ASYMPTOTIC STABILITY FOR DYNAMIC EQUATIONS ON TIME SCALES

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We examine the conditions of asymptotic stability of second-order linear dynamic equations on time scales. To establish asymptotic stability we prove the stability estimates by using integral representations of the solutions via asymptotic solutions, error estimates, and calculus on time scales.

1. Main result

In this paper, we examine asymptotic stability of second-order dynamic equation on a time scale $\mathbb{T}$,

$$L[y(t)] = y^{\nabla\nabla} + p(t)y^{\nabla}(t) + q(t)y(t) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where y^{∇} is nabla derivative (see [4]).

Exponential decay and stability of solutions of dynamic equations on time scales were investigated in recent papers [1, 5–7, 11, 12] using Lyapunov's method. We use different approaches based on integral representations of solutions via asymptotic solutions and error estimates developed in [2, 8–10].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

For $t \in \mathbb{T}$ we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \forall t \in \mathbb{T}. \tag{1.2}$$

The backward graininess function $\nu : \mathbb{T} \to [0,\infty]$ is defined by

$$\nu(t) = t - \rho(t). \tag{1.3}$$

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If $\rho(t) < t$ or $\nu > 0$, we say that t is left scattered. If $t > \inf(\mathbb{T})$ and $\rho(t) = t$, then t is called left dense. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$ define the nabla derivative of f at t denoted $f^\nabla(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^\nabla(t)(\rho - s) \right| \le \varepsilon \left| \rho(t) - s \right| \quad \forall s \in U. \tag{1.4}$$

We assume $\sup \mathbb{T} = \infty$. For some positive $t_0 \in \mathbb{T}$ denote $\mathbb{T}_\infty \equiv \mathbb{T} \bigcap [t_0, \infty)$.

Equation (1.1) is called asymptotically stable if every solution $y(t)$ of (1.1) and its nabla derivative approach zero as t approaches infinity. That is,

$$\lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} y^\nabla(t) = 0. \tag{1.5}$$

We establish asymptotic stability of dynamic equations on time scales by using calculus on time scales [3, 4] and integral representations of solutions via asymptotic solutions [8].

A function $f :\in \mathbb{T} \to \mathbb{R}$ is called ld-continuous ($C_{\mathrm{ld}}(\mathbb{T})$) provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$.

By $L_{\mathrm{ld}}(\mathbb{T})$ we denote a class of functions $f : \mathbb{T} \to \mathbb{R}$ that are ld-continuous on $\mathbb{T}$ and Lebesgue nabla integrable on $\mathbb{T}$. $C^2_{\mathrm{ld}}(\mathbb{T})$ is the class of functions for which second nabla derivatives exist and are ld-continuous on $\mathbb{T}$.

$$\mathbb{R}^+_\nu = \left\{K : \mathbb{T} \longrightarrow \mathbb{R},\ K(t) \ge 0,\ 1 - \nu K(t) > 0,\ K \in C_{\mathrm{ld}}(\mathbb{T})\right\}. \tag{1.6}$$

We assume that $p, q \in C_{\mathrm{ld}}(\mathbb{T}_\infty)$.

From a given function $\theta \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ we construct a function

$$k(t) = \frac{\theta^\nabla(t)}{2\theta^2(t)}. \tag{1.7}$$

For $\nu > 0$ we choose $\theta_1(t)$ as a solution of the quadratic equation

$$\nu\theta_1^2 - 2\theta_1(1 + \nu\theta) + 2\theta - \frac{p + \nu q + 2k\theta}{1 - 2k\theta\nu} = 0, \tag{1.8}$$

or

$$\theta_1 = \theta + \frac{1}{\nu} + \sqrt{D}, \quad D = \theta^2 + \frac{1 + \nu p + \nu^2 q}{(1 - 2k\theta\nu)\nu^2}. \tag{1.9}$$

If $\nu = 0$, then (1.8) turns into a linear equation and $\theta_1(t)$ is defined by the formula

$$\theta_1(t) = \theta(t) - \frac{\theta'(t)}{2\theta(t)} - \frac{p(t)}{2}, \tag{1.10}$$

Note that (1.8) is a version of Abel's formula for a dynamic equation (1.1), and (1.10) is Abel's formula for the corresponding differential equation.

Define auxiliary functions

$$\theta_2(t) = \theta_1(t) - 2\theta(t), \qquad \Psi(t) = \begin{pmatrix} \hat{e}_{\theta_1}(t,t_0) & \hat{e}_{\theta_2}(t,t_0) \\ \theta_1 \hat{e}_{\theta_1}(t,t_0) & \theta_2 \hat{e}_{\theta_2}(t,t_0) \end{pmatrix}, \tag{1.11}$$

$$\mathrm{Hov}_j(t) = -q - p\theta_j - \theta_j^2 - \theta_j^{\nabla}(1 - \nu\theta_j), \quad j = 1,2, \tag{1.12}$$

$$K(s) = \left(\frac{1}{|1 - \nu\theta_1(s)|} + \frac{1}{|1 - \nu\theta_2(s)|} \right) K_1(s), \tag{1.13}$$

$$K_1(s) = \left| \frac{(\theta_1 \mathrm{Hov}_2 - \theta_2 \mathrm{Hov}_1)(s)}{4\theta(s)\theta(\rho(s))} \right|, \tag{1.14}$$

$$Q_{jk}(t) = \|1 - \nu\Psi^{-1}\Psi^{\nabla}(t)\| \left| \frac{\mathrm{Hov}_j(t)\hat{e}_{\theta_k}(t,t_0)}{\theta(t)\hat{e}_{\theta_j}(t,t_0)} \right|, \quad j,k = 1,2, \tag{1.15}$$

where $\hat{e}_\theta(t,t_0)$ is the nabla exponential function on a time scale, and $\|\cdot\|$ is the Euclidean matrix norm $\|A\| = \sqrt{\sum_{k,j=1}^n A_{kj}^2}$.

Note that θ_1 and θ_2 can be used to form approximate solutions y_1 and y_2 of (1.1) in the form $y_j(t) = \hat{e}_{\theta_j}(t,t_0)$, $j = 1,2$. Also, from the given approximate solutions y_1 and y_2 the function $\theta = (\theta_1 - \theta_2)/2$ can be constructed.

