4}$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote the usual inner product and norm in $\mathbb{R}^{2N}$, respectively.

It is easy to see that $(E_T, \langle\cdot,\cdot\rangle)$ is a finite dimensional Hilbert space with dimension $2NT$, which can be identified with $\mathbb{R}^{2NT}$. For convenience, we identify $x \in E_T$ with $x = (x^\tau(1), x^\tau(2), \dots, x^\tau(T))^\tau$, where $x(n) = \binom{x_1(n)}{x_2(n)} \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}(1,T)$, and $(\cdot)^\tau$ is the transpose of a vector or a matrix.

Define another norm in E_T as

$$\|x\|_r = \left(\sum_{n=1}^{T} |x(n)|^r \right)^{1/r}, \quad \forall x \in E_T \tag{2.5}$$

for $r > 1$. Obviously, $\|x\|_2 = \|x\|$ and $(E_T, \|\cdot\|)$ is equivalent to $(E_T, \|\cdot\|_r)$. Hence there exist $C_1 > 0$ and $C_2 \geq C_1 > 0$ such that

$$C_1\|x\|_r \leq \|x\| \leq C_2\|x\|_r, \quad \forall x \in E_T. \tag{2.6}$$

Let $C_1 = T^{-1}$, $C_2 = T$, one can see that the above inequality holds. In fact, define $\|x\|_\infty = \sup_{n\in\mathbb{Z}(1,T)} |x(n)|$, since T is a positive integer and $r > 1$, one can get that

$$\|x\|_\infty \leq \|x\|_r \leq T^{1/r}\|x\|_\infty \leq T\|x\|_\infty. \tag{2.7}$$

Then we can obtain

$$\begin{gathered} \|x\|_\infty \leq \|x\| \leq \sqrt{T}\|x\|_\infty \leq T\|x\|_\infty \leq T\|x\|_r, \\ T^{-1}\|x\|_r \leq \|x\|_\infty \leq \|x\|. \end{gathered} \tag{2.8}$$

For $T > 0$, we define the functional $F(x)$ on E_T as

$$F(x) = \frac{1}{2}\sum_{n=1}^{T}(J\Delta Lx(n-1), x(n)) + \sum_{n=1}^{T} H(n, Lx(n)), \quad \forall x \in E_T. \tag{2.9}$$

Then we have $F \in C^1(E_T, \mathbb{R})$ and

$$\begin{aligned} \langle F'(x), y\rangle &= \sum_{n=1}^{T}(J\Delta Lx(n-1), y(n)) + \sum_{n=1}^{T}(\nabla H(n, Lx(n)), Ly(n)) \\ &= \sum_{n=1}^{T}(J\Delta x(n), Ly(n)) + \sum_{n=1}^{T}(\nabla H(n, Lx(n)), Ly(n)) \end{aligned} \tag{2.10}$$

for all $x, y \in E_T$. Obviously, for any $x \in E_T$, $F'(x) = 0$ if and only if

$$J\Delta x(n) + \nabla H(n, Lx(n)) = 0 \tag{2.11}$$

for all $n \in \mathbb{Z}(1,T)$. Therefore, the problem of finding the T-periodic solution for (1.1) is reduced to the one of seeking the critical point of functional F.

Next, we construct a variational structure by using the operator theory which is different from the one in [9, 10, 13].

Consider the eigenvalue problem

$$J\Delta Lx(n-1) = \lambda x(n), \qquad x(n+T) = x(n). \tag{2.12}$$

Setting

$$A(\lambda) = \begin{pmatrix} I_N & \lambda I_N \\ -\lambda I_N & (1-\lambda^2)I_N \end{pmatrix}, \tag{2.13}$$

then the problem (2.12) is equivalent to

$$x(n+1) = A(\lambda)x(n), \qquad x(n+T) = x(n). \tag{2.14}$$

As we all know, the solution of problem (2.14) is denoted by $x(n) = \mu^n C$ with $C = x(0) \in \mathbb{R}^{2N}$, where μ is the eigenvalue of $A(\lambda)$ and $\mu^T = 1$. Then it follows from $\mu_k^T = 1$ and $|A(\lambda_k) - \mu_k I_{2N}| = 0$ that $\mu_k = \exp(k\omega i)$ with $\omega = 2\pi/T$ and $\lambda_k = 2\sin(k\pi/T)$ with $\lambda_{T-k} = \lambda_k$ for all $k \in \mathbb{Z}(-[T/2],[T/2])$, where $[\cdot]$ is Gauss function.

Now we give some lemmas which will be important in the proofs of the results of the paper.

