

ON THE EXISTENCE OF A POSITIVE SOLUTION FOR A SEMILINEAR ELLIPTIC EQUATION IN THE UPPER HALF SPACE

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Received 14 September 2001

We show the existence of a nontrivial solution to the semilinear elliptic equation $-\Delta u + u = b(x)|u|^{p-2}u$, $u > 0$, $u \in H_0^1(\mathbb{R}_+^N)$ under some suitable conditions.

1. Introduction

In this paper, we study the following problem:

$$-\Delta u + u = b(x)|u|^{p-2}u, \quad u > 0, u \in H_0^1(\mathbb{R}_+^N), \quad (1.1)$$

where $2 < p < 2^*$, $2^* = 2N/(N-2)$ when $N > 2$ and $2^* = +\infty$ when $N = 2$, $\mathbb{R}_+^N = \{(x_1, x_2, \dots, x_N) \mid x_N > 0\}$ is the upper half space of \mathbb{R}^N . We put the following conditions on b :

(B₁) $b \in C(\mathbb{R}_+^N)$, and

$$1 = \inf_{x \in \mathbb{R}_+^N} b(x) = \lim_{|x| \rightarrow +\infty, x \in \mathbb{R}_+^N} b(x); \quad (1.2)$$

(B₂) there exists $\delta \in (0, 1)$, such that

$$\lim_{|x| \rightarrow \infty} (b(x) - 1)e^{(1-\delta)|x|} = +\infty. \quad (1.3)$$

Our main result is the following result.

THEOREM 1.1. *If b satisfies (B₁) and (B₂), then (1.1) has a nontrivial solution.*

Similar problems have been studied extensively. In [3], Esteban and Lions showed that $-\Delta u + u = |u|^{p-2}u$, $u \in H_0^1(\mathbb{R}_+^N)$ does not have nontrivial solution.

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However, with the help of the potential $b(x)$ and the comparison technique derived in [2], we proved the existence of a nontrivial solution to problem (1.1).

2. Proof of Theorem 1.1

Let

$$\varphi(u) = \int_{\mathbb{R}_+^N} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}_+^N} b(x) |u|^p dx. \quad (2.1)$$

LEMMA 2.1. Denote $S_p = \inf_{u \in H_0^1(\mathbb{R}_+^N), \|u\|_p=1} \|u\|^2$. Then

$$\inf_{u \in H_0^1(\mathbb{R}_+^N), \|u\|_p=1} \|u\|^2 = S_p. \quad (2.2)$$

LEMMA 2.2. If $0 < c < c^* := (1/2 - 1/p) S_p^{p/(p-2)}$, then φ satisfies $(PS)_c$ condition.

One can see [2] or [5] for the proof of Lemmas 2.1 and 2.2.

LEMMA 2.3. The equation $-\Delta u + u = |u|^{p-2} u$ has a unique positive solution w in \mathbb{R}^N and satisfies the following conditions:

- (i) $w > 0$ in \mathbb{R}^N , $w \in C^\infty(\mathbb{R}^N)$.
- (ii) $w(x) = w(|x|)$.
- (iii) $w(r) r^{(N-1)/2} \exp(r) \rightarrow C > 0$ and $w'(r) r^{(N-1)/2} \exp(r) \rightarrow -C$, where $r = |x| \rightarrow +\infty$.

Proof. See [1, 4]. □

Proof of Theorem 1.1. By Lemma 2.2, we need only to prove that the Mountain Pass value of φ ,

$$c = \inf \left\{ \left(\frac{1}{2} - \frac{1}{p} \right) \left(\frac{\|v\|^2}{\left(\int_{\mathbb{R}_+^N} b(x) |v|^p dx \right)^{2/p}} \right)^{p/(p-2)} ; v \in H_0^1(\mathbb{R}_+^N), v \neq \theta \right\}, \quad (2.3)$$

satisfies that $0 < c < c^*$.

Set

$$\psi_n(x) = \begin{cases} 1, & |x| \leq n-1, \\ n-|x|, & n-1 < |x| \leq n, \\ 0, & |x| > n. \end{cases} \quad (2.4)$$

Let $x_n = (0, 0, \dots, 0, n)$, $u_n = \psi_n(\cdot - x_n) w(\cdot - x_n)$. Then $u_n \in H_0^1(\mathbb{R}_+^N)$, $n = 1, 2, \dots$

Notice that

$$\begin{aligned}
\|u_n\|^2 &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla(\psi_n(\cdot - x_n)w(\cdot - x_n))|^2 dx + \int_{\mathbb{R}^N} |\psi_n(\cdot - x_n)w(\cdot - x_n)|^2 dx \\
&= \int_{B(\theta, n)} |\nabla(\psi_n \cdot w)|^2 dx + \int_{B(\theta, n)} |\psi_n \cdot w|^2 dx \\
&= \int_{B(\theta, n-1)} |\nabla \omega|^2 dx + \int_{B(\theta, n-1)} \omega^2 dx \\
&\quad + \int_{B_{n-1, n}} \psi_n^2 \cdot |\nabla \omega|^2 dx + \int_{B_{n-1, n}} |\nabla \psi_n|^2 \cdot \omega^2 dx \\
&\quad + 2 \int_{B_{n-1, n}} \omega \cdot \psi_n \cdot \nabla \omega \cdot \nabla \psi_n dx + \int_{B_{n-1, n}} \psi_n^2 \cdot \omega^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla \omega|^2 dx + \int_{\mathbb{R}^N} \omega^2 dx \\
&\quad + \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\nabla \omega|^2 dx + \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\omega|^2 dx \\
&\quad + \int_{B_{n-1, n}} \psi_n^2 \cdot |\nabla \omega|^2 dx + 2 \int_{B_{n-1, n}} \omega \cdot \psi_n \cdot \nabla \omega \cdot \nabla \psi_n dx \\
&\quad + \int_{B_{n-1, n}} |\nabla \psi_n|^2 \cdot \omega^2 dx + \int_{B_{n-1, n}} \psi_n^2 \cdot \omega^2 dx,
\end{aligned} \tag{2.5}$$

where $B_{n-1, n} = B(\theta, n) \setminus B(\theta, n-1)$.

