USING SUPERMODELS IN QUANTUM OPTICS

NICOLE GARBERS AND ANDREAS RUFFING

Received 27 January 2006; Revised 11 April 2006; Accepted 12 April 2006

Starting from supersymmetric quantum mechanics and related supermodels within Schrödinger theory, we review the meaning of self-similar superpotentials which exhibit the spectrum of a geometric series. We construct special types of discretizations of the Schrödinger equation on time scales with particular symmetries. This discretization leads to the same type of point spectrum for the referred Schrödinger difference operator than in the self-similar superpotential case, hence exploiting an isospectrality situation. A discussion is opened on the question of how the considered energy sequence and its generalizations serve the understanding of coherent states in quantum optics.

Copyright © 2006 N. Garbers and A. Ruffing. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Items like "coherent states" or "squeezed states" can nowadays be found in many recent articles on *quantum optics*. The fact that the Nobel Prize in Physics 2005 has been awarded to pioneers on this area, like R. Glauber, gives insight how active this area is.

The kind of physical states behind coherent states or squeezed states are the so-called nonclassical states. They are minimal uncertainty states. These properties are essential for an efficient signal transmission in the quantum world. The theory of coherent states in physics has been developed all over the last decades, among others by Glauber, Klauder, and Sudarshan.

Coherent states play a major role in *laser physics*. The mathematical modeling in laser physics allows three different approaches to coherent states: first by the method of translation operators, second by the method of ladder operators, and third by the method of minimal uncertainty. Nonclassical states like *squeezed laser fields* are very important for applications: the experimental methods when dealing with squeezed laser fields include for instance the so-called *self-homodyne tomography*. The mathematical modeling in self-homodyne tomography allows a tomographical reconstruction of the Wigner function, belonging to a set of probability densities of fluctuations in different field amplitudes.

Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2006, Article ID 72768, Pages 1–14
DOI 10.1155/ADE/2006/72768

Squeezed laser states are—like coherent states—states of minimal uncertainty. From the viewpoint of statistics, semiconductor lasers have a *super-Poisson distribution* up to the *pumping level*, that is, a distribution which goes beyond the non-normalized Poisson distribution. But also the so-called *sub-Poisson distribution* has a particular meaning in *multiboson systems*: the definition of coherent states is directly related to solutions of Stieltjes moment problems. In [4], Penson and Solomon could show that q-discretizations of orthogonality measures, solving the moment problems, allow to investigate multiboson coherent states which could not so far be understood in the conventional quantum setting. The nature of these states is sub-Poissonian. Regarding the statistical properties of different physical quantum states in quantum optics, one also refers to *super-Poisson states* and *sub-Poisson states*.

Apart from the development in quantum optics, there are also great achievements visible in mathematical physics which support the theoretical understanding of the described phenomena. On the one hand, the mathematical frame for coherent states is steadily developed throughout analysis, on the other hand one can see the development that *supersymmetric quantum mechanics* and *discrete Schrödinger theory* become related to essential problems in quantum optics.

In this article, we are going to exploit the connections between *self-similar supermodels* within supersymmetric quantum mechanics, so-called *basic versions* of coherent states, and their relations to *discretized Schrödinger equations on time scales*. In Sections 2 and 3, we review some fundamental facts on supersymmetric Schrödinger operators where we are going to exploit a generalized *supermodel definition* at the end of Section 4. We are going to represent generalized supermodels through their creation and annihilation operators and characterize them as solutions to a new type of discretization for Schrödinger operators in Section 4. Finally, in Section 5, we are going to establish the connection of the obtained results and representations to basic items of coherent state theory in quantum optics.

2. Schrödinger equations and superpotentials

Supersymmetry is one of the most powerful tools being applied to problems of theoretical physics. In the last years, there were great achievements especially on the area of supersymmetry in quantum mechanics. For an excellent contribution to the topic see for instance the articles by Robnik [7] and Robnik and Liu [8]. The stationary one-dimensional version of Schrödinger's equation (λ being a fixed value in $\mathbb{R}$) is

$$-\psi''(x)+V(x)\psi(x)=\lambda\psi(x), \quad x\in\mathbb{R}. \tag{2.1}$$

It can with some general success be factorized by using the concept of so-called superpotentials.

In Schrödinger theory, the following scenario is of particular interest. Given two Schrödinger equations with different potentials V_1 and V_2. Under some circumstances, it is possible to write them in the form

$$B^+B\varphi=\lambda\varphi, \qquad BB^+\psi=\mu\psi, \tag{2.2}$$

where B and B^+ are formally adjoint to each other, being defined on some common domain in $\mathscr{L}^2(\mathbb{R})$. Let us shortly review the method of how to address the stated factorization problem.