Lemma 2.1. *Set* $H_k = \{x \in E_T : J\Delta Lx(n-1) = \lambda_k x(n)$ *for all* $k \in \mathbb{Z}(-[T/2],[T/2])\}$. *Then*

$$H_k \perp H_j, \quad \forall k, j \in \mathbb{Z}\left(-\left[\frac{T}{2}\right],\left[\frac{T}{2}\right]\right),\ k \neq j, \tag{2.15}$$

$$E_T = \bigoplus_{k=-[T/2]}^{[T/2]} H_k. \tag{2.16}$$

Proof. For all $x \in H_k$, $y \in H_j$, we have

$$\begin{aligned}\lambda_k \langle x, y\rangle &= \sum_{n=1}^{T} (\lambda_k x(n), y(n)) = \sum_{n=1}^{T} (J\Delta Lx(n-1), y(n)) \\ &= \sum_{n=1}^{T} (x(n), J\Delta Ly(n-1)) = \lambda_j \langle x, y\rangle.\end{aligned} \tag{2.17}$$

Since $\lambda_k \neq \lambda_j$, we have $\langle x, y\rangle = 0$, that is, $H_k \perp H_j$, then (2.15) holds.

Next we consider the elements of H_k for all $k \in \mathbb{Z}(-[T/2],[T/2])$.

Case 1. For all $x \in H_0$, it follows from $\mu_0 = 1$ that

$$H_0 = \{x \in E_T : x(n) \equiv x(0) = C \in \mathbb{R}^{2N}\}, \tag{2.18}$$

and $\dim H_0 = 2N$.

Case 2. T is even. For $k = [T/2] = T/2$, it follows from $\lambda_{T/2} = 2$, $\mu_{T/2} = -1$, and $(A(2) + I_N)C = 0$ that $C = (\rho^\tau, -\rho^\tau)^\tau$ with $\rho \in \mathbb{R}^N$. Therefore,

$$H_{[T/2]} = \{x \in E_T : x(n) = (-1)^n (\rho^\tau, -\rho^\tau)^\tau,\ \rho \in \mathbb{R}^N\}, \tag{2.19}$$

and $\dim H_{[T/2]} = N$. Similarly, for $k = -[T/2] = -T/2$, we have

$$H_{-[T/2]} = \{x \in E_T : x(n) = (-1)^n (\rho^\tau, \rho^\tau)^\tau,\ \rho \in \mathbb{R}^N\}, \tag{2.20}$$

and $\dim H_{-[T/2]} = N$.

T is odd. Similarly, for $k = [T/2] = (T-1)/2$, we have

$$H_{[T/2]} = \left\{x \in E_T : x(n) = \exp\left(\frac{n(T-1)\pi i}{T}\right)\left(\rho^\tau, -\exp\left(-\frac{\pi i}{2T}\right)\rho^\tau\right)^\tau,\ \rho \in \mathbb{C}^N\right\}, \tag{2.21}$$

and $\dim H_{[T/2]} = 2N$. For $k = -[T/2] = -(T-1)/2$, we have

$$H_{-[T/2]} = \left\{ x \in E_T : x(n) = \exp\left(-\frac{n(T-1)\pi i}{T} \right) \left(\rho^\tau, \exp\left(\frac{\pi i}{2T} \right) \rho^\tau \right)^\tau, \ \rho \in \mathbb{C}^N \right\}, \tag{2.22}$$

and $\dim H_{-[T/2]} = 2N$.

Case 3. For $k \in \mathbb{Z}(1, [T/2]-1) \cup \mathbb{Z}(-[T/2]+1, -1)$, it follows from $\lambda_k = 2\sin(k\pi/T)$, $\mu_k = \exp(2k\pi i/T)$, and $(A(\lambda_k) - \mu_k I_{2N})C = 0$ that

$$H_k = \left\{ x \in E_T : x(n) = \exp\left(\frac{2kn\pi i}{T} \right) \left(\rho^\tau, -\exp\left(-\left(\frac{\pi}{2} - \frac{k\pi}{T} \right) i \right) \rho^\tau \right)^\tau, \ \rho \in \mathbb{C}^N \right\}, \tag{2.23}$$

and $\dim H_k = 2N$.

Thus, from Cases 1, 2, and 3, we have

$$\dim \bigoplus_{k=-[T/2]}^{[T/2]} H_k = 2N + 2\left(\left[\frac{T}{2} \right] - 1 \right) \times 2N + N + N = 2NT \tag{2.24}$$

when T is even, and

$$\dim \bigoplus_{k=-[T/2]}^{[T/2]} H_k = 2N + 2\left[\frac{T}{2} \right] \times 2N = 2NT \tag{2.25}$$

when T is odd.

Since $\dim E_T = 2NT$ and $\bigoplus_{k=-[T/2]}^{[T/2]} H_k \subseteq E_T$, $E_T = \bigoplus_{k=-[T/2]}^{[T/2]} H_k$. Lemma 2.1 is completed. □

Let $E_T^0 = H_0$, $E_T^+ = \bigoplus_{k=1}^{[T/2]} H_k$, and $E_T^- = \bigoplus_{k=-[T/2]}^{-1} H_k$, then it is easy to obtain the following lemma.