From [Lemma 2.3](#) we know that, when n is big enough,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\nabla \omega|^2 dx &= C_1 \int_{n-1}^{+\infty} |\omega'(r)|^2 \cdot r^{N-1} dr \\
&\leq C'_1 \int_{n-1}^{+\infty} e^{-2r} dr = \frac{C'_1}{e^{2(n-1)}}.
\end{aligned} \tag{2.6}$$

Similarly, when n is big enough, we get

$$\int_{\mathbb{R}^N \setminus B(\theta, n-1)} \omega^2 dx \leq \frac{C'_2}{e^{2(n-1)}}. \tag{2.7}$$

So

$$\|u_n\|^2 = \|\omega\|^2 + O\left(\frac{1}{e^{2(n-1)}}\right). \tag{2.8}$$

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Let $a(x) = b(x) - 1$. Then

$$\begin{aligned}
 \int_{\mathbb{R}_+^N} b(x) |u_n|^p dx &= \int_{\mathbb{R}_+^N} |u_n|^p dx + \int_{\mathbb{R}_+^N} a(x) |u_n|^p dx \\
 &= \int_{B(\theta, n-1)} |\omega|^p dx + \int_{B(\theta, n-1)} a(x+x_n) |\omega|^p dx \\
 &\quad + \int_{B_{n-1, n}} a(x+x_n) |\psi_n(x) \cdot \omega(x)|^p dx \tag{2.9} \\
 &= \int_{\mathbb{R}^N} |\omega|^p dx + \int_{B(\theta, n-1)} a(x+x_n) |\omega|^p dx \\
 &\quad + \int_{B_{n-1, n}} a(x+x_n) |\psi_n(x) \cdot \omega(x)|^p dx + \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\omega|^p dx.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_{\mathbb{R}^N \setminus B(\theta, n-1)} |\omega|^p dx &= o\left(\frac{1}{e^{2(n-1)}}\right), \\
 \int_{B_{n-1, n}} a(x+x_n) |\psi_n(x) \cdot \omega(x)|^p dx &= o\left(\frac{1}{e^{2(n-1)}}\right). \tag{2.10}
 \end{aligned}$$

From (B₂), we see that there exists a positive constant λ , such that

$$a(x) \geq \frac{\lambda}{e^{(1-\delta)|x|}}. \tag{2.11}$$

So there exists a positive constant C_3 , such that

$$\int_{\mathbb{R}_+^N} b(x) |u_n|^p dx \geq \int_{\mathbb{R}^N} |\omega|^p dx + \frac{C_3}{e^{2(1-\delta)(n-1)}}. \tag{2.12}$$

Then

$$\begin{aligned}
 &\left(\int_{\mathbb{R}_+^N} b(x) |u_n|^p dx\right)^{2/p} \\
 &\geq \left(\int_{\mathbb{R}^N} |\omega|^p dx + \frac{C_3}{e^{2(1-\delta)(n-1)}}\right)^{2/p} \\
 &= \left(\int_{\mathbb{R}^N} |\omega|^p dx\right)^{2/p} \left(1 + \frac{C_4}{e^{2(1-\delta)(n-1)}}\right)^{2/p} \\
 &= \left(\int_{\mathbb{R}^N} |\omega|^p dx\right)^{2/p} \left(1 + \frac{C_5}{e^{2(1-\delta)(n-1)}} + o\left(\frac{1}{e^{2(1-\delta)(n-1)}}\right)\right). \tag{2.13}
 \end{aligned}$$

Thus

$$\frac{\|u_n\|^2}{\left(\int_{\mathbb{R}_+^N} b(x)|u_n|^p dx\right)^{2/p}} = \frac{\|\omega\|^2}{\left(\int_{\mathbb{R}^N} \omega^p dx\right)^{2/p}} - \frac{C_6}{e^{2(1-\delta)(n-1)}} + o\left(\frac{1}{e^{2(1-\delta)(n-1)}}\right). \quad (2.14)$$

So

$$\frac{\|u_n\|^2}{\left(\int_{\mathbb{R}_+^N} b(x)|u_n|^p dx\right)^{2/p}} < \frac{\|\omega\|^2}{\left(\int_{\mathbb{R}^N} \omega^p dx\right)^{2/p}}, \quad (2.15)$$

when n is big enough.

Thus, we come to the conclusion that

$$c < \left(\frac{1}{2} - \frac{1}{p}\right) S_p^{p/(p-2)}. \quad (2.16)$$

□

Acknowledgement

This work was supported in part by the National Natural Science Foundation and the Fujian Province Natural Science Foundation of China.

References

- [1] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.
- [2] W. Y. Ding and W. M. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Rational Mech. Anal. **91** (1986), no. 4, 283–308.
- [3] M. J. Esteban and P.-L. Lions, *Existence and nonexistence results for semilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. Edinburgh Sect. A **93** (1982/83), no. 1-2, 1–14.
- [4] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.
- [5] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser, Massachusetts, 1996.

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