The first step is the construction of a so-called *superpotential* W such that one can express the *partner potentials* V_1, V_2 as follows:

$$V_1 = \frac{1}{2}(W^2 - \sqrt{2}W'), \qquad V_2 = \frac{1}{2}(W^2 + \sqrt{2}W'). \tag{2.3}$$

The superpotential W is fixed by assuming that the potential V_1 allows 0 as an eigenvalue, the corresponding eigenfunction φ being positive. This leads to the condition

$$-\varphi''(x) + \frac{1}{2}(W^2(x) - \sqrt{2}W'(x))\varphi(x) = 0, \quad x \in \mathbb{R}. \tag{2.4}$$

A solution to this equation is given by

$$W(x) = -\sqrt{2}(\ln\varphi)'(x), \quad x \in \mathbb{R}. \tag{2.5}$$

The aimed factorization is now achieved by the equalities

$$H_1 = B^+B, \qquad H_2 = BB^+, \tag{2.6}$$

where the differential operators H_1, H_2 are specified by the *supersymmetric ladder operators*

$$B := \frac{1}{\sqrt{2}}\left(W + \sqrt{2}\frac{d}{dx}\right), \qquad B^+ := \frac{1}{\sqrt{2}}\left(W - \sqrt{2}\frac{d}{dx}\right). \tag{2.7}$$

To illustrate this formalism, let us consider the two potentials

$$V_1(x) = \frac{x^2}{4} - \frac{1}{2}, \quad V_2(x) = \frac{x^2}{4} + \frac{1}{2}, \quad x \in \mathbb{R}. \tag{2.8}$$

As for the superpotential W, we obtain just $W(x) = \sqrt{2}x$, leading to the well-understood conventional ladder operator formalism

$$\begin{gathered} H_1 = B^+B, \qquad H_2 = BB^+, \\ B = \frac{1}{\sqrt{2}}\left(x + \sqrt{2}\frac{d}{dx}\right), \qquad B^+ = \frac{1}{\sqrt{2}}\left(x - \sqrt{2}\frac{d}{dx}\right). \end{gathered} \tag{2.9}$$

The key message is now that one can determine the point spectrum of H_1, H_2 completely by using the operators B, B^+. Further, more illustrative example is the so-called Rosen-Morse potential, in which a real parameter y occurs:

$$V_1(x,y) = y^2 - \frac{y(y+1)}{\cosh^2(x)}, \quad x \in \mathbb{R}. \tag{2.10}$$

Assuming that the equation

$$-\varphi''(x) + V_1(x,y)\varphi(x) = 0 \tag{2.11}$$

has a solution $\varphi \in \mathcal{L}^2(\mathbb{R})$, we are first led to the superpotential

$$W(x) = \sqrt{2}y \tanh(x), \quad x \in \mathbb{R}, \tag{2.12}$$

as well as to the potential

$$V_2(x, y) = y^2 - \frac{y(y-1)}{\cosh^2(x)}, \quad x \in \mathbb{R}. \tag{2.13}$$

The difference of the two partner potentials is given by

$$V_2(x, y) - V_1(x, y-1) = \frac{2y-1}{2}, \quad x \in \mathbb{R}. \tag{2.14}$$

Generalizing the observations made so far, the two potentials V_1, V_2 are called *form-invariant* if the following identity holds for different values of x, y_1, y_2, the expression $R(y_1)$ being a continuous function of y_1:

$$V_2(x, y_1) = V_1(x, y_2) + R(y_1). \tag{2.15}$$

According to the above construction, the pair B, B^+ specifies two different Schrödinger equations, which together are referred to as a *supersymmetric model* or just as a *supermodel.*

3. Self-similar supermodels

In many important applications, it follows from the defining equation for form invariance,

$$V_2(x, y_1) = V_1(x, y_2) + R(y_1), \tag{3.1}$$

that $y_2 = y_1 + h$, where h is a real constant. A completely new class of form-invariant potentials has been proposed in [3], where potentials were constructed whose parameters are related to each other by

$$y_2 = q y_1, \quad 0 < q < 1. \tag{3.2}$$

In order to make apparent what kind of possible point spectrum is generated by the property (3.2), we follow the basic outline in [2], where the superpotential is expanded as follows:

$$W(x, y) = \sum_{j=0}^{\infty} g_j(x) y^j \tag{3.3}$$

for some suitable parameter, $y \in \mathbb{R}$. The function R from (3.3) is assumed to be given by an analytic ansatz

$$R(y) = \sum_{j=0}^{\infty} R_j y^j. \tag{3.4}$$

Plugging this ansatz into formula (3.1) yields

$$\begin{aligned} R(y) &= V_2(x,y) - V_1(x,qy) \\ &= W^2(x,y) - W^2(x,qy) + \sqrt{2}\big((\partial_x W)(x,y) + (\partial_x W)(x,qy)\big). \end{aligned} \tag{3.5}$$