Lemma 2.2.

$$\begin{gathered}
\sum_{n=1}^{T} (J\Delta L x(n-1), x(n)) = 0, \quad \forall x \in E_T^0, \\
\lambda_1 \|x\|^2 \le \sum_{n=1}^{T} (J\Delta L x(n-1), x(n)) \le \lambda_{[T/2]} \|x\|^2, \quad \forall x \in E_T^+, \\
-\lambda_{[T/2]} \|x\|^2 \le \sum_{n=1}^{T} (J\Delta L x(n-1), x(n)) \le -\lambda_1 \|x\|^2, \quad \forall x \in E_T^-,
\end{gathered} \tag{2.26}$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{[T/2]}$.

3. Proofs of the main theorems

Proof of Theorem 1.1. Let $F(x)$ be defined as (2.9), clearly, $F \in C^1(E_T, \mathbb{R})$.

We will first show that F satisfies the Palais-Smale condition, that is, any sequence $\{x^{(k)}\} \subset E_T$ for which $|F(x^{(k)})| \leq M_2$ with constant $M_2 > 0$ and $F'(x^{(k)}) \to 0$ $(k \to \infty)$ possesses a convergent subsequence in E_T. Recall that E_T is identified with $\mathbb{R}^{2NT}$. Consequently, in order to prove that F satisfies Palais-Smale condition, we only need to prove that $\{x^{(k)}\}$ is bounded.

Suppose that $\{x^{(k)}\}$ is unbounded, then we can assume, going to a subsequence if necessary, that $\|x^{(k)}\| \to \infty$ as $k \to \infty$.

Let $x^{(k)} = u^{(k)} + v^{(k)} + w^{(k)} = y^{(k)} + w^{(k)}$, where $u^{(k)} \in E_T^+$, $v^{(k)} \in E_T^-$, $w^{(k)} \in E_T^0$ with $w^{(k)}(n) \equiv C^{(k)}$ for all $n \in \mathbb{Z}$.

In view of (H_2), we have

$$\begin{aligned}
&\left| \sum_{n=1}^{T} [H(n, Lx^{(k)}(n)) - H(n, Lw^{(k)}(n))] \right| \\
&\quad \leq \sum_{n=1}^{T} \int_0^1 |\nabla H(n, Lw^{(k)}(n) + sLy^{(k)}(n))| \, |Ly^{(k)}(n)| \, ds \\
&\quad \leq \sum_{n=1}^{T} \int_0^1 [M_0 |Lw^{(k)}(n) + sLy^{(k)}(n)|^{\alpha} + M_1] \, |Ly^{(k)}(n)| \, ds \\
&\quad \leq 2M_0 \sum_{n=1}^{T} (|Lw^{(k)}(n)|^{\alpha} + |Ly^{(k)}(n)|^{\alpha}) |Ly^{(k)}(n)| + \sum_{n=1}^{T} M_1 |Ly^{(k)}(n)| \\
&\quad \leq \frac{2M_0^2}{\lambda_1} \sum_{n=1}^{T} |Lw^{(k)}(n)|^{2\alpha} + \frac{M_0 \lambda_1}{2M_0} \sum_{n=1}^{T} |Ly^{(k)}(n)|^2 \\
&\qquad + 2M_0 \sum_{n=1}^{T} |Ly^{(k)}(n)|^{\alpha+1} + \sum_{n=1}^{T} M_1 |Ly^{(k)}(n)| \\
&\quad \leq \frac{2M_0^2 T}{\lambda_1} |C^{(k)}|^{2\alpha} + \frac{\lambda_1}{2} \|y^{(k)}\|^2 + \frac{2M_0}{C_1^{\alpha+1}} \|y^{(k)}\|^{\alpha+1} + M_1 \sqrt{T} \|y^{(k)}\|.
\end{aligned} \tag{3.1}$$

By using the same method, we can obtain

$$\left| \sum_{n=1}^{T} (\nabla H(n, Lx^{(k)}(n)), Lu^{(k)}(n)) \right| \leq \frac{2M_0^2}{\lambda_1 C_1^{2\alpha}} \|x^{(k)}\|^{2\alpha} + \frac{\lambda_1}{2} \|u^{(k)}\|^2 + M_1 \sqrt{T} \|u^{(k)}\|, \tag{3.2}$$

$$\left| \sum_{n=1}^{T} (\nabla H(n, Lx^{(k)}(n)), Lv^{(k)}(n)) \right| \leq \frac{2M_0^2}{\lambda_1 C_1^{2\alpha}} \|x^{(k)}\|^{2\alpha} + \frac{\lambda_1}{2} \|v^{(k)}\|^2 + M_1 \sqrt{T} \|v^{(k)}\|. \tag{3.3}$$