Theorem 1.1. *Assume there exists a function $\theta(t) \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ such that $Q_{jk} \in \mathbb{R}^+_{\mathrm{ld}}$, $1 - \nu\theta_j \neq 0$ for all $t \in \mathbb{T}_\infty$, $k,j = 1,2$,*

$$\lim_{t\to\infty} \hat{e}_{Q_{jk}}(t,t_0) < \infty. \tag{1.16}$$

Then (1.1) is asymptotically stable if and only if the condition

$$\lim_{t\to\infty} \left| \theta_j^{k-1} \hat{e}_{\theta_j}(t,t_0) \right| = 0, \quad k,j = 1,2, \tag{1.17}$$

is satisfied.

We can simplify condition (1.16) under additional monotonicity condition (1.19) below.

Theorem 1.2. *Assume there exists a function $\theta(t) \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ such that $K \in \mathbb{R}^+_{\mathrm{ld}}$, $1 - \nu\theta_j \neq 0$ for all $t \in \mathbb{T}_\infty$, the conditions*

$$\lim_{t\to\infty} |\hat{e}_{\theta_j}(t,t_0)| = 0, \quad j = 1,2, \tag{1.18}$$

$$2\Re[\theta_j(t)] \leq \nu(t)|\theta_j(t)|^2, \quad t \in \mathbb{T}_\infty,\ j = 1,2, \tag{1.19}$$

$$\lim_{t\to\infty} \hat{e}_K(t,t_0) < \infty, \quad t \in \mathbb{T}_\infty, \tag{1.20}$$

are satisfied.

Then every solution of (1.1) approaches zero as $t \to \infty$.

Corollary 1.3. *Assume there exists a function $\theta(t) \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ such that $K_1 \in \mathbb{R}^+_{\mathrm{ld}}$, $1 - \nu\theta_j \neq 0$ for all $t \in \mathbb{T}_\infty$, conditions (1.18), (1.19), and*

$$\lim_{t\to\infty} \widehat{e}_{K_1}(t,t_0) < \infty, \quad t \in \mathbb{T}_\infty, \text{ where } K_1 \text{ is defined by } (1.14), \tag{1.21}$$

are satisfied.

Then every solution of (1.1) approaches zero as $t \to \infty$.

The next two lemmas from [1, 12] are useful tools for checking condition (1.18).

Lemma 1.4. *Let $M(t)$ be a complex-valued function such that for all $t \in \mathbb{T}_\infty, 1 - M(t)\nu(t) \neq 0$, then*

$$\lim_{t\to\infty} \widehat{e}_{M(t)}(t,t_0) = 0 \tag{1.22}$$

if and only if

$$\lim_{T\to\infty} \int_{t_0}^{T} \lim_{p\searrow\nu(s)} \frac{\mathrm{Log}\,|1 - pM(s)|}{-p} \nabla s = -\infty. \tag{1.23}$$

The following lemma gives simpler sufficient conditions of decay of nabla exponential function.

Lemma 1.5. *Assume $M \in C_{\mathrm{ld}}(\mathbb{T})$, and for some $\varepsilon > 0$,*

$$\lim_{t\to\infty} \int_{t_0}^{t} \Re[M(s)]\nabla s = -\infty \quad \text{if } \nu = 0, \tag{1.24}$$

$$|1 - M\nu(t)| \geq e^{\varepsilon} > 1, \quad \int_{t_0}^{\infty} \frac{\nabla s}{\nu(s)} = \infty \quad \text{if } \nu > 0. \tag{1.25}$$

Then (1.22) is satisfied.

Remark 1.6 [1]. The first condition (1.25), for $\nu > 0$, means that the values of $M(t)$ are located in the the exterior of the ball with center $1/\nu_*$ and radius $1/\nu_*$,

$$\left\{ z : \left| z - \frac{1}{\nu_*} \right| > \frac{1}{\nu_*} \right\}, \quad \nu_* = \inf[\nu(t)], \tag{1.26}$$

and it may be written in the form

$$2\Re[M(t)] < \nu(t)|M(t)|^2. \tag{1.27}$$

Remark 1.7. In view of Lemma 1.5, conditions (1.18) and (1.19) of Theorem 1.2 can be replaced by

$$\begin{gathered} \int_{t_0}^{\infty} \frac{ds}{\nu(s)} = \infty, \quad \text{for } \nu > 0, \\ 2\Re[\theta_j(t)] < \nu(t)|\theta_j(t)|^2, \quad t \in \mathbb{T}_\infty,\ j = 1,2. \end{gathered} \tag{1.28}$$

Remark 1.8. In order to apply Theorem 1.2 for the study of exponential stability of a dynamic equation (1.1), one can replace condition (1.18) by the necessary and sufficient condition of exponential stability of an exponential function on a time scale given in [12].

Example 1.9. Consider the Euler equation

$$y^{\nabla\nabla} + \frac{ay^{\nabla}}{\rho(t)} + \frac{by(t)}{t\rho(t)} = 0, \quad a, b \in \mathbb{R}, \tag{1.29}$$

on the time scale $\mathbb{T}_\infty \subset (0, \infty)$. We assume that the regressivity condition

$$t\rho(t) + at\nu(t) + b\nu^2(t) \neq 0, \quad \forall t \in \mathbb{T}_\infty, \tag{1.30}$$

is satisfied. Suppose λ_1 and λ_2 are two distinct roots of the associated characteristic equations

$$\lambda^2 + (a-1)\lambda + b = 0, \qquad \lambda_{1,2} = \frac{1 - a \pm \sqrt{(1-a)^2 - 4b}}{2}. \tag{1.31}$$

If

$$2\Re[\lambda_j] < \frac{\nu(t)}{t}|\lambda_j|^2, \quad \int_{t_0}^{\infty} \frac{\nabla s}{\nu(s)} = \infty, \quad j = 1, 2, \tag{1.32}$$

then from Theorem 1.2 it follows that all solutions of (1.29) approach zero as $t \to \infty$.