Inserting now the expansion (3.4) for the function R, one obtains, by comparing the coefficients,

$$R_n = \frac{1}{2}\sum_{i=1}^{\infty}(1+q^{n-i})(1-q^i)g_i g_{n-i} + \sqrt{2}(1+q^n)g_n', \quad n \in \mathbb{N}, \tag{3.6}$$

and the value R_0 being given by $R_0 = g_0'$. With the abbreviations

$$r_n := \frac{R_n}{1-q^n}, \quad d_n := \frac{1-q^n}{1+q^n}, \quad n \in \mathbb{N}, \tag{3.7}$$

one is led to nonlinear integral equations, given by

$$g_n(x) = \frac{d_n}{\sqrt{2}}\int_a^x \left(2r_n - \sum_{i=1}^{n-1} g_i(t)g_{n-i}(t)\right)dt, \quad x \in \mathbb{R},\ n \in \mathbb{N}, \tag{3.8}$$

where restrictions of the solutions of these equations are put by the conditions

$$R_0 = 0, \qquad g_0(x) = 0, \qquad r_n = z\delta_{n1}, \quad n \in \mathbb{N}, \tag{3.9}$$

δ_{n1} denoting the Kronecker symbol and z being a positive parameter. This nonlinear integral equation allows now the solutions

$$R(y) = R_1 y = R, \qquad g_n(x) = \frac{1}{\sqrt{2}}\beta_n x^{2n-1}, \quad x \in \mathbb{R},\ n \in \mathbb{N}, \tag{3.10}$$

where the coefficients β_n are fixed by the recurrence formula

$$\beta_1 = \frac{2R_1}{1+q}, \qquad \beta_0 = 0, \qquad \beta_n = -\frac{d_n}{2n-1}\sum_{i=1}^{n-1}\beta_i\beta_{n-i}, \quad n \in \mathbb{N}\setminus\{1\}. \tag{3.11}$$

The superpotential now reads

$$W(x,y) = \sum_{j=1}^{\infty}\beta_j y^j\left(\frac{x}{\sqrt{2}}\right)^{2j-1}, \quad x \in \mathbb{R}. \tag{3.12}$$

The formal ground-state, belonging to V_1, is given by the formula

$$\psi_0(x,y) = Ce^{-\sum_{j=1}^{\infty}(\beta_j/2j)y^j(x/\sqrt{2})^{2j}}, \quad x \in \mathbb{R}. \tag{3.13}$$

Direct calculation leads to the formula

$$W(x,y_2) = \sqrt{q}\,W(\sqrt{q}x, y_1), \quad x \in \mathbb{R}, \tag{3.14}$$

showing some self-similar property. This type of superpotential is therefore also referred to by the name *self-similar superpotential*. Applying a generalized version of the ladder operator formalism, one is led to the energies of the operator H_1, being given by

$$\lambda_n = R\sum_{j=0}^{n-1} q^j = R\frac{1-q^n}{1-q}, \quad n \in \mathbb{N}_0. \tag{3.15}$$

Let us now arrive at an interesting isospectrality scenario.

4. Isospectral supermodels and strip discretizations

We now address the general question of how to construct discrete Schrödinger operators, that is, Schrödinger difference operators whose wave functions are defined on a nonempty closed set $\Omega \subset \mathbb{R}$ with Lebesgue measure $\mu(\Omega) > 0$, leading to the same point spectrum (3.15).

We address this question in context of Schrödinger q-difference equations, where we study piecewise continuous solutions to these equations, having support on some kind of strip structures which are generated by the symmetries of the lattice $\{+q^n, -q^n \mid n \in \mathbb{Z}\}$. This approach seems to fit naturally to the given point spectrum (3.15) and might be of importance for applications and numerical investigations of the underlying eigenvalue and spectral problems.

Let us now elucidate in some detail the philosophy of using the framework of q-difference operators for discretizing the Schrödinger equation. As indicated, we restrict our investigations to subsets of the real axis which we will call homogeneous q-strip discretizations or just strip discretizations. To do so, we have to provide the tools that help us in formulating the special boundary conditions.

Let us refer throughout the sequel to a parameter $0 < q < 1$, as it was motivated by the investigation of self-similar superpotentials in the previous section.

Definition 4.1 (strip discretization). Let $\Omega \subseteq \mathbb{R} \setminus \{0\}$ be a nonempty closed set with Lebesgue measure $\mu(\Omega) > 0$ as well as

$$\forall x \in \Omega, \quad qx \in \Omega, \quad q^{-1}x \in \Omega, \quad -x \in \Omega. \tag{4.1}$$

We call the time scale Ω a homogeneous strip discretization or just *strip discretization* of the configuration space. The Hilbert space of the strip discretization is introduced by the requirement

$$\mathcal{L}^2(\Omega) := \{f \in \mathcal{L}^2(\mathbb{R}) \mid f = f \circ \chi_\Omega\}, \tag{4.2}$$

and the scalar product of two functions $f, g \in \mathcal{L}^2(\Omega)$ is introduced by

$$(f,g)_\Omega := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\chi_\Omega(x)\,dx = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx, \tag{4.3}$$

using the *characteristic function* χ_Ω of the time scale Ω. By construction, it is clear that $\mathcal{L}^2(\Omega)$ is a Hilbert space over $\mathbb{C}$, being a proper subspace of the square-integrable functions themselves, that is, of $\mathcal{L}^2(\mathbb{R})$. In order to proceed, let us first review some facts on the Schrödinger equation with quadratic potential, given by