It follows from inequality (3.2) and

$$\langle F'(x), y \rangle = \sum_{n=1}^{T} (J\Delta Lx(n-1), y(n)) + \sum_{n=1}^{T} (\nabla H(n, Lx(n)), Ly(n)), \quad \forall x, y \in E_T \tag{3.4}$$

that

$$\begin{aligned}
\lambda_1 \|u^{(k)}\|^2 &\le \sum_{n=1}^{T} \left(J\Delta L x^{(k)}(n-1), u^{(k)}(n)\right) \\
&= \langle F'(x^{(k)}), u^{(k)}\rangle - \sum_{n=1}^{T} \left(\nabla H(n, Lx^{(k)}(n)), Lu^{(k)}(n)\right) \\
&\le \|u^{(k)}\| + \frac{2M_0^2}{\lambda_1 C_1^{2\alpha}} \|x^{(k)}\|^{2\alpha} + \frac{\lambda_1}{2}\|u^{(k)}\|^2 + M_1\sqrt{T}\|u^{(k)}\|
\end{aligned} \tag{3.5}$$

for sufficiently large k. That is,

$$\frac{\lambda_1}{2}\|u^{(k)}\|^2 - \left(1 + M_1\sqrt{T}\right)\|u^{(k)}\| \le \frac{2M_0^2}{\lambda_1 C_1^{2\alpha}}\|x^{(k)}\|^{2\alpha} \tag{3.6}$$

for k large enough. Since $\|x^{(k)}\| \to \infty$ as $k \to \infty$, we can assume that $\|x^{(k)}\| \ge 1$ for sufficiently large k. Therefore, for sufficiently large k, from the above inequality (3.6), there exists a constant $M_3 > 0$ such that

$$\|u^{(k)}\| \le M_3 \|x^{(k)}\|^{\alpha}. \tag{3.7}$$

In fact, if (3.7) is false, then there exists some subsequence of $\{x^{(k)}\}$, still denoted by $\{x^{(k)}\}$, such that

$$\frac{\|x^{(k)}\|^{\alpha}}{\|u^{(k)}\|} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{3.8}$$

Thanks to the inequality (3.6), one has

$$\frac{\lambda_1}{2} \le \frac{2M_0^2}{\lambda_1 C_1^{2\alpha}} \left(\frac{\|x^{(k)}\|^{\alpha}}{\|u^{(k)}\|}\right)^2 + \frac{1 + M_1\sqrt{T}}{\|u^{(k)}\|} \tag{3.9}$$

for k large enough. Obviously, the above two inequalities imply that $\|x^{(k)}\|$ is bounded for sufficiently large k, which is contradictory with the assumption that $\|x^{(k)}\| \to \infty$ as $k \to \infty$.

Therefore, (3.7) is true, and then we have

$$\frac{\|u^{(k)}\|}{\|x^{(k)}\|} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{3.10}$$

Similarly, from inequality (3.3) and equality (2.10), there exists a constant $M_3' > 0$ such that

$$\|v^{(k)}\| \le M_3' \|x^{(k)}\|^{\alpha} \tag{3.11}$$

for sufficiently large k, and then

$$\frac{\|v^{(k)}\|}{\|x^{(k)}\|} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{3.12}$$

It follows from (3.10) and (3.12) that

$$\frac{||w^{(k)}||}{||x^{(k)}||} \longrightarrow 1, \quad k \longrightarrow \infty, \tag{3.13}$$

and then (3.7) and (3.11) mean that there exists $M_4 > 0$ such that

$$||y^{(k)}|| = ||u^{(k)}|| + ||v^{(k)}|| \leq 2M_4 T^{\alpha/2} |C^{(k)}|^{\alpha} \tag{3.14}$$

for sufficiently large k. Therefore, from (3.1), we have

$$\begin{aligned} &\left| \sum_{n=1}^{T} [H(n, Lx^{(k)}(n)) - H(n, Lw^{(k)}(n))] \right| \\ &\quad \leq \left(\frac{2M_0^2 T}{\lambda_1} + 2\lambda_1 M_4^2 T^{\alpha} \right) |C^{(k)}|^{2\alpha} + \frac{2^{\alpha+2} M_0 M_4^{\alpha+1} T^{\alpha(\alpha+1)/2}}{C_1^{\alpha+1}} |C^{(k)}|^{\alpha(\alpha+1)} \\ &\quad\quad + 2M_1 M_4 T^{(\alpha+1)/2} |C^{(k)}|^{\alpha}. \end{aligned} \tag{3.15}$$

Then there exists $M_5 > 0$ such that

$$|C^{(k)}|^{-2\alpha} \left| \sum_{n=1}^{T} [H(n, Lx^{(k)}(n)) - H(n, Lw^{(k)}(n))] \right| \leq M_5 \tag{3.16}$$

as $|C^{(k)}| \to \infty$.