To check the conditions of Theorem 1.2 we set

$$\theta = \frac{\lambda_1 - \lambda_2}{2t} = \frac{\sqrt{(1-a)^2 - 4b}}{2t}. \tag{1.33}$$

In view of

$$\theta^{\nabla} = \frac{\lambda_2 - \lambda_1}{2t\rho(t)}, \tag{1.34}$$

we have

$$\begin{gathered} 2k\theta = \frac{\theta^{\nabla}}{\theta} = -\frac{1}{\rho(t)}, \qquad 1 - 2k\theta\nu = 1 + \frac{\nu}{\rho(t)} = \frac{t}{\rho}, \\ D = \theta^2 + \frac{1 + \nu p + \nu^2 q}{(1 - 2k\theta\nu)\nu^2} = \theta^2 + \frac{1 + a\nu/\rho + b\nu^2/t\rho}{t\nu^2/\rho} = \left(\frac{1-a}{2t} - \frac{1}{\nu}\right)^2. \end{gathered} \tag{1.35}$$

Hence from (1.9), (1.11) we get

$$\theta_1 = \theta + \frac{1}{\nu} + \sqrt{D} = \theta + \frac{1}{\nu} + \frac{1-a}{2t} - \frac{1}{\nu} = \frac{\lambda_1}{t}, \qquad \theta_2 = \frac{\lambda_2}{t}. \tag{1.36}$$

By direct calculations from (1.12) we get

$$\mathrm{Hov}_j = \frac{-b - (a-1)\lambda_j - \lambda_j^2}{t\rho} = 0, \quad j = 1, 2. \tag{1.37}$$

So

$$K_1(t) = \left| \frac{(\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1)(s)}{4\theta(s)\theta(\rho(s))} \right| \equiv 0, \quad K(t) \equiv 0, \tag{1.38}$$

and condition (1.20) is satisfied. Conditions (1.18), (1.19) follow from (1.32) and Lemma 1.5 (with $M = \lambda_j/t$).

If $\mathbb{T} = \mathbb{R}$, then $\nu = 0$, $\hat{e}_{\theta_j}(t,t_0) = (t/t_0)^{\lambda_j}$, $j = 1,2$, and condition (1.32) becomes

$$\Re(2\lambda_j) = \Re\Big(1 - a \pm \sqrt{(a-1)^2 - 4b}\Big) < 0. \tag{1.39}$$

If $\mathbb{T} = \mathbb{Z}$, then $\nu = 1$ and $\rho = t+1$. From [4] exact solutions of (1.29) are $\hat{e}_{\lambda_j/t}(t,t_0) = \Gamma(t+1)\Gamma(t_0+1-\lambda_j)/\Gamma(t+1-\lambda_j)\Gamma(t_0+1)$, $j = 1,2$, and condition (1.32) becomes

$$2\Re\Big[1 \pm \sqrt{(a-1)^2 - 4b}\Big] < \frac{\big|1 \pm \sqrt{(a-1)^2 - 4b}\big|^2}{t}. \tag{1.40}$$

Example 1.10. Consider the linear dynamic equation on a time scale

$$y^{\nabla\nabla}(t) + \frac{a y^{\nabla}(t)}{\rho(t)} + \frac{tby(t)}{\rho(t)(1+t^2)} = 0. \tag{1.41}$$

Choosing θ again as in (1.33) we have (1.36) and

$$\begin{aligned} \theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1 &= \theta_1\theta_2\Big(1 - \frac{\nu\theta^{\nabla}}{\theta}\Big) - \theta_1^{\nabla} + \frac{\theta^{\nabla}}{\theta}\theta_1 - q \\ &= \frac{b}{t\rho} - q = \frac{b}{t\rho} - \frac{tb}{\rho(1+t^2)} = \frac{b}{\rho t(1+t^2)}. \end{aligned} \tag{1.42}$$

Thus

$$K_1(t) = \frac{|b|}{(1+t^2)(\lambda_1 - \lambda_2)^2}. \tag{1.43}$$

From Theorem 1.2 it follows that all solutions of (1.41) approach zero as $t \to \infty$, provided that conditions (1.32) and (1.21) are satisfied.

For the time scales $\mathbb{T} = R$, condition (1.21) is satisfied. For the time scale $\mathbb{T} = \mathbb{Z}$ with $t_0 = 1$, we have $\nu \equiv 1$, and condition (1.21) is satisfied also since

$$\int_{t_0}^{\infty} \frac{\operatorname{Log}(1 - \nu(s)K_1(s))}{-\nu(s)} \nabla s = -\sum_{n=1}^{\infty} \operatorname{Log}(1 - K_1(n)) \leq -\sum_{n=1}^{\infty} \operatorname{Log}(1 - Cn^{-2}) < \infty. \tag{1.44}$$

2. Method of integral representations of solutions

Lemma 2.1 (Gronwall's inequality). *Assume $y, f \in C_{\mathrm{ld}}(\mathbb{T} \bigcap (t,b))$, $K \in \mathbb{R}^{+}_{\mathrm{ld}} y, f, K \geq 0$. Then*

$$y(t) \leq f(t) + \int_t^b K(s)y(s)\nabla s \quad \forall t \in \mathbb{T} \bigcap (t,b) \tag{2.1}$$

implies for all $t \in \mathbb{T} \bigcap (t,b)$ *that*

$$y(t) \leq f(t) + \int_t^b \hat{e}_K(\rho(s),t)K(s)f(s)\nabla s. \tag{2.2}$$

Proof. From $K(t) \geq 0$ it follows that

$$K_2(t) \equiv \frac{K(t)}{1+\nu(t)K(t)} \geq 0, \qquad 1 - K_2(t)\nu(t) = \frac{1}{1+K(t)\nu(t)} > 0, \tag{2.3}$$

and from [4, Theorem 3.22] we have

$$\hat{e}_{K_2}(b,t) > 0. \tag{2.4}$$

Denote

$$M(t) \equiv \int_t^b K(s)y(s)\nabla s, \qquad \ominus K_2 \equiv \frac{-K_2}{1-K_2\nu}. \tag{2.5}$$

