ON THE DIFFERENCE EQUATION $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$

E. M. ELABBASY, H. EL-METWALLY, AND E. M. ELSAYED

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We investigate some qualitative behavior of the solutions of the difference equation $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$, $n = 0,1,\ldots$, where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants.

1. Introduction

In this paper we deal with some properties of the solutions of the difference equation

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0,1,\ldots, \tag{1.1}$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants.

Recently, there has been a lot of interest in studying the global attractivity, boundedness character, and the periodic nature of nonlinear difference equations. For some results in this area, see, for example, [1–13], we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and the function f has continuous partial derivatives on I^{k+1}, where $I^{k+1} = I \times I \times \cdots \times I$ $(k+1-\text{times})$. Then, for initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0,1,\ldots, \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

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A point $\overline{x} \in I$ is called an equilibrium point of (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.3}$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.1 (stability). (i) The equilibrium point $\overline{x}$ of (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, with

$$\begin{gathered} |x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta, \\ |x_n - \overline{x}| < \epsilon \quad \forall n \geq -k. \end{gathered} \tag{1.4}$$

(ii) The equilibrium point $\overline{x}$ of (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of (1.2) and there exists $\gamma > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, with

$$\begin{gathered} |x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma, \\ \lim_{n \to \infty} x_n = \overline{x}. \end{gathered} \tag{1.5}$$

(iii) The equilibrium point $\overline{x}$ of (1.2) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$,

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{1.6}$$

(iv) The equilibrium point $\overline{x}$ of (1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of (1.2).

(v) The equilibrium point $\overline{x}$ of (1.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.7}$$

Now assume that the characteristic equation associated with (1.7) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + \cdots + p_{k-1} \lambda + p_k = 0, \tag{1.8}$$

where $p_i = \partial f(\overline{x}, \overline{x}, \ldots, \overline{x})/\partial x_{n-i}$.

Theorem 1.2 [9]. *Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \ldots$, and $k \in \{0, 1, 2, \ldots\}$. Then*

$$\sum_{i=1}^{k} |p_i| < 1 \tag{1.9}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + \cdots + p_k y_n = 0, \quad n = 0, 1, \ldots. \tag{1.10}$$

Corollary 1.3 [9]. *Assume that f is a C^1 function and let $\overline{x}$ be an equilibrium of (1.2). Then the following statements are true.*

(a) *If all roots of the polynomial equation (1.8) lie in the open unite disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of (1.2) is asymptotically stable.*

(b) *If at least one root of (1.8) has absolute value greater than one, then the equilibrium $\overline{x}$ of (1.2) is unstable.*

Remark 1.4. The condition (1.9) implies that all the roots of the polynomial equation (1.8) lie in the open unite disk $|\lambda| < 1$.

Consider the following equation:

$$x_{n+1} = f(x_n, x_{n-1}). \tag{1.11}$$

The following theorem will be useful for the proof of our main results in this paper.

Theorem 1.5 [10]. *Let $[a,b]$ be an interval of real numbers and assume that*

$$f : [a,b]^2 \longrightarrow [a,b] \tag{1.12}$$

is a continuous function satisfying the following properties.

(a) *$f(x,y)$ is nondecreasing in $x \in [a,b]$ for each $y \in [a,b]$, and is nonincreasing in $y \in [a,b]$ for each $x \in [a,b]$.*

(b) *If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system*

$$m = f(m,M), \qquad M = f(M,m), \tag{1.13}$$

then

$$m = M. \tag{1.14}$$

Then (1.11) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of (1.11) converges to $\overline{x}$.

2. Periodic solutions

In this section we study the existence of periodic solutions of (1.1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions.

Theorem 2.1. *Equation (1.1) has positive prime period-two solutions if and only if*

$$(c+d)(a+1) > 4d, \quad ac \neq d, \, c > d. \tag{2.1}$$

Proof. First suppose that there exists a prime period-two solution

$$\dots, p, q, p, q, \dots \tag{2.2}$$

of (1.1). We will prove that condition (2.1) holds.