By using Lemma 2.2 and the boundedness of $F(x^{(k)})$, we have

$$\begin{aligned} M_2 \geq F(x^{(k)}) &= \frac{1}{2} \sum_{n=1}^{T} [(J\Delta Lx^{(k)}(n-1), x^{(k)}(n)) + H(n, Lx^{(k)}(n))] \\ &= \frac{1}{2} \sum_{n=1}^{T} (J\Delta Lu^{(k)}(n-1), u^{(k)}(n)) + \frac{1}{2} \sum_{n=1}^{T} (J\Delta Lv^{(k)}(n-1), v^{(k)}(n)) \\ &\quad + \sum_{n=1}^{T} [H(n, Lx^{(k)}(n)) - H(n, Lw^{(k)}(n))] + \sum_{n=1}^{T} H(n, Lw^{(k)}(n)) \\ &\geq \frac{\lambda_1}{2} ||u^{(k)}||^2 - \frac{\lambda_{[T/2]}}{2} ||v^{(k)}||^2 + \sum_{n=1}^{T} [H(n, Lx^{(k)}(n)) - H(n, Lw^{(k)}(n))] \\ &\quad + \sum_{n=1}^{T} H(n, Lw^{(k)}(n)). \end{aligned} \tag{3.17}$$

It follows from (3.14) and (3.16), by multiplying $|C^{(k)}|^{-2\alpha}$ with both sides of above inequality, that there exists $M_6 > 0$ such that

$$|LC^{(k)}|^{-2\alpha} \sum_{n=1}^{T} H(n, LC^{(k)}) = |C^{(k)}|^{-2\alpha} \sum_{n=1}^{T} H(n, Lw^{(k)}(n)) \leq M_6 \tag{3.18}$$

as $|C^{(k)}| \to \infty$. This is a contradiction with (H_3), consequently, $\|x^{(k)}\|$ is bounded. Thus we conclude that the Palais-Smale condition is satisfied.

In order to use the saddle point theorem (see [20, Theorem 4.6]), we only need to verify the following:

(F_1) $F(x) \to -\infty$ as $\|x\| \to \infty$ in $X_1 = E_T^-$,

(F_2) $F(x) \to +\infty$ as $\|x\| \to \infty$ in $X_2 = E_T^0 \oplus E_T^+$.

In fact, for $v \in E_T^-$, there exists $M_7 > 0$ such that

$$\begin{aligned} F(v) &= \frac{1}{2}\sum_{n=1}^{T}(J\Delta Lv(n-1), v(n)) + \sum_{n=1}^{T}[H(n, Lv(n)) - H(n,0)] + \sum_{n=1}^{T} H(n,0) \\ &\le -\frac{\lambda_1}{2}\|v\|^2 + \sum_{n=1}^{T}\int_0^1 |\nabla H(n, sLv(n))| \cdot |Lv(n)|\,ds + \sum_{n=1}^{T} H(n,0) \\ &\le -\frac{\lambda_1}{2}\|v\|^2 + \frac{M_0}{C_1^{\alpha+1}}\|v\|^{\alpha+1} + M_1\sqrt{T}\|v\| + M_7 \longrightarrow -\infty \end{aligned} \tag{3.19}$$

as $\|v\| \to \infty$. Thus (F_1) is verified.

Next, for all $u + w \in E_T^+ \oplus E_T^0$, we have

$$\begin{aligned} &F(u+w) \\ &\quad = \frac{1}{2}\sum_{n=1}^{T}(J\Delta Lu(n-1), u(n)) + \sum_{n=1}^{T}[H(n, Lu(n) + Lw(n)) - H(n, Lw(n))] \\ &\qquad + \sum_{n=1}^{T} H(n, Lw(n)) \\ &\quad \ge \frac{\lambda_1}{2}\|u\|^2 - \sum_{n=1}^{T}\int_0^1 |\nabla H(n, Lw(n) + sLu(n))| \cdot |Lu(n)|\,ds + \sum_{n=1}^{T} H(n, Lw(n)) \\ &\quad \ge \frac{\lambda_1}{4}\|u\|^2 - \frac{M_0}{C_1^{\alpha+1}}\|u\|^{\alpha+1} - M_1\sqrt{T}\|u\| - \frac{4M_0^2T}{\lambda_1}|C|^{2\alpha} + \sum_{n=1}^{T} H(n, LC). \end{aligned} \tag{3.20}$$