Then

$$y(t) \leq f(t) + M(t), \tag{2.6}$$

or

$$M^{\nabla} = -K(t)y(t) \geq -K(t)\big(f(t)+M(t)\big), \tag{2.7}$$

which implies that

$$M^{\nabla} + K(t)M(t) \geq -K(t)f(t). \tag{2.8}$$

Multiplying the last inequality by $-1/\hat{e}_{K_2}(b,\rho(t)) < 0$, and in view of

$$\frac{\hat{e}^{\nabla}_{K_2}(b,t)}{\hat{e}_{K_2}(b,t)} = \frac{\hat{e}^{\nabla}_{\ominus K_2}(t,b)}{\hat{e}_{\ominus K_2}(t,b)} = \ominus K_2(t) = -K(t), \tag{2.9}$$

we have

$$\left(\frac{M(t)}{\hat{e}_{K_2}(b,t)}\right)^{\nabla} = \frac{M^{\nabla} - \big(\hat{e}^{\nabla}_{K_2}(b,t)/\hat{e}_{K_2}(b,t)\big)M}{\hat{e}^{\rho}_{K_2}(b,t)} = \frac{M^{\nabla}+KM}{\hat{e}^{\rho}_{K_2}(b,t)}. \tag{2.10}$$

Hence

$$-\left(\frac{M(t)}{\hat{e}_{K_2}(b,t)}\right)^{\nabla} \leq \frac{Kf(t)}{\hat{e}^{\rho}_{K_2}(b,t)}. \tag{2.11}$$

Integrating over (t,b) we have

$$\begin{gathered} \frac{M(t)}{\hat{e}_{K_2}(b,t)} - M(b) \leq \int_t^b \frac{K(s)f(s)\nabla s}{\hat{e}^{\rho}_{K_2}(b,s)}, \\ M(t) \leq \hat{e}_{K_2}(b,t)\int_t^b \frac{K(s)f(s)\nabla s}{\hat{e}_{K_2}(b,\rho(s))} = \int_t^b \hat{e}_{K_2}(\rho(s),t)K(s)f(s)f\nabla s, \end{gathered} \tag{2.12}$$

or

$$y(t) \le f(t) + M(t) \le f(t) + \int_t^b \hat{e}_{K_2}(\rho(s),t)K(s)f(s)\nabla s. \tag{2.13}$$

From this inequality and in view of

$$\hat{e}_{K_2}(s,t) \le \hat{e}_K(s,t), \tag{2.14}$$

(2.2) follows.

The last inequality is trivial for $\nu = 0$ because

$$0 \le K_2(s) \le K(s). \tag{2.15}$$

For $\nu > 0$ we also have

$$\begin{aligned}\hat{e}_{K_2}(s,t) &= \exp\int_t^s \frac{\operatorname{Log}(1-K_2\nu(z))\nabla z}{-\nu} \\ &\le \exp\int_t^s \frac{\operatorname{Log}(1-K\nu(z))\nabla z}{-\nu} = \hat{e}_K(s,t).\end{aligned} \tag{2.16}$$

□

Consider the system of ordinary differential equations

$$a^\nabla(t) = A(t)a(t), \quad t \in \mathbb{T}_\infty, \tag{2.17}$$

where $a(t)$ is an n-vector function and $A(t) \in C_{\text{ld}}(T,\infty)$ is an $n\times n$ matrix function. Suppose we can find the exact solutions of the system

$$\psi^\nabla(t) = A_1(t)\psi(t), \quad t \in \mathbb{T}_\infty, \tag{2.18}$$

with the matrix function A_1 close to the matrix function A, which means that condition (2.21) is satisfied. Let $\Psi(t)$ be the $n\times n$ fundamental matrix of the auxiliary system (2.18). If the matrix function A_1 is regressive and ld-continuous, the matrix $\Psi(t)$ exists (see [6]). Then solutions of (2.17) can be represented in the form

$$a(t) = \Psi(t)(C+\varepsilon(t)), \tag{2.19}$$

where $a(t)$, $\varepsilon(t)$, C are the n-vector columns: $a(t) = \text{column}(a_1(t),\dots,a_n(t))$, $\varepsilon(t) = \text{column}(\varepsilon_1(t),\dots,\varepsilon_n(t))$, $C = \text{column}(C_1,\dots,C_n)$; C_k are arbitrary constants. We can consider (2.19) as a definition of the error vector function $\varepsilon(t)$.

Denote

$$H(t) \equiv (\Psi - \nu\Psi^\nabla)^{-1}(A\Psi - \Psi^\nabla)(t). \tag{2.20}$$

Theorem 2.2. *Assume there exists a matrix function $\Psi(t) \in C^1_{\text{ld}}(\mathbb{T}_\infty)$ such that $\|H\| \in \mathbb{R}^+_{\text{ld}}$, the matrix function $\Psi - \nu\Psi^\nabla$ is invertible, and*

$$\hat{e}_{\|H\|}(\infty,t) = \exp\int_t^\infty \lim_{m\to\nu(s)} \frac{\operatorname{Log}(1-m\|H(s)\|)\nabla s}{-m} < \infty. \tag{2.21}$$

Then every solution of (2.17) can be represented in form (2.19) and the error vector function $\varepsilon(t)$ *can be estimated as*

$$||\varepsilon(t)|| \le \|C\| \left(\hat{e}_{\|H\|}(\infty, t) - 1\right), \tag{2.22}$$

where $\|\cdot\|$ *is the Euclidean vector (or matrix) norm* $\|C\| = \sqrt{C_1^2 + \cdots + C_n^2}$.

Remark 2.3. From (2.22) the error $\varepsilon(t)$ is small when the expression

$$\int_t^\infty \lim_{m \searrow \nu(s)} \left(\frac{\operatorname{Log}\left(1 - m||(\Psi - \nu\Psi^\nabla)^{-1}(A - A_1)\Psi(s)||\right)}{-m} \right) \nabla s \tag{2.23}$$

is small.

Proof of Theorem 2.2. Let $a(t)$ be a solution of (2.17). The substitution $a(t) = \Psi(t)u(t)$ transforms (2.17) into

$$u^\nabla(t) = H(t)u(t), \quad t > T, \tag{2.24}$$

where H is defined by (2.20). By integration we get

$$u(t) = C - \int_t^b H(s)u(s)\nabla s, \quad b > t > T, \tag{2.25}$$

where the constant vector C is chosen as in (2.19).