We see from (1.1) that

$$
\begin{aligned}
p &= aq - \frac{bq}{cq - dp}, \\
q &= ap - \frac{bp}{cp - dq}.
\end{aligned} \tag{2.3}
$$

Then

$$cpq - dp^2 = acq^2 - adpq - bq, \tag{2.4}$$

$$cpq - dq^2 = acp^2 - adpq - bp. \tag{2.5}$$

Subtracting (2.5) from (2.4) gives

$$d(q^2 - p^2) = ac(q^2 - p^2) - b(q - p). \tag{2.6}$$

Since $p \neq q$, it follows that

$$p + q = \frac{b}{ac - d}. \tag{2.7}$$

Again, adding (2.4) and (2.5) yields

$$2cpq - d(p^2 + q^2) = ac(p^2 + q^2) - 2adpq - b(p + q). \tag{2.8}$$

It follows by (2.7), (2.8), and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \forall p, q \in \mathbb{R}, \tag{2.9}$$

that

$$pq = \frac{b^2 d}{(ac - d)^2 (c + d)(a + 1)}. \tag{2.10}$$

Now it is clear from (2.7) and (2.10) that p and q are the two positive distinct roots of the quadratic equation

$$(ac - d)t^2 - bt + \frac{b^2 d}{(ac - d)(c + d)(a + 1)} = 0 \tag{2.11}$$

and so

$$b^2 > \frac{4b^2 d}{(c + d)(a + 1)}. \tag{2.12}$$

Therefore, inequality (2.1) holds.

Second, suppose that inequality (2.1) is true. We will show that (1.1) has a prime period-two solution.

Assume that

$$p = \frac{b+\alpha}{2(ac-d)}, \qquad q = \frac{b-\alpha}{2(ac-d)}, \tag{2.13}$$

where $\alpha = \sqrt{b^2 - 4b^2 d/((c+d)(a+1))}$.

From inequality (2.1) it follows that α is a real positive number, therefore, p and q are distinct positive real numbers.

Set

$$x_{-1} = p, \qquad x_0 = q. \tag{2.14}$$

We show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$.

It follows from (1.1) that

$$\begin{aligned} x_1 &= aq - \frac{bq}{cq-dp} = \frac{acq^2 - adpq - bq}{cq-dp} \\ &= \frac{ac\left[(b-\alpha)/(2(ac-d))\right]^2 - ad\left[b^2 d/((ac-d)^2(c+d)(a+1))\right] - b\left[(b-\alpha)/(2(ac-d))\right]}{c\left[(b-\alpha)/(2(ac-d))\right] - d\left[(b+\alpha)/(2(ac-d))\right]}. \end{aligned} \tag{2.15}$$

Multiplying the denominator and numerator by $4(ac-d)^2$ gives

$$x_1 = \frac{2b^2 d - (4ab^2 cd + 4ab^2 d^2)/((c+d)(a+1)) - 2bd\alpha}{2(ac-d)\{cb - bd - (c+d)\alpha\}}. \tag{2.16}$$

Multiplying the denominator and numerator by $\{cb - bd + (c+d)\alpha\}\{(c+d)(a+1)\}$ we get

$$x_1 = \frac{\left[(4b^3 d^3 + 4b^3 cd^2 - 4ab^3 c^2 d - 4ab^3 cd^2) + (4b^2 cd^2 + 4b^2 d^3 - 4ab^2 c^2 d - 4ab^2 cd^2)\alpha\right]}{2(ac-d)\{4b^2 cd^2 + 4b^2 d^3 - 4ab^2 c^2 d - 4ab^2 cd^2\}}. \tag{2.17}$$

Dividing the denominator and numerator by $\{4b^2 cd^2 + 4b^2 d^3 - 4ab^2 c^2 d - 4ab^2 cd^2\}$ gives

$$x_1 = \frac{b+\alpha}{2(ac-d)} = p. \tag{2.18}$$

Similarly as before one can easily show that

$$x_2 = q. \tag{2.19}$$

Then it follows by induction that

$$x_{2n} = q, \qquad x_{2n+1} = p \quad \forall n \geq -1. \tag{2.20}$$

Thus (1.1) has the positive prime period two solution

$$\ldots, p, q, p, q, \ldots, \tag{2.21}$$

where p and q are the distinct roots of the quadratic equation (2.11) and the proof is complete. □

3. Local stability of the equilibrium point

In this section we study the local stability character of the solutions of (1.1).