Since $1 \le \alpha + 1 < 2$,

$$\frac{\lambda_1}{4}\|u\|^2 - \frac{2M_0}{C_1^{\alpha+1}}\|u\|^{\alpha+1} - M_1\sqrt{T}\|u\| \longrightarrow +\infty, \quad \|u\| \longrightarrow \infty. \tag{3.21}$$

By (H_3) we have

$$\begin{aligned} &|LC|^{-2\alpha}\left[\sum_{n=1}^{T} H(n, LC) - \frac{4M_0^2T}{\lambda_1}|C|^{2\alpha}\right] \\ &\qquad = |LC|^{-2\alpha}\sum_{n=1}^{T} H(n, LC) - \frac{4M_0^2T}{\lambda_1} \longrightarrow +\infty, \quad |C| \longrightarrow \infty. \end{aligned} \tag{3.22}$$

Then we have

$$\begin{aligned}&\sum_{n=1}^{T} H(n,LC) - \frac{4M_0^2T}{\lambda_1}|C|^{2\alpha}\\&\quad= |LC|^{2\alpha}|LC|^{-2\alpha}\left[\sum_{n=1}^{T} H(n,LC) - \frac{4M_0^2T}{\lambda_1}|C|^{2\alpha}\right] \longrightarrow +\infty, \quad |C| \longrightarrow \infty.\end{aligned} \tag{3.23}$$

Since $\|u+w\| \to \infty$ is equivalent to $\|u\|^2 + T|C|^2 \to \infty$, we have

$$F(u+w) \longrightarrow +\infty, \qquad \|u+w\| \longrightarrow \infty, \tag{3.24}$$

which implies that (F_2) is verified. Then the proof of Theorem 1.1 is finished. □

Proof of Corollary 1.3. Let $G(x) = -F(x)$, by a similar argument to the proof of Theorem 1.1, we can prove that G satisfies the Palais-Smale condition and $G(x) \to +\infty$ as $\|x\| \to \infty$ in $X_2 = E_T^0 \oplus E_T^-$ and $G(x) \to -\infty$ as $\|x\| \to \infty$ in $X_1 = E_T^+$. Corollary 1.3 is completed. □

Proof of Theorem 1.5. As we all know, a deformation lemma can be proved with the weaker (C) condition which is introduced in [21] replacing the usual Palais-Smale condition, and the saddle point theorem holds true under (C) condition.

First, we prove that F satisfied (C) condition, that is, any sequence $\{x^{(k)}\} \subset E_T$ for which $F(x^{(k)})$ is bounded and $(1+\|x^{(k)}\|)\|F'(x^{(k)})\| \to 0$ $(k \to \infty)$ possesses a convergent subsequence in E_T.

Then there exists constant $C_3 > 0$ such that

$$|F(x^{(k)})| \le C_3, \qquad (1+\|x^{(k)}\|)\|F'(x^{(k)})\| \le C_3. \tag{3.25}$$

Thus

$$\begin{aligned}-3C_3 &\le -(1+\|x^{(k)}\|)\|F'(x^{(k)})\| - 2|F(x^{(k)})|\\&\le \langle F'(x^{(k)}), x^{(k)}\rangle - 2F(x^{(k)})\\&= \sum_{n=1}^{T}\left[(\nabla H(n,x^{(k)}(n)), Lx^{(k)}(n)) - 2H(n,Lx^{(k)}(n))\right].\end{aligned} \tag{3.26}$$

Consequently, by (H_5) and (3.26), $\|x^{(k)}\|$ is bounded.

In fact, if $\|x^{(k)}\|$ is unbounded, without loss of generality, there exist integer $n_1 > 0$ and constant $C_4 > 0$ such that $|x^{(k)}(n)| \to \infty$ for all $T \ge n > n_1$ and $|x^{(k)}(n)| \le C_4$ for all $1 \le n \le n_1$.

When $T \ge n > n_1$, by (H_5), one can obtain

$$(\nabla H(n,x^{(k)}(n)), Lx^{(k)}(n)) - 2H(n,Lx^{(k)}(n)) \longrightarrow -\infty. \tag{3.27}$$

When $1 \le n \le n_1$, by the differential of $H(n,z)$ in z, there exists constant $C_5 > 0$ such that

$$|(\nabla H(n,x^{(k)}(n)), Lx^{(k)}(n)) - 2H(n,Lx^{(k)}(n))| \le C_5. \tag{3.28}$$

Then we have

$$\sum_{n=1}^{T}\left[\left(\nabla H\left(n,x^{(k)}(n)\right),Lx^{(k)}(n)\right)-2H\left(n,Lx^{(k)}(n)\right)\right]\longrightarrow-\infty,\tag{3.29}$$

which is contrary to (3.26), so $\|x^{(k)}\|$ is bounded.