$$-\psi''(x)+x^2\psi(x)=\lambda\psi(x),\quad x\in\mathbb{R}. \tag{4.4}$$

The following structure is one the most familiar facts within mathematical physics: let the sequence of functions $(\psi_n)_{n\in\mathbb{N}_0}$ be recursively given by the requirement

$$\psi_{n+1}(x):=-\psi_n'(x)+x\psi_n(x),\quad x\in\mathbb{R},\ n\in\mathbb{N}_0, \tag{4.5}$$

where $\psi_0:\mathbb{R}\to\mathbb{R}$, $x\mapsto\psi_0(x):=e^{-(1/2)x^2}$. We then have $\psi_n\in\mathscr{L}^2(\mathbb{R})\cap C^2(\mathbb{R})$ for $n\in\mathbb{N}_0$ and moreover

$$-\psi_n''(x)+x^2\psi_n(x)=(2n+1)\psi_n(x),\quad x\in\mathbb{R},\ n\in\mathbb{N}_0. \tag{4.6}$$

This result reflects the conventional *ladder operator formalism*. We now develop a result in discrete Schrödinger theory on strip structures which turns out to be a q-analog of the just described continuous situation. Let us therefore state in a next step some more useful tools for the strip discretization approach.

Definition 4.2. Let $\Omega\subseteq\mathbb{R}\setminus\{0\}$ be a nonempty closed set with the property $\mu(\Omega)>0$ as well as

$$\forall x\in\Omega,\quad qx\in\Omega,\quad q^{-1}x\in\Omega,\quad -x\in\Omega. \tag{4.7}$$

Let for any $f:\Omega\to\mathbb{R}$ the right-shift, respectively, left-shift operations be defined by

$$(Rf)(x):=f(qx),\quad (Lf)(x):=f(q^{-1}x),\quad x\in\Omega, \tag{4.8}$$

respectively. The right-hand, respectively, left-hand q-difference operations will for any $f:\Omega\to\mathbb{R}$ be given by

$$(D_qf)(x):=\frac{f(qx)-f(x)}{qx-x},\quad (D_{q^{-1}}f)(x):=\frac{f(q^{-1}x)-f(x)}{q^{-1}x-x},\quad x\in\Omega. \tag{4.9}$$

Let moreover $\alpha>0$ and let

$$g:\Omega\longrightarrow\mathbb{R}^+,\quad x\longmapsto g(x):=\frac{\sqrt{\varphi(qx)}-\sqrt{\varphi(x)}}{\sqrt{\varphi(x)}(q-1)x}=\frac{\sqrt{1+\alpha(1-q)x^2}-1}{qx-x}, \tag{4.10}$$

where the positive even continuous function $\varphi:\Omega\to\mathbb{R}^+$ is chosen as a solution to the q-difference equation

$$\varphi(qx)=\left(1+\alpha(1-q)x^2\right)\varphi(x),\quad x\in\Omega. \tag{4.11}$$

We are now able to define *discrete ladder operators* on strip structures.

The *creation operation* $A_q^\dagger$ and, respectively, *annihilation operation* A_q are introduced by their actions on any $\psi : \Omega \to \mathbb{R}$ as follows:

$$A_q^\dagger \psi = \big(- D_q + g(X)R\big)\psi, \qquad A_q\psi = q^{-1}\big(LD_q + Lg(X)\big)\psi. \tag{4.12}$$

We refer to the *discrete Schrödinger equation* with an oscillator potential on Ω by

$$q^{-1}\big(- D_q + g(X)R\big)\big(LD_q + Lg(X)\big)\psi = \lambda\psi. \tag{4.13}$$

The following result reveals that the discrete Schrödinger equation with an oscillator potential on Ω shows similar properties than its classical analog does.

Theorem 4.3. *Let the time scale Ω be a strip discretization in the sense of Definition 4.1 and let the function φ be specified like in Definition 4.2, satisfying the q-difference equation (4.11) on Ω,*

$$\varphi(qx) = \big(1 + \alpha(1 - q)x^2\big)\varphi(x), \quad \varphi(x) = \varphi(-x) > 0, \quad x \in \Omega. \tag{4.14}$$

For $n \in \mathbb{N}_0$, the functions $\psi_n : \Omega \to \mathbb{R}$, given by $\psi_n(x) := ((A_q^\dagger)^n\sqrt{\varphi})(x)$, $x \in \Omega$, are well defined in $\mathcal{L}^2(\Omega)$ and solve the q-Schrödinger equation (4.13) in the following sense:

$$q^{-1}\big(- D_q + g(X)R\big)\big(LD_q + Lg(X)\big)\psi_n = \frac{\alpha}{q}\frac{q^n - 1}{q - 1}\psi_n. \tag{4.15}$$