The equilibrium points of (1.1) are given by the relation

$$\overline{x} = a\overline{x} - \frac{b\overline{x}}{c\overline{x} - d\overline{x}}. \tag{3.1}$$

If $(c-d)(a-1) > 0$, then the only positive equilibrium point of (1.1) is given by

$$\overline{x} = \frac{b}{(c-d)(a-1)}. \tag{3.2}$$

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = au - \frac{bu}{cu - dv}. \tag{3.3}$$

Therefore,

$$\begin{aligned} \frac{\partial f(u,v)}{\partial u} &= a + \frac{bdv}{(cu-dv)^2}, \\ \frac{\partial f(u,v)}{\partial v} &= -\frac{bdu}{(cu-dv)^2}. \end{aligned} \tag{3.4}$$

Then we see that

$$\begin{aligned} \frac{\partial f(\overline{x},\overline{x})}{\partial u} &= a + \frac{d(a-1)}{(c-d)} = p_0, \\ \frac{\partial f(\overline{x},\overline{x})}{\partial v} &= -\frac{d(a-1)}{(c-d)} = p_1. \end{aligned} \tag{3.5}$$

Then the linearized equation of (1.1) about $\overline{x}$ is

$$y_{n+1} - p_0 y_{n-1} - p_1 y_n = 0. \tag{3.6}$$

Theorem 3.1. *Assume that*

$$|ac - d| + |ad - d| < |c - d|. \tag{3.7}$$

Then the equilibrium point of (1.1) is locally asymptotically stable.

Proof. Suppose that

$$|ac-d|+|ad-d|<|c-d|, \tag{3.8}$$

then

$$\left|a+\frac{d(a-1)}{(c-d)}\right|+\left|-\frac{d(a-1)}{(c-d)}\right|<1. \tag{3.9}$$

Thus

$$|p_1|+|p_0|<1. \tag{3.10}$$

It is followed by Theorem 1.2 that (3.6) is asymptotically stable. The proof is complete. □

4. Global attractor of the equilibrium point of (1.1)

In this section we investigate the global attractivety character of solutions of (1.1).

Theorem 4.1. *The equilibrium point $\overline{x}$ of (1.1) is a global attractor if $c \neq d$.*

Proof. We can easily see that the function $f(u,v)$ which is defined by (3.3) is increasing in u and decreasing in v.

Suppose that (m,M) is a solution of the system

$$m=f(m,M), \qquad M=f(M,m). \tag{4.1}$$

Then it results

$$\frac{1}{cm-dM}=\frac{1}{cM-dm}, \tag{4.2}$$

that is, $M=m$. It follows by Theorem 1.5 that $\overline{x}$ is a global attractor of (1.1) and then the proof is complete. □

5. Special case of (1.1)

In this section we study the following special case of (1.1):

$$x_{n+1}=x_n-\frac{x_n}{x_n-x_{n-1}}, \tag{5.1}$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers with $x_{-1}, x_0 \in \mathbb{R}/\{0\}$, and $x_{-1} \neq x_0$.

5.1. The solution form of (5.1). In this section we give a specific form of the solutions of (5.1).

THEOREM 5.1. *Let $\{x_n\}_{n=-1}^{\infty}$ be the solution of (5.1) satisfying $x_{-1} = k$, $x_0 = h$ with $k \neq h$, $k, h \in \mathbb{R}/\{0\}$. Then for $n = 0, 1, \ldots,$*

$$
\begin{aligned}
x_{2n-1} &= k + n\left(h - k - (n-1) - \frac{h}{h-k}\right), \\
x_{2n} &= h + n\left(h - k - n - \frac{h}{h-k}\right).
\end{aligned}
\tag{5.2}
$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$
\begin{aligned}
x_{2n-3} &= k + (n-1)\left(h - k - (n-2) - \frac{h}{h-k}\right), \\
x_{2n-2} &= h + (n-1)\left(h - k - (n-1) - \frac{h}{h-k}\right).
\end{aligned}
\tag{5.3}
$$

Now, it follows from (5.1) that

$$
\begin{aligned}
x_{2n-1} &= x_{2n-2} - \frac{x_{2n-2}}{x_{2n-2} - x_{2n-3}} \\
&= h + (n-1)\left(h - k - (n-1) - \frac{h}{h-k}\right) \\
&\quad - \frac{h+(n-1)(h-k-(n-1)-h/(h-k))}{(h+(n-1)(h-k-(n-1)-h/(h-k))) - (k+(n-1)(h-k-(n-2)-h/(h-k)))} \\
&= h + (n-1)\left(h - k - (n-1) - \frac{h}{h-k}\right) - \frac{h+(n-1)(h-k-(n-1)-h/(h-k))}{h-k-(n-1)}.
\end{aligned}
\tag{5.4}
$$