Then as a consequence in finite dimensional space E_T, $\{x^{(k)}\}$ has a convergent subsequence and thus (C) condition is verified.

Next we show that F satisfies (F_1) and (F_2).

By (H_4), there exists $C_6>0$ such that

$$\left|H(n,z)\right|\leq\frac{\lambda_1}{4}|z|^2+C_6,\quad\forall(n,z)\in\mathbb{Z}\times\mathbb{R}^{2N}.\tag{3.30}$$

Then

$$\begin{aligned}F(v)&=\frac{1}{2}\sum_{n=1}^{T}\left(J\Delta Lv(n-1),v(n)\right)+\sum_{n=1}^{T}H\left(n,Lv(n)\right)\\&\leq-\frac{\lambda_1}{2}\|v\|^2+\frac{\lambda_1}{4}\|v\|^2+TC_6\longrightarrow-\infty\end{aligned}\tag{3.31}$$

as $\|v\|\to\infty$ for $v\in X_1=E_T^-$. Therefore, (F_1) is verified.

Conditions (H_4) and (H_5) imply that $H(n,z)\to+\infty$ as $|z|\to\infty$ for all $n\in\mathbb{Z}(1,T)$.

In fact, let $s>1$, from (H_5), for all $\epsilon>0$ there exists constant $C_7>0$ such that

$$\left(\nabla H(n,sz),sz\right)-2H(n,sz)\leq-\frac{1}{\epsilon},\quad\forall|z|\geq C_7,\tag{3.32}$$

then we have

$$\begin{aligned}\frac{d}{ds}\left(\frac{H(n,sz)}{s^2}\right)&=\frac{\left(\nabla H(n,sz),sz\right)-2H(n,sz)}{s^3}\\&\leq-\frac{1}{\epsilon s^3}=\frac{d}{ds}\left(\frac{1}{2\epsilon s^2}\right),\quad\forall|z|\geq C_7.\end{aligned}\tag{3.33}$$

By integrating both sides of the above inequality from 1 to s, we can obtain

$$\frac{H(n,sz)}{s^2}-H(n,z)\leq\frac{1}{2\epsilon s^2}-\frac{1}{2\epsilon},\quad\forall|z|\geq C_7.\tag{3.34}$$

By (H_4), we have

$$\frac{H(n,sz)}{s^2}\longrightarrow0,\quad s\longrightarrow\infty.\tag{3.35}$$

Then

$$H(n,z)\geq\frac{1}{2\epsilon},\quad\forall|z|\geq C_7.\tag{3.36}$$

From the arbitrariness of ϵ, one can conclude that

$$H(n,z) \longrightarrow +\infty, \quad |z| \longrightarrow \infty \tag{3.37}$$

for all $n \in \mathbb{Z}(1,T)$.

Thus, thanks to Lemma 2.2, one has

$$F(u+w) \geq \sum_{n=1}^{T} H(n, Lu(n) + Lw(n)) \longrightarrow +\infty \tag{3.38}$$

as $\|u+w\| \to \infty$ for $u+w \in X_2 = E_T^+ \oplus E_T^0$, which implies that (F_2) is verified. The proof of Theorem 1.5 is finished. □

Proof of Corollary 1.6. Let $G(x) = -F(x)$, $X_1 = E_T^+$, and $X_2 = E_T^- \oplus E_T^0$, by a similar argument to the proof of Theorem 1.5, we can prove that Corollary 1.6 holds. □

Proof of Corollary 1.7. By (H_4'), for all $\varepsilon > 0$ there exist $\theta \in (0,1)$, $R_2 > 0$, and $C_8 > 0$ such that

$$\begin{aligned} H(n,z) &= H(n,0) + \int_0^1 (\nabla H(n,\theta z), z)\, d\theta \\ &\leq \int_0^1 |\nabla H(n,\theta z)| \cdot |z|\, d\theta + C_8 \\ &\leq \int_0^1 \varepsilon\theta |z|^2 d\theta + C_8 \\ &\leq \varepsilon |z|^2 + C_8, \quad |z| \geq \frac{R_2}{\theta} > R_2, \end{aligned} \tag{3.39}$$

which implies that (H_4) holds. Then it follows from Theorem 1.5 and Corollary 1.6 that Corollary 1.7 holds. □

Proof of Corollary 1.8. From (H_7), there exists $C_9 > 0$ such that

$$H(n,z) > 0, \quad \forall |z| \geq C_9,\ \forall n \in \mathbb{Z}(1,T). \tag{3.40}$$

Setting $R_2 = \max\{R_1, C_9\}$, by (H_6), we have

$$\left(\frac{\nabla H(n,z)}{H(n,z)}, \frac{z}{|z|} \right) \leq \frac{\beta}{|z|}, \quad \forall n \in \mathbb{Z}(1,T),\ |z| \geq R_2. \tag{3.41}$$