Moreover, the linear maps A_q, $A_q^\dagger$ act as ladder operators on the functions $(\psi_n)_{n\in\mathbb{N}_0}$ in the following sense ($n \in \mathbb{N}_0$, $\psi_{-1} := 0$):

$$A_q^\dagger\psi_n = \psi_{n+1}, \qquad A_q\psi_n = \frac{\alpha}{q}\frac{q^n - 1}{q - 1}\psi_{n-1}, \qquad \psi_n(x) = H_n^q(x)\psi_0(x), \quad x \in \Omega, \tag{4.16}$$

where for $n \in \mathbb{N}_0$, the functions $H_n^q : \Omega \to \mathbb{R}$ are given by

$$H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha\frac{q^n - 1}{q - 1}H_{n-1}^q(x) = 0, \qquad H_0^q(x) = 1, \qquad H_1^q(x) = \alpha x. \tag{4.17}$$

These recurrence relations apply for $x \in \Omega$ and $n \in \mathbb{N}_0$, where $\psi_{-1} := 0$, $H_{-1}^q := 0$ is set. There exists the general observation

$$\big(A_q^\dagger\psi_m, \psi_n\big)_\Omega = \big(\psi_m, A_q\psi_n\big)_\Omega, \quad m,n \in \mathbb{N}_0, \tag{4.18}$$

and the functions $(\psi_n)_{n\in\mathbb{N}_0}$ constitute an orthonormal system in $\mathcal{L}^2(\Omega)$.

Proof. Let us for $\varphi \in C(\mathbb{R})$ first consider the equation

$$\varphi(qx)x^n = \big(1 + \alpha(1 - q)x^2\big)\varphi(x)x^n, \quad x \in \Omega,\ n \in \mathbb{N}_0, \tag{4.19}$$

which directly follows from (4.11). Using standard arguments, one can show that the function φ fulfiling (4.11) is in $\mathscr{L}^1(\mathbb{R})$. This implies

$$\int_{-\infty}^{\infty} \varphi(qx)x^n\chi_\Omega(x)dx = \int_{-\infty}^{\infty} (1+\alpha(1-q)x^2)\varphi(x)x^n\chi_\Omega(x)dx, \quad n \in \mathbb{N}_0. \tag{4.20}$$

Using the substitution rule to the left-hand side, this directly implies

$$\int_{-\infty}^{\infty} \varphi(t)t^n q^{-n}\chi_\Omega(q^{-1}t)q^{-1}dt = \int_{-\infty}^{\infty} (1+\alpha(1-q)x^2)\varphi(x)x^n\chi_\Omega(x)dx, \quad n \in \mathbb{N}_0. \tag{4.21}$$

Because of (4.7) we have $\chi_\Omega(q^{-1}t) = \chi_\Omega(t)$ for any $t \in \mathbb{R}$ and, therefore, (4.21) is equivalent to

$$\int_{-\infty}^{\infty} \varphi(t)t^n q^{-n}\chi_\Omega(t)q^{-1}dt = \int_{-\infty}^{\infty} (1+\alpha(1-q)x^2)\varphi(x)x^n\chi_\Omega(x)dx, \quad n \in \mathbb{N}_0. \tag{4.22}$$

Using the abbreviation $\mu_n(\Omega) := \int_\Omega x^n\varphi(x)dx$ for $n \in \mathbb{N}_0$ we obtain the following result:

$$\mu_{2n+2}(\Omega) = \frac{q^{-2n-1}-1}{\alpha(1-q)}\mu_{2n}(\Omega), \quad \mu_{2n+1}(\Omega) = 0, \quad n \in \mathbb{N}_0. \tag{4.23}$$

We have shown earlier [1] that any probability density ψ which generates moments of type (4.23) yields an orthogonality measure to the polynomials $(H_n^q)_{n\in\mathbb{N}_0}$ which are for $k \in \mathbb{N}$ fixed through the recurrence relation

$$H_{k+1}^q(x) - \alpha q^k x H_k^q(x) + \alpha\frac{q^k-1}{q-1}H_{k-1}^q(x) = 0, \qquad H_0^q(x) = 1, \qquad H_1^q(x) = \alpha x, \tag{4.24}$$

the variable x being chosen in a suitable integration support. As a consequence of this general result, we obtain the following orthogonality relation:

$$\int_{-\infty}^{\infty} H_m^q(x)H_n^q(x)\varphi(x)\chi_\Omega(x)dx = \nu_n(\Omega)\delta_{mn}, \quad m,n \in \mathbb{N}_0, \tag{4.25}$$

with a sequence of positive numbers $(\nu_n(\Omega))_{n\in\mathbb{N}_0}$. Direct calculations and induction show

$$\psi_n(x) := \left((A_q^\dagger)^n\sqrt{\varphi}\circ\chi_\Omega\right)(x) = \left(H_n^q(X)\sqrt{\varphi}\circ\chi_\Omega\right)(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0. \tag{4.26}$$