Estimating $u(t)$ we have

$$||u(t)|| \le \|C\| + \int_t^b ||H(s)|| \cdot ||u(s)||\nabla s. \tag{2.26}$$

From

$$\begin{gathered} \left(\hat{e}_K(t, c)\right)^\nabla = K\hat{e}_K(t, c), \\ \left(\hat{e}_K(c, t)\right)^\nabla = \left(\frac{1}{\hat{e}_K(t, c)}\right)^\nabla = \frac{-K}{\hat{e}_K^\rho(t, c)} = -K\hat{e}_K(c, \rho(t)), \end{gathered} \tag{2.27}$$

by integration we get

$$\int_a^b K(s)\hat{e}_K(s, c)\nabla s = \hat{e}_K(b, c) - \hat{e}_K(a, c), \tag{2.28}$$

$$\int_a^b K(s)\hat{e}_K(c, \rho(s))\nabla s = \hat{e}_K(c, a) - \hat{e}_K(c, b). \tag{2.29}$$

Using Gronwall's inequality (2.2) from (2.26) we get

$$\begin{aligned} ||u(t)|| &\le \|C\| \left(1 + \int_t^b \|H\| \hat{e}_{\|H\|}(\rho(s), t)\nabla s\right) \\ &\le \|C\| \left(1 + \int_t^b \|H\| \hat{e}_{\|H\|}(s, t)\nabla s\right). \end{aligned} \tag{2.30}$$

In view of (2.28),

$$\|u(t)\| \leq \|C\|\widehat{e}_{\|H\|}(b,t). \tag{2.31}$$

From representation (2.19) and expression (2.25) we have

$$\varepsilon(t) = \Psi^{-1}a - C = u - C = -\int_t^b H(s)u(s)\nabla s. \tag{2.32}$$

Then using (2.31) we obtain the estimate given by (2.22):

$$\begin{aligned}\|\varepsilon(t)\| &\leq \int_t^b \|Hu\|\nabla s \leq \|C\| \int_t^b \|H(s)\|\widehat{e}_{\|H\|}(b,s)\nabla s \\ &\leq \|C\| \int_t^b \|H(s)\|\widehat{e}_{\|H\|}(b,\rho(s))\nabla s = \|C\|(\widehat{e}_{\|H\|}(b,t)-1).\end{aligned} \tag{2.33}$$

□

Theorem 2.4. *Let $y_1, y_2 \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ be the complex-valued functions such that $\|H\| \in \mathbb{R}^+_{\mathrm{ld}}$, and*

$$\widehat{e}_{\|H\|}(\infty,t) < \infty, \tag{2.34}$$

where

$$B_{kj}(t) \equiv \frac{y_k(t)Ly_j(t)}{W(y_1,y_2)}, \quad Ly \equiv y^{\nabla\nabla} + p(t)y^{\nabla} + q(t)y, \quad j = 1,2, \tag{2.35}$$

$$\Psi = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1^{\nabla}(t) & y_2^{\nabla}(t) \end{pmatrix}, \qquad W(y_1,y_2) = y_2^{\nabla}(t)y_1(t) - y_1^{\nabla}(t)y_2(t), \tag{2.36}$$

$$H(t) = \left(1 - \nu\Psi^{-1}\Psi^{\nabla}\right)^{-1} \begin{pmatrix} B_{21}(t) & B_{22}(t) \\ -B_{11}(t) & -B_{12}(t) \end{pmatrix}. \tag{2.37}$$

Then every solution of (1.1) can be written in the form

$$y(t) = [C_1 + \varepsilon_1(t)]y_1(t) + [C_2 + \varepsilon_2(t)]y_2(t), \tag{2.38}$$

$$y^{\nabla}(t) = [C_1 + \varepsilon_1(t)]y_1^{\nabla}(t) + [C_2 + \varepsilon_2(t)]y_2^{\nabla}(t), \tag{2.39}$$

where C_1, C_2 are arbitrary constants, and the error function satisfies the estimate

$$\|\varepsilon(t)\| \leq \|C\|\left(-1 + \widehat{e}_{\|H\|}(\infty,t)\right). \tag{2.40}$$

Proof of Theorem 2.4. We can rewrite (1.1) in the form

$$v^{\nabla}(t) = A(t)v(t), \tag{2.41}$$

where

$$v(t) = \begin{pmatrix} y(t) \\ y^{\nabla}(t) \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}. \tag{2.42}$$

Now we apply Theorem 2.2 to the system (2.41). By direct calculations from (2.20) we get (2.37), and condition (2.21) of Theorem 2.2 follows from (2.34).

From Theorem 2.2 it follows that

$$v(t) = \Psi(t)\big(C + \varepsilon(t)\big). \tag{2.43}$$

Representations (2.38), (2.39), and estimates (2.40) follow from Theorem 2.2. □

Proof of Theorem 1.1. We are looking for solutions of (1.1) in the form

$$y_j(t) \equiv \hat{e}_{\theta_j}(t,t_0) = \exp\left(\int_{t_0}^{t} \lim_{m\searrow\nu(\tau)} \frac{\operatorname{Log}\big(1 - m\theta_j(\tau)\big)}{-m}\nabla\tau\right), \quad j = 1,2, \tag{2.44}$$

where the functions θ_j are defined by (1.8) and (1.11).

From (2.44) (see [4]) we have

$$\begin{gathered} y_1^{\nabla}(t) = \theta_1(t)y_1(t), \qquad y_2^{\nabla}(t) = \theta_2(t)y_2(t), \\ \frac{W[y_1,y_2]}{y_1y_2} = \frac{y_1y_2^{\nabla} - y_2y_1^{\nabla}}{y_1y_2} = \theta_2 - \theta_1 = -2\theta, \\ \frac{Ly_j}{y_j} = \theta_j^2 + (1 - \nu\theta_j)\theta_j^{\nabla} + p\theta_j + q, \quad j = 1,2. \end{gathered} \tag{2.45}$$

By direct calculations

$$\begin{gathered} B_{12}(t) = \frac{y_1Ly_2}{W(y_1,y_2)} = \frac{\operatorname{Hov}_2(t)}{2\theta(t)}, \qquad B_{21}(t) = \frac{y_2Ly_1}{W(y_1,y_2)} = \frac{\operatorname{Hov}_1(t)}{2\theta(t)}, \\ B_{11}(t) = \frac{y_1Ly_1}{W(y_1,y_2)} = \frac{\operatorname{Hov}_1(t)}{2\theta(t)}\frac{\hat{e}_{\theta_1}(t,t_0)}{\hat{e}_{\theta_2}(t,t_0)}, \\ B_{22}(t) = \frac{y_2Ly_2}{W(y_1,y_2)} = \frac{\operatorname{Hov}_2(t)}{2\theta(t)}\frac{\hat{e}_{\theta_2}(t,t_0)}{\hat{e}_{\theta_1}(t,t_0)}. \end{gathered} \tag{2.46}$$

In view of (1.16) condition (2.34) of Theorem 2.4 is satisfied. From Theorem 2.4 and (2.40) it follows that $|\varepsilon_j(t)| \le C$, $j = 1,2$. From (1.17) we get $y_j(t) \to 0$, $y_j^{\nabla}(t) \to 0$, $t \to \infty$. So asymptotic stability of (1.1) follows from representations (2.38) and (2.39).