Multiplying the denominator and numerator by $(h-k)$ we get

$$
\begin{aligned}
x_{2n-1} &= k + (n-1)\left(h - k - (n-1) - \frac{h}{h-k}\right) - \frac{(h+(n-1)(h-k))}{(h-k)} + (h-k) \\
&= k + (n-1)\left(h - k - (n-1) - \frac{h}{h-k}\right) + (h-k) - (n-1) - \frac{h}{(h-k)},
\end{aligned}
\tag{5.5}
$$

then we have

$$
x_{2n-1} = k + n\left(h - k - (n-1) - \frac{h}{h-k}\right).
\tag{5.6}
$$

Also, we get from (5.1)

$$\begin{aligned} x_{2n} &= x_{2n-1} - \frac{x_{2n-1}}{x_{2n-1} - x_{2n-2}} \\ &= k + n\left(h - k - (n-1) - \frac{h}{h-k}\right) + \frac{k + n(h - k - (n-1) - h/(h-k))}{(n-1) + h/(h-k)}. \end{aligned} \tag{5.7}$$

Multiplying the denominator and numerator by $(h-k)$ we get

$$\begin{aligned} x_{2n} = k &+ n\left(h - k - (n-1) - \frac{h}{h-k}\right) \\ &+ \frac{(k(h-k) + n(h-k)^2) - (n(n-1)(h-k) + nh)}{(n-1)(h-k) + h}. \end{aligned} \tag{5.8}$$

Thus we obtain

$$x_{2n} = h + n\left(h - k - n - \frac{h}{h-k}\right). \tag{5.9}$$

Hence, the proof is complete. □

Remark 5.2. It is easy to see that every solution of (5.1) is unbounded.

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References

[1] E. Camouzis, R. DeVault, and G. Papaschinopoulos, *On the recursive sequence*, Advances in Difference Equations **2005** (2005), no. 1, 31–40.
[2] C. Çinar, *On the difference equation* $x_{n+1} = x_{n-1}/-1 + x_n x_{n-1}$, Applied Mathematics and Computation **158** (2004), no. 3, 813–816.
[3] ———, *On the positive solutions of the difference equation* $x_{n+1} = ax_{n-1}/1 + bx_n x_{n-1}$, Applied Mathematics and Computation **156** (2004), no. 2, 587–590.
[4] ———, *On the positive solutions of the difference equation* $x_{n+1} = x_{n-1}/1 + x_n x_{n-1}$, Applied Mathematics and Computation **150** (2004), no. 1, 21–24.
[5] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, *On the periodic nature of some max-type difference equations*, International Journal of Mathematics and Mathematical Sciences **2005** (2005), no. 14, 2227–2239.
[6] ———, *On the Difference Equation* $x_{n+1} = \alpha x_{n-k}/\beta + \gamma \prod_{i=0}^{k} x_{n-i}$, to appear in Journal of Concrete and Applicable Mathematics.
[7] H. El-Metwally, E. A. Grove, and G. Ladas, *A global convergence result with applications to periodic solutions*, Journal of Mathematical Analysis and Applications **245** (2000), no. 1, 161–170.
[8] H. El-Metwally, E. A. Grove, G. Ladas, and H. D. Voulov, *On the global attractivity and the periodic character of some difference equations*, Journal of Difference Equations and Applications **7** (2001), no. 6, 837–850.
[9] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Mathematics and Its Applications, vol. 256, Kluwer Academic, Dordrecht, 1993.
[10] M. R. S. Kulenovic and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC, Florida, 2001.

[11] M. Migda, A. Musielak, and E. Schmeidel, *On a class of fourth-order nonlinear difference equations*, Advances in Difference Equations **2004** (2004), no. 1, 23–36.
[12] Ch. G. Philos and I. K. Purnaras, *An asymptotic result for some delay difference equations with continuous variable*, Advances in Difference Equations **2004** (2004), no. 1, 1–10.
[13] T. Sun, H. Xi, and L. Hong, *On the system of rational difference equations $x_{n+1} = f(x_n, y_{n-k})$, $y_{n+1} = f(y_n, x_{n-k})$*, Advances in Difference Equations **2006** (2006), Article ID 16949, 7 pages.

E. M. Elabbasy: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail address: emelabbasy@mans.edu.eg

H. El-Metwally: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail address: helmetwally@mans.edu.eg

E. M. Elsayed: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail address: emelsayed@mans.edu.eg