Let us from now on—without any restriction—refer to the special parameter choice $\alpha = 1$. The functions $(\psi_n)_{n\in\mathbb{N}_0}$ constitute an orthonormal system in $\mathscr{L}^2(\Omega)$. Let us show next that the ladder property (4.16) is fulfiled. The first equation in (4.16) is trivial due to the definition of the functions $(\psi_n)_{n\in\mathbb{N}_0}$. We remember that the function g is specified like in Definition 4.2. We obtain in the sense of the multiplication operator notation

$$(LD_q + Lg(X))(X^n\psi_0) = LD_qX^n\psi_0 + Lg(X)X^n\psi_0, \quad n \in \mathbb{N}_0, \tag{4.27}$$

which yields

$$(LD_q + Lg(X))(X^n\psi_0) = L\left(\frac{q^n-1}{q-1}X^{n-1}R\psi_0 + X^n D_q\psi_0\right) + Lg(X)X^n\psi_0, \quad n \in \mathbb{N}_0. \tag{4.28}$$

This may be rewritten as

$$(LD_q + Lg(X))(X^n\psi_0) = \frac{q^n-1}{q-1}q^{-n+1}X^{n-1}\psi_0 + LX^n(D_q\psi_0 + g\psi_0), \quad n \in \mathbb{N}_0. \tag{4.29}$$

Using now however the formulas in (4.10) for the function g, we obtain $(D_q\psi_0 + g\psi_0) = 0$ and therefore

$$(LD_q + Lg(X))(X^n\psi_0) = \frac{q^n-1}{q-1}q^{-n+1}X^{n-1}\psi_0, \quad n \in \mathbb{N}. \tag{4.30}$$

For $m \in \mathbb{N}_0$, the first $m+1$ polynomials of the sequence $(H_n^q(X))_{n\in\mathbb{N}_0}$ can uniquely be generated by linear combinations of the first $m+1$ monomials of the sequence $(X^n)_{n\in\mathbb{N}_0}$. We therefore conclude as

$$A_q\psi_n = \sum_{j=0}^{n-1} c_j^n\psi_j, \quad n \in \mathbb{N}, \tag{4.31}$$

with uniquely defined real numbers c_j^n, where $j = 0,\dots,n-1$ with $n \in \mathbb{N}$. Applying again standard substitution techniques to the scalar product integral (4.3), we can derive for any functions $f,g \in \mathcal{L}^2(\Omega)$ which are both in the algebraic span of the functions $(\psi_n)_{n\in\mathbb{N}_0}$ the following relation:

$$(A_q^\dagger f, g)_\Omega = (f, A_q g)_\Omega. \tag{4.32}$$

In particular, this result implies

$$(A_q^\dagger\psi_m, \psi_n)_\Omega = (\psi_m, A_q\psi_n)_\Omega, \quad m,n \in \mathbb{N}_0. \tag{4.33}$$

Using the first equation in (4.16) and because of the fact that the functions $(\psi_n)_{n\in\mathbb{N}_0}$ constitute an orthogonal system in $\mathcal{L}^2(\Omega)$, the second relation in (4.16) follows from standard methods of calculating the norms of the functions $(\psi_n)_{n\in\mathbb{N}_0}$.

Equation (4.15) now follows immediately from the first two relations in (4.16). Taking all the steps of the proof together, this finally confirms the statements of Theorem 4.3.

Let us interpret the obtained results in context of quantum mechanical supermodels.

First, we have obtained the desired isospectrality result, that is, we have found a rich class of discrete Schrödinger operators showing in (4.15) the same point spectrum that we obtain from the self-similar superpotentials in (3.15). This gives us an important tool at hand to extend the definition of a *supermodel for purposes of quantum optics.*

As the discrete ladder operator formalism that we have revealed and orthogonal eigensystems for self-similar superpotentials lead to the same point spectrum, fixed by (4.15), both type of solutions may be considered as two different representations of one and the same formal orthogonal system. □

This is the reason why we introduce the following general definition for a *self-similar supermodel*.

Definition 4.4. Let for $0 < q < 1$ the bounded sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ of eigenvalues

$$\lambda_n := \frac{q^n - 1}{q - 1}, \quad n \in \mathbb{N}_0, \tag{4.34}$$

be given, and let moreover the subspace $l^2(\lambda) \subseteq l^2(\mathbb{N}_0)$ be canonically introduced as follows:

$$\left\{\psi = \sum_{n=0}^{\infty} c_n e_n \in l^2(\mathbb{N}_0) \mid \sum_{n=1}^{\infty} \lambda_n \lambda_{n-1} |c_n|^2 < \infty\right\}. \tag{4.35}$$

Let the pair of adjoint operators

$$A^\dagger : D_{\max}(A^\dagger) \subseteq l^2(\lambda) \longrightarrow l^2(\mathbb{N}_0), \qquad A : D_{\max}(A) \subseteq l^2(\lambda) \longrightarrow l^2(\mathbb{N}_0) \tag{4.36}$$

be given, being fixed by their actions on the standard orthogonal basis vectors of $l^2(\lambda) \subseteq l^2(\mathbb{N}_0)$ as follows:

$$A^\dagger e_n := \sqrt{\lambda_{n+1}} e_{n+1}, \quad A e_n := \sqrt{\lambda_n} e_n, \quad n \in \mathbb{N}_0. \tag{4.37}$$

Then the triple $(A, A^\dagger, l^2(\lambda))$ is called *self-similar supermodel* in Fock space.