Now we prove that if one of (1.17) is not satisfied, then there exists asymptotically unstable solution $y(t)$.

Assume for contradiction that (1.5) is satisfied and, for example, the first condition of (1.17) is not satisfied. Then there exists the sequence $t_n \to \infty$ such that

$$\lim_{t_n\to\infty} |y_1(t_n)| = \lambda_1 > 0. \tag{2.47}$$

There exists the subsequence $t_{n_j} \equiv t_m$ of the sequence t_n such that

$$\lim_{t_m\to\infty} |y_2(t_m)| = \lambda_2 \ge 0. \tag{2.48}$$

From Theorem 2.4 any solution $y(t)$ of (1.1) can be represented in the form (2.38) with some constants C_1, C_2, or

$$y(t_m) = [C_1 + \varepsilon_1(t_m)]y_1(t_m) + [C_2 + \varepsilon_2(t_m)]y_2(t_m), \tag{2.49}$$

where from (2.40) we have

$$|\varepsilon_j(t)| \leq \|C\|(\widehat{e}_{\|H\|}(\infty,t) - 1) \longrightarrow 0, \tag{2.50}$$

as $t = t_m \to \infty$.

From representation (2.49) it follows that λ_1, λ_2 must be finite numbers. Otherwise, the left side of the representation vanishes and the right side approaches infinity when t_m approaches infinity. Choosing $C_1 = 1$, $C_2 = 0$ we obtain from (2.49), as $t \to \infty$,

$$0 = \lambda_1 + \lambda_1 \lim_{t_k \to \infty} \varepsilon_1(t_k) + \lambda_2 \lim_{t_k \to \infty} \varepsilon_2(t_k) = \lambda_1, \tag{2.51}$$

which contradicts the assumption $\lambda_1 > 0$. □

Lemma 2.5. *If $1 - \nu\theta_j(t) \neq 0$ for all $t \in \mathbb{T}_\infty$, and*

$$2\Re[\theta_j(t)] \leq \nu(t)|\theta_j(t)|^2, \quad t \in \mathbb{T}_\infty,\ j = 1,2, \tag{2.52}$$

then the functions $|y_j(t)|$ are nonincreasing. That is,

$$|y_j(t)| \leq |y_j(\tau)| \quad \text{whenever } t_0 \leq \tau \leq t. \tag{2.53}$$

Proof. If $\nu \equiv 0$, then the functions $|y_j|$ (see (2.44)) are nonincreasing in view of (2.52) and

$$\frac{|y_j(t)|^\nabla}{|y_j(t)|} = \frac{|y_j(t)|'}{|y_j(t)|} = \Re[\theta_j] \leq 0. \tag{2.54}$$

If $\nu > 0$, then from (2.52) it follows that

$$|1 - \nu\theta_j| = \sqrt{(1 - \nu\Re[\theta_j])^2 + (\nu\Im[\theta_j])^2} \geq 1, \quad j = 1,2, \tag{2.55}$$

$$\frac{\operatorname{Log}|1 - \nu(t)\theta_j(t)|}{-\nu(t)} \leq 0. \tag{2.56}$$

Hence the functions $|y_j(t)| = \exp(\int_{t_0}^t (\operatorname{Log}|1 - \nu\theta_j(\tau)|/-\nu(\tau))\nabla\tau)$ are nonincreasing. □

Proof of Theorem 1.2. In view of (1.12), (2.44) by direct calculations we have

$$\frac{y_1^\nabla y_2^{\nabla\nabla} - y_2^\nabla y_1^{\nabla\nabla}}{W[y_1,y_2]} = q(t) + \frac{\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1}{2\theta}. \tag{2.57}$$

It is easy to check that y_1, y_2 are exact solutions of

$$y^{\nabla\nabla} + \frac{y_1^{\nabla\nabla} y_2 - y_2^{\nabla\nabla} y_1}{W[y_1,y_2]} y^\nabla(t) + \frac{y_1^\nabla y_2^{\nabla\nabla} - y_2^\nabla y_1^{\nabla\nabla}}{W[y_1,y_2]} y(t), \tag{2.58}$$

or

$$y^{\nabla\nabla} + p(t)y^{\nabla}(t) + \left(q(t) + \frac{\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1}{2\theta}\right)y(t) = 0, \tag{2.59}$$

in view of (1.8) and (2.57).

Indeed from

$$y_1^{\nabla\nabla} + py_1^{\nabla} + qy_1 = 0, \qquad y_2^{\nabla\nabla} + py_2^{\nabla} + qy_2 = 0 \tag{2.60}$$

we have

$$p = \frac{y_1^{\nabla\nabla} y_2 - y_2^{\nabla\nabla} y_1}{W[y_1, y_2]}, \qquad q = \frac{y_2^{\nabla\nabla} y_1^{\nabla} - y_1^{\nabla\nabla} y_2^{\nabla}}{W[y_1, y_2]}, \tag{2.61}$$