Then

$$\frac{d \ln H(n,z)}{d|z|} \leq \frac{\beta}{|z|}, \quad \forall n \in \mathbb{Z}(1,T),\ |z| \geq R_2, \tag{3.42}$$

which implies

$$\frac{d}{d|z|} \left(\ln H(n,z) - \beta \ln |z| \right) \leq 0, \quad \forall n \in \mathbb{Z}(1,T),\ |z| \geq R_2. \tag{3.43}$$

Let $I = \max\{\ln H(n,z) - \beta \ln |z| : |z| = R_2\}$, by (3.43),

$$\ln H(n,z) - \beta \ln |z| \le I, \quad \forall n \in \mathbb{Z}(1,T),\ |z| \ge R_2. \tag{3.44}$$

That is,

$$0 < H(n,z) \le C_{10}|z|^{\beta}, \quad \forall n \in \mathbb{Z}(1,T),\ |z| \ge R_2, \tag{3.45}$$

where $C_{10} = e^I$. Thus we have

$$0 < \frac{H(n,z)}{|z|^2} \le \frac{C_{10}|z|^{\beta}}{|z|^2}, \quad \forall n \in \mathbb{Z}(1,T),\ |z| \ge R_2. \tag{3.46}$$

Since $\beta \in (0,2)$, from above inequality, we can conclude that

$$\frac{H(n,z)}{|z|^2} \longrightarrow 0 \tag{3.47}$$

as $|z| \to \infty$, which implies (H_4).

Since $\beta \in (0,2)$, it follows from (H_6) and (H_7) that

$$(\nabla H(n,z), z) - 2H(n,z) \le (\beta - 2)H(n,z) \longrightarrow -\infty \tag{3.48}$$

as $|z| \to \infty$ for all $n \in \mathbb{Z}(1,T)$, which implies (H_5). □

Then the result of Corollary 1.8 holds by using Theorem 1.5.

Finally, we give two examples to illustrate our conclusions.

Example 3.1. Consider the system (1.1) with

$$H(n,z) = |z|^{4/3} + (e(n),z), \quad n \in \mathbb{Z},\ z \in \mathbb{R}^{2N}, \tag{3.49}$$

where $e(n+T) = e(n) \in \mathbb{R}^{2N}$.

Let $\bar{e} = \max_{n\in\mathbb{Z}(1,T)} |e(n)|$, $\alpha = 1/3$, then we have

$$\begin{gathered} \left| \nabla H(n,z) \right| \le \frac{4}{3}|z|^{1/3} + \bar{e}, \quad \forall (n,z) \in \mathbb{Z} \times \mathbb{R}^{2N}, \\ |z|^{-2/3}\sum_{n=1}^{T} H(n,z) = \sum_{n=1}^{T} |z|^{2/3} + \sum_{n=1}^{T} |z|^{-2/3}(e(n),z) \ge T\left(|z|^{2/3} - \bar{e}|z|^{1/3}\right) \longrightarrow +\infty \end{gathered} \tag{3.50}$$

as $|z| \to \infty$.

Thus it follows from Theorem 1.1 that (1.1) with H as defined in (3.49) possesses at least one T-periodic solution.

Example 3.2. Consider the system (1.1) with

$$H(n,z) = \left(g(n) + |z|\right)\ln\left(1 + |z|^2\right) + \left(h(n),z\right), \quad \forall (n,z) \in \mathbb{Z} \times \mathbb{R}^{2N}, \tag{3.51}$$

where $g(n+T) = g(n) \in \mathbb{R}^{2N}$, $g(n) > 0$, and $h(n+T) = h(n) \in \mathbb{R}^{2N}$ for all $n \in \mathbb{Z}$.

It is easy to see that $H(n,z)/|z|^2 \to 0$ as $|z| \to \infty$, which implies that condition (H_4) holds.

At last, we have

$$\begin{aligned}
&(\nabla H(n,z),z) - 2H(n,z) \\
&\quad = -\left(2g(n)+|z|\right)\ln\left(1+|z|^2\right) + \frac{2\left(g(n)+|z|\right)|z|^2}{1+|z|^2} - (h(n),z) \\
&\quad \le -\left(2g(n)+|z|\right)\ln\left(1+|z|^2\right) + 2\left(g(n)+|z|\right) - (h(n),z) \longrightarrow -\infty
\end{aligned} \tag{3.52}$$

as $|z| \to \infty$. So (H_5) holds.

Thus, it follows from Theorem 1.5 that (1.1) with H as defined in (3.51) possesses at least one T-periodic solution.

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Xiaoqing Deng: College of Mathematics and Econometrics, Hunan University, Changsha 410082, China; Department of Information, Hunan Business College, Changsha 410205, China
Email address: dengxq924@163.com