As a direct consequence, we see that A and $A^\dagger$ fulfil on a common maximum domain of $l^2(\lambda) \subseteq l^2(\mathbb{N}_0)$ the commutation relation

$$AA^\dagger - qA^\dagger A = E, \tag{4.38}$$

the operator E denoting the identity map on the maximal common domain of A, $A^\dagger$.

By Definition 4.4, we give an abstract formulation of the ladder operator formalism for which the self-similar supermodels from Section 3 and the discrete Schrödinger models from Section 4 so far are two different realizations. Therefore, an eigenvalue distribution of type (4.34) has indeed a physical interpretation in Schrödinger theory. We now may ask what is the impact of this type of spectrum for coherent state theory within quantum optics. A discussion on this topic will be started by the next section.

5. Discussion of applications to coherent state theory

In the introduction, we have mentioned the role of super-Poisson states respectively, sub-Poisson states in quantum optics. A mathematical model for distinguishing between super-Poisson states and sub-Poisson states is given by the Mandel functional, see for instance [5, 6]. The characterization of Mandel's functional is associated with Jacobi operators in $l^2(\mathbb{N}_0)$. The physical states of the quantum system are modelled by the elements of $l^2(\mathbb{N}_0)$.

One first considers the $\mathbb{C}$-linear subspace $l_1 \subseteq l^2(\mathbb{N}_0)$ which is described by all elements $\psi = \sum_{n=0}^{\infty} c_n e_n \in l^2(\mathbb{N}_0)$ with the property $\sum_{n=1}^{\infty} n|c_n|^2 < \infty$. Let in this context $(e_n)_{n\in\mathbb{N}_0}$ be a standard orthonormal basis of $l^2(\mathbb{N}_0)$. Moreover, let

$$l_2 \subseteq l_1 \subseteq l^2(\mathbb{N}_0) \tag{5.1}$$

denote the particular $\mathbb{C}$-linear subspace which is defined by all the elements $\psi = \sum_{n=0}^{\infty} c_n e_n \in l^2(\mathbb{N}_0)$ with the property $\sum_{n=1}^{\infty} n(n-1)|c_n|^2 < \infty$. For an arbitrary element

$$\psi = \sum_{n=1}^{\infty} c_n e_n \in l_2, \tag{5.2}$$

the nonlinear *Mandel functional* $f : l_2 \to \mathbb{R}$ is defined by

$$f(\psi) := \frac{1}{\sum_{n=1}^{\infty} n |c_n|^2} \sum_{n=1}^{\infty} n(n-1) |c_n|^2 - \sum_{n=1}^{\infty} n |c_n|^2. \tag{5.3}$$

Mandel's functional now allows to distinguish between super-Poisson states and sub-Poisson states of the quantum system. On super-Poisson states $\psi \in l_2$, we have $f(\psi) > 0$, and on sub-Poisson states $\psi \in l_2$, we have $f(\psi) < 0$. States which are associated with the Poisson distribution itself, are characterized by a vanishing Mandel functional.

Let now $U \subseteq \mathbb{C}$ be a connected set and let

$$\Psi_U := \{\psi_z \in l^2(\mathbb{N}_0) \mid z \in U\}. \tag{5.4}$$

Ψ_U is modeling quantum states on which a continuous label z from U has been fixed.

The elements of Ψ_U are referred to by the name *coherent states* if

$$z_n \longrightarrow z \Longrightarrow ||\psi_{z_n} - \psi_z||_{l^2(\mathbb{N}_0)} \longrightarrow 0 \tag{5.5}$$

for any convergent sequence $(z_n)_{n\in\mathbb{N}}$ in U and if they allow a decomposition of the identity map in the following sense:

$$\forall \varphi \in l^2(\mathbb{N}_0) : \int_U w(|z|^2)\psi_z^*(\varphi)\psi_z dz = \varphi. \tag{5.6}$$

In this context, $w : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a suitable weight function and ψ_z^* denotes a canonically constructed dual form on $l^2(\mathbb{N}_0)$, its construction being inherited from the structure of Ψ_U. Physical states which arise in the simplest quantum systems, for instance, are modeled by

$$\psi_z = e^{-(1/2)|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} e_n, \quad z \in U = \mathbb{C}. \tag{5.7}$$