or

$$\begin{gathered}
p = \frac{(\theta_1 y_1)^{\nabla} y_2 - (\theta_2 y_2)^{\nabla} y_1}{(\theta_2 - \theta_1) y_1 y_2} = \frac{\theta_1^{\nabla} + \theta_1^2 - \nu\theta_1^{\nabla}\theta_1 - \theta_2^{\nabla} - \theta_2^2 + \nu\theta_2^{\nabla}\theta_2}{\theta_2 - \theta_1}, \\
q = \frac{(\theta_2 y_2)^{\nabla}\theta_1 y_1 - (\theta_1 y_1)^{\nabla}\theta_2 y_2}{(\theta_2 - \theta_1) y_1 y_2} = \frac{\theta_1(\theta_2^{\nabla} + \theta_2^2 - \nu\theta_2^{\nabla}\theta_2) - \theta_2(\theta_1^{\nabla} + \theta_1^2 - \nu\theta_1^{\nabla}\theta_1)}{\theta_2 - \theta_1}, \\
p = -\theta_1 - \theta_2 - \frac{\theta^{\nabla}}{\theta} + \nu\theta_2^{\nabla} + \nu\theta_1\frac{\theta^{\nabla}}{\theta}, \qquad q = \theta_1\theta_2\left(1 - \nu\frac{\theta^{\nabla}}{\theta}\right) + \theta_1\frac{\theta^{\nabla}}{\theta} - \theta_1^{\nabla}.
\end{gathered} \tag{2.62}$$

Excluding θ_1^{∇} we have

$$\begin{aligned}
\nu q + p &= \nu\theta_1\theta_2\left(1 - \nu\frac{\theta^{\nabla}}{\theta}\right) + \nu\theta_1\frac{\theta^{\nabla}}{\theta} - \nu\theta_1^{\nabla} - \theta_1 - \theta_2 - \frac{\theta^{\nabla}}{\theta} + \nu\theta_2^{\nabla} + \nu\theta_1\frac{\theta^{\nabla}}{\theta} \\
&= \nu\theta_1\theta_2\left(1 - \nu\frac{\theta^{\nabla}}{\theta}\right) + 2\nu\theta_1\frac{\theta^{\nabla}}{\theta} - 2\nu\theta^{\nabla} - \theta_1 - \theta_2 - \frac{\theta^{\nabla}}{\theta} \\
&= \nu\theta_1(\theta_1 - 2\theta)(1 - 2k\theta\nu) + 2\nu\theta_1 2k\theta - 2\nu 2k\theta^2 - 2\theta_1 + 2\theta - 2k\theta \\
&= \nu\theta_1^2(1 - 2k\theta\nu) - 2\theta_1(\nu\theta - \nu^2 2k\theta^2 + 1 - 2k\theta\nu) + 2\theta(1 - 2k\theta\nu) - 2k\theta.
\end{aligned} \tag{2.63}$$

Thus we get (1.8):

$$\nu q + p = \nu\theta_1^2(1 - 2k\theta\nu) - 2\theta_1(1 + \nu\theta)(1 - 2k\theta\nu) + 2\theta(1 - 2k\theta\nu) - 2k\theta. \tag{2.64}$$

So the solutions of (1.1) or the nonhomogeneous equation

$$y^{\nabla\nabla} + p(t)y^{\nabla}(t) + \left(q(t) + \frac{\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1}{2\theta}\right)y(t) = \left(\frac{\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1}{2\theta}\right)y(t) \tag{2.65}$$

may be written in the form (see [3, 4])

$$
\begin{aligned}
y - C_1 y_1 - C_2 y_2 &= \int_{t_0}^{t} \frac{y_1(\rho(\tau))y_2(t) - y_2(\rho(\tau))y_1(t)}{W[y_1,y_2](\rho(\tau))}\left(\frac{\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1}{2\theta}\right)(\tau)y(\tau)\nabla\tau \\
&= -\int_{t_0}^{t}\left(\frac{y_2(t)}{y_2(\rho(\tau))} - \frac{y_1(t)}{y_1(\rho(\tau))}\right)\frac{(\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1)(\tau)y(\tau)\nabla\tau}{4\theta(\rho(\tau))\theta(\tau)}.
\end{aligned}
\tag{2.66}
$$

Using Lemma 2.5 and the formula $y(\rho(\tau)) = y(\tau) - \nu(\tau)y^{\nabla}(\tau)$ we get the estimate

$$
\begin{aligned}
|y(t)| \le{} & |C_1 y_1(t) + C_2 y_2(t)| \\
&+ \int_{t_0}^{t}\left(\frac{|y_1(\tau)|}{|y_1(\rho(\tau))|} + \frac{|y_2(\tau)|}{|y_2(\rho(\tau))|}\right)\frac{|(\theta_1 \operatorname{Hov}_2 - \theta_2 \operatorname{Hov}_1)(\tau)y(\tau)|\nabla\tau}{4|\theta(\rho(\tau))\theta(\tau)|},
\end{aligned}
\tag{2.67}
$$

or

$$
|y(t)| \le |C_1 y_1(t) + C_2 y_2(t)| + \int_{t_0}^{t} K(\tau)|y(\tau)|\nabla\tau, \tag{2.68}
$$

where $K(\tau)$ is defined in (1.13).

Further we need a different version of Gronwall's inequality from the one in Lemma 2.1. Note that delta version of this Gronwall's inequality was proved in [3].

Lemma 2.6. *Assume $y, f \in C_{\mathrm{ld}}(\mathbb{T})$, $f, y \ge 0$, $K \in \mathbb{R}_{\nu}^{+}$. Then*

$$
y(t) \le f(t) + \int_{t_0}^{t} K(s)y(s)\nabla s \quad \forall t \in \mathbb{T}_{\infty}, \tag{2.69}
$$

implies

$$
y(t) \le f(t) + \int_{t_0}^{t} \hat{e}_K(t,\rho(s))K(s)f(s)\nabla s \quad \forall t \in \mathbb{T}_{\infty}. \tag{2.70}
$$

Proof. Denote

$$
z(t) \equiv \int_{t_0}^{t} K(s)y(s)\nabla s, \tag{2.71}
$$

then

$$
y(t) \le f(t) + z(t), \qquad z^{\nabla}(t) - K(t)z(t) = K(t)y(t) - K(t)z(t) \le Kf(t). \tag{2.72}
$$