In the sense of the above-given definition, these functions indeed are coherent states. They are states which are eigenvectors of the linear map being fixed by

$$A : D_{\max}(A) \subseteq l^2(\mathbb{N}_0) \longrightarrow l^2(\mathbb{N}_0), \qquad Ae_n := \sqrt{n}e_{n-1}, \quad n \in \mathbb{N}, \tag{5.8}$$

on the maximal possible definition range for the $\mathbb{C}$-linear map A. The map A is modeling the transition from a particular energy level of the quantum system to its lower neighbor. In this way, the coherent states which are fixed by (4.26) are stable towards energy perturbations. This is one of their most important properties, being fundamental for the respective applications in quantum optics. This stability is mathematically modeled by the following eigenvalue equation:

$$A\varphi_z = z\varphi_z, \quad z \in \mathbb{C}. \tag{5.9}$$

The definition of a so-called *squeezed state* is given by the variance $v_Q(\psi)$ of a linear operator Q in $\psi \in l^2(\mathbb{N}_0)$. Let Q and Q^2 be defined on a common dense domain $\Delta \subseteq l^2(\mathbb{N}_0)$. The nonlinear variance functional $v_Q : \Delta \to \mathbb{C}$ is expressed in terms of the canonical scalar product as follows:

$$v_Q(\psi) := (\psi, Q^2\psi) - (\psi, Q\psi)^2, \quad \psi \in \Delta. \tag{5.10}$$

Let now A^* be the adjoint of the operator A in (5.8). Let moreover the following operators be given, sharing the joint definition ranges of A, respectively, A^* in $l^2(\mathbb{N}_0)$:

$$P := \frac{-i}{\sqrt{2}}(A - A^*), \qquad X := \frac{1}{\sqrt{2}}(A + A^*). \tag{5.11}$$

One has then for all elements ψ of a common definition range Δ:

$$\textit{Heisenberg uncertainty relation} \quad v_P(\psi)v_X(\psi) \geq \frac{1}{4}, \quad \psi \in \Delta. \tag{5.12}$$

In this situation, one calls

$$\psi \in \Delta \quad \textit{squeezed state:} \Longleftrightarrow v_X(\psi) < \frac{1}{2}. \tag{5.13}$$

Thus it is a state which has only a small freedom of variation in the spatial sense.

Starting from a suitable sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ of positive numbers modeling the energies of a given quantum system, on can investigate generalizations of the Mandel functional $f_\lambda : l^2(\lambda) \to \mathbb{R}$ and study their properties:

$$f_\lambda(\psi) := \frac{1}{\sum_{n=1}^{\infty} \lambda_n |c_n|^2} \sum_{n=1}^{\infty} \lambda_n\lambda_{n-1} |c_n|^2 - \sum_{n=1}^{\infty} \lambda_n |c_n|^2, \quad \psi \in l^2(\lambda). \tag{5.14}$$

The subspace $l^2(\lambda) \subseteq l^2(\mathbb{N}_0)$ is canonically given as a generalization of the situation (5.1),

$$\left\{\psi = \sum_{n=0}^{\infty} c_n e_n \in l^2(\mathbb{N}_0) \mid \sum_{n=1}^{\infty} \lambda_n\lambda_{n-1} |c_n|^2 < \infty\right\}. \tag{5.15}$$

The following question is of particular interest, given a special energy sequence $(\lambda_n)_{n\in\mathbb{N}_0}$. How often do the *super-Poisson distribution*, respectively, the *sub-Poisson distribution* occur? Discussing this question for the energy sequence of the self-similar supermodel will therefore be the detailed subject of a different article.

References

[1] K. Ey and A. Ruffing, *The moment problem of discrete q-Hermite polynomials on different time scales*, Dynamic Systems and Applications **13** (2004), no. 3-4, 409–418.
[2] H. Kalka and G. Soff, *Supersymmetrie*, Teubner, Stuttgart, 1997.
[3] A. Khare and U. P. Sukhatme, *New shape-invariant potentials in supersymmetric quantum mechanics*, Journal of Physics. A **26** (1993), no. 18, L901–L904.
[4] K. A. Penson and A. I. Solomon, *New generalized coherent states*, Journal of Mathematical Physics **40** (1999), no. 5, 2354–2363.
[5] C. Quesne, K. A. Penson, and V. M. Tkachuk, *Maths-type q-deformed coherent states for* $q > 1$, Physics Letters. A **313** (2003), no. 1-2, 29–36.
[6] ———, *Geometrical and physical properties of maths-type q-deformed coherent states for* $0 < q < 1$ *and* $q > 1$, preprint, 2003.
[7] M. Robnik, *Supersymmetric quantum mechanics based on higher excited states*, Journal of Physics. A **30** (1997), no. 4, 1287–1294.
[8] M. Robnik and J. Liu, *Supersymmetric quantum Mechanics based on higher excited states II: a few new examples of isospectral partner potentials*, preprint CAMTP, 1997.

Nicole Garbers: Fakultät für Mathematik, Technische Universität München, Boltzmannstraße 3, Garching 85747, Germany
E-mail address: ngarbers@web.de

Andreas Ruffing: Fakultät für Mathematik, Technische Universität München, Boltzmannstraße 3, Garching 85747, Germany
E-mail address: ruffing@ma.tum.de