From $K \in \mathbb{R}_{\nu}^{+}$ follows $\hat{e}_K > 0$ (see [4, Theorem 3.22]), and

$$
\left(\frac{z}{\hat{e}_K}\right)^{\nabla} = \frac{z^{\nabla} - Kz}{\hat{e}_K^{\rho}} \le \frac{Kf}{\hat{e}_K^{\rho}}, \tag{2.73}
$$

by integration

$$
\frac{z(t)}{\hat{e}_K(t,t_0)} - z(t_0) = \frac{z(t)}{\hat{e}_K(t,t_0)} \le \int_{t_0}^{t} \frac{K(s)f(s)\nabla s}{\hat{e}_K(\rho(s),t_0)}. \tag{2.74}
$$

or

$$z(t) \leq \int_{t_0}^{t} K(s)\hat{e}_K(t,t_0)\hat{e}_K(t_0,\rho(s))f(s)\nabla s = \int_{t_0}^{t} K(s)\hat{e}_K(t,\rho(s))f(s)\nabla s. \tag{2.75}$$

Thus

$$y \leq f + z \leq f + \int_{t_0}^{t} K(s)\hat{e}_K(t,\rho(s))f(s)\nabla s. \tag{2.76}$$

□

Using Gronwall's inequality (2.70) in (2.68) we get the stability estimate

$$|y(t)| \leq |C_1 y_1(t) + C_2 y_2(t)| + \int_{t_0}^{t} \hat{e}_K(t,\rho(s))|C_1 y_1(s) + C_2 y_2(s)|K(s)\nabla s. \tag{2.77}$$

From (1.18) it follows that

$$\lim_{t\to\infty} |C_1 y_1(t) + C_2 y_2(t)| = 0. \tag{2.78}$$

Or for any $\varepsilon > 0$ there exists t_0 such that for $t > t_0$ we have

$$|C_1 y_1(t) + C_2 y_2(t)| < \varepsilon. \tag{2.79}$$

Hence it follows from (2.77) that

$$|y(t)| \leq \varepsilon\left(1 + \int_{t_0}^{t} \hat{e}_K(t,\rho(s))K(s)\nabla s\right). \tag{2.80}$$

From (2.29)

$$\int_{t_0}^{t} K(s)\hat{e}_K(t,\rho(s))\nabla s = \hat{e}_K(t,t_0) - \hat{e}_K(t,t), \tag{2.81}$$

and so

$$|y(t)| \leq \varepsilon\hat{e}_K(t,t_0) \leq C\varepsilon, \tag{2.82}$$

from which it follows that $y(t) \to 0$ as $t \to \infty$. □

Corollary 1.3 follows from Theorem 1.2 because of inequality $K(s) \leq K_1(s)$ which follows from (2.55).

Proof of Lemma 1.4. Proof follows from the Hilger's representation of exponential function on a time scale (see [6, Theorem 7.4(iii)])

$$|\hat{e}_{M(t)}(T,t_0)| = \exp\int_{t_0}^{T} \lim_{p\searrow\nu(s)} \frac{\operatorname{Log}|1-pM(s)|}{-p}\nabla s. \tag{2.83}$$

□

Proof of Lemma 1.5. From (1.25) it follows that $1 - M\nu \neq 0$ and $\hat{e}_M$ exists. For $\nu > 0$ from (1.25) we have

$$|\hat{e}_{M(t)}(t,t_0)| = \exp\int_{t_0}^{t} \frac{\operatorname{Log}|1 - M\nu(s)|}{-\nu(s)}\nabla s \leq \exp\int_{t_0}^{t} \frac{\varepsilon\nabla s}{-\nu(s)} \longrightarrow 0, \quad t \longrightarrow \infty. \tag{2.84}$$

□

For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}. \tag{2.85}$$

The graininess function $\mu : \mathbb{T} \to [0, \infty]$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{2.86}$$

A function $f :\in \mathbb{T} \to \mathbb{R}$ is called rd-continuous ($C_{\mathrm{rd}}(\mathbb{T})$) provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

By $L_{\mathrm{rd}}(\mathbb{T})$ we denote a class of functions $f : \mathbb{T} \to \mathbb{R}$ that are rd-continuous on $\mathbb{T}$ and Lebesgue integrable on $\mathbb{T}$.

Assume y^{Δ} is a delta (Hilger) derivative, and $e_{\lambda}(t,t_0)$ is a delta exponential function.

Lemma 2.7 (see [1, 12]). *Assume $K \in C_{\mathrm{rd}}(\mathbb{T})$, and for some $\varepsilon > 0$*

$$0 < |1 + K\mu(t)| \leq 1 - \varepsilon < 1, \qquad \int_{t_0}^{\infty} \frac{\Delta s}{\mu(s)} = \infty \quad \text{if } \mu > 0, \tag{2.87}$$

$$\int_{t_0}^{\infty} \Re[K(s)]\nabla s = -\infty \quad \text{if } \mu = 0. \tag{2.88}$$

Then

$$\lim_{t\to\infty} e_K(t,t_0) = 0. \tag{2.89}$$

Remark 2.8. The first condition (2.87) means (see [1]) that the values of $K(t)$ are located in the interior of the annulus with center $-1/\mu^*$ and radius $1/\mu^*$,

$$S_{\mu^*} = \left\{z : 0 < \left|z + \frac{1}{\mu^*}\right| < \frac{1}{\mu^*}\right\}, \quad \mu^* = \sup[\mu(t)], \tag{2.90}$$

and it may be written in the form

$$2\Re[K(t)] < -\mu(t)|K(t)|^2. \tag{2.91}$$

Proof. It is enough to prove the lemma for the case $\mu > 0$.

From $|1 + K\mu(t)| > 0$ it follows that e_K exists. From $\operatorname{Log}|1 - \varepsilon| < -\varepsilon$ for $0 < \varepsilon < 1$ and from (2.87) for $\mu > 0$ we have

$$|\hat{e}_{K(t)}(t,t_0)| = \exp\int_{t_0}^{t} \frac{\operatorname{Log}|1 + K\mu(s)|}{\mu(s)}\Delta s \leq \exp\int_{t_0}^{t} \frac{-\varepsilon\Delta s}{\mu(s)} \longrightarrow 0, \quad t \longrightarrow \infty. \tag{2.92}$$

□

Remark 2.9. Comparing Lemmas 2.7 and 1.5 we see that nabla exponential functions approach zero in the larger region (see (1.25)) in the complex plane than delta exponential functions (see (2.87)). Thus asymptotic stability conditions for nabla dynamic equations should be less restrictive than for delta dynamic equations.

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