UNIFORM ASYMPTOTIC NORMAL STRUCTURE, THE UNIFORM SEMI-OPIAL PROPERTY AND FIXED POINTS OF ASYMPTOTICALLY REGULAR UNIFORMLY LIPSCHITZIAN SEMIGROUPS. PART I

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Abstract. In this paper we introduce the uniform asymptotic normal structure and the uniform semi-Opial properties of Banach spaces. This part is devoted to a study of the spaces with these properties. We also compare them with those spaces which have uniform normal structure and with spaces with $WCS(X) > 1$.

1. Introduction

Normal structure is one of the basic concepts in metric fixed point theory. It was introduced by Brodskii and Milman [6] and applied in Kirk's well-known fixed point theorem [24]. Asymptotic normal structure appeared for the first time in a paper by Baillon and Schöneberg [4] in which they generalized Kirk's theorem. The semi-Opial property was considered in the context of the fixed point property in product spaces [25]. To study more carefully the geometric structure of Banach spaces Bynum [9] introduced the normal structure coefficient $N(X)$ which was applied by Casini and Maluta [10] to obtain a fixed point theorem for uniformly lipschitzian mappings. This result has been recently improved by Domínguez Benavides [15] . In his paper he used both $N(X)$ and the weakly convergent sequence coefficient $WCS(X)$ [9]. In the first part of the present paper we introduce new geometric coefficients: the asymptotic normal structure and the semi-Opial coefficients. In the second part of our paper we apply them to the fixed point theory of uniformly lipschitzian nonlinear semigroups.

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2. The asymptotic normal structure and the semi-Opial coefficients

Let $(X, \|\cdot\|)$ be a Banach space. As we mentioned in the Introduction, Bynum [9] introduced the coefficient $N(X)$ related to normal structure. Namely, he defined $N(X)$ as the biggest constant k such that

$$k \cdot r(C) \leq diam(C)$$

for each nonempty bounded convex set $C \subset X$, where $diam(C)$ denotes the diameter of C and $r(C)$ is the Chebyshev radius of C with respect to itself, i.e.,

$$r(C) = \inf_{y \in C} \sup_{x \in C} \|x - y\|.$$

If $\{x_n\}_{n \geq 1}$ is a bounded sequence in $(X, \|\cdot\|)$ and $\{x_{n_i}\}_{i \geq 1}$ is a subsequence, then we denote by $r_a\left(\{x_{n_i}\}_{i \geq 1}\right)$ the asymptotic radius for the norm $\|\cdot\|$ of this subsequence with respect to the set $\overline{conv}\left(\{x_n\}_{n \geq 1}\right)$ (the closure in the norm $\|\cdot\|$ of the convex hull of the whole sequence $\{x_n\}_{n \geq 1}$), i.e.,

$$r_a\left(\{x_{n_i}\}_{i \geq 1}\right) =$$

$$= \inf\left\{ r_a\left(x, \{x_{n_i}\}_{i \geq 1}\right) = \limsup_i \|x - x_{n_i}\| : x \in \overline{conv}\left(\{x_n\}_{n \geq 1}\right)\right\}.$$

Throughout this paper we will use the following notation:

$$\alpha_k = diam_{\|\cdot\|}\left(\{x_n\}_{n \geq k}\right), \qquad diam_a(\{x_n\}) = \lim_k \alpha_k = \alpha.$$

One can consider (see [9] and [2]) the following weakly convergent sequence coefficient:

$$WCS(X) = \sup\{k : k \cdot r_a(\{x_n\}) \leq diam(\{x_n\})$$

$$\text{for every weakly convergent sequence } \{x_n\} \text{ in } X\} =$$

$$= \sup\left\{k : k \cdot \limsup_n \|x_n\| \leq diam(\{x_n\}) \right.$$

$$\left. \text{for every weakly null sequence } \{x_n\} \text{ in } X \right\}.$$

Let us observe that in the above definition of $WCS(X)$, $diam(\{x_n\})$ can be replaced by $diam_a(\{x_n\})$ and that our definition is a little different from the one in common use.

We always have

$$1 \leq N(X) \leq WCS(X),$$

and for some Banach spaces (see e.g. [15]) the strict inequalities

$$1 < N(X) < WCS(X)$$

are valid.

Recall that a bounded sequence $\{x_n\}_{n \geq 1}$ with $x_n - x_{n+1} \to 0$ is called asymptotically regular.

We say that X has asymptotic normal structure (with respect to the weak topology) [4], ANS (respectively, w-ANS) for short, if for each bounded closed (weakly compact) and convex subset C of X consisting of more than one point and each asymptotically regular sequence $\{x_n\}$ in C, there is a point $x \in C$ such that

$$\liminf_n \|x - x_n\| < diam\,(C)$$

(see also [1, 2, 7, 8, 19, 20, 26, 30, 36]).

Recall that a Banach space is said to have the semi-Opial (weak semi-Opial) property [8, 25], SO (w-SO) for short, if for each bounded nonconstant asymptotically regular sequence $\{x_n\}$ (with a weakly compact convex hull), there exists a subsequence $\{x_{n_i}\}$, weakly convergent to x, such that

$$\liminf_i \|x - x_{n_i}\| < diam\,(\{x_n\})\,.$$

Let us observe that in Examples 1 and 5 on page 461 in [25] the authors use, in fact, the weak semi-Opial property. Similarly in Theorem 4 in [25] we can assume that $(X_2, \|\cdot\|)$ has the weak semi-Opial property.

A Banach space X is said to satisfy the Opial condition [32] (respectively, the nonstrict Opial condition [22]) if whenever a sequence $\{x_n\}$ in X converges weakly to x, then

$$\liminf_n \|x - x_n\| < \liminf_n \|y - x_n\|$$

$$\left(\liminf_n \|x - x_n\| \le \liminf_n \|y - x_n\|\right)$$

for every $y \in X \setminus \{x\}$.

For more information about the connections between the above mentioned geometric properties of Banach spaces (and other ones) see [1, 2, 3, 13, 14, 18, 19, 20, 27, 29, 33, 34, 35, 37, 38, 39, 40].

We now define the asymptotic normal structure coefficient by

$$\sup\left\{k : k \cdot \inf_{\{x_{n_i}\}_{i\ge1}} r_a\left(\{x_{n_i}\}_{i\ge1}\right) \le diam_a\left(\{x_n\}\right)\right.$$
$$\left.\text{for each bounded sequence } \{x_n\}_{n\ge1} \text{ with } x_n - x_{n+1} \to 0\right\}.$$

We denote it by $AN\,(X)$.

If in the definition of $AN\,(X)$ we add the condition that the sequence $\{x_n\}_{n\ge1}$ has a weakly compact $\overline{conv}\left(\{x_n\}_{n\ge1}\right)$, then we get the asymptotic normal structure coefficient with respect to the weak topology, w-$AN\,(X)$, for short. In other words,

$$w\text{-}AN\,(X) = \sup\left\{k : k \cdot \inf_{\{x_{n_i}\}_{i\ge1}} r_a\left(\{x_{n_i}\}_{i\ge1}\right) \le diam_a\left(\{x_n\}\right)\right.$$
$$\text{for each sequence } \{x_n\}_{n\ge1} \text{ such that}$$

$$\left.\overline{conv}\left(\{x_n\}_{n\geq 1}\right) \text{ is weakly compact and } x_n - x_{n+1} \to 0\right\}.$$

The semi-Opial coefficient with respect to the weak topology, w-SOC for short, is defined as follows:

$$w\text{-}SOC\,(X) = \sup\left\{k : k \cdot \inf_{\{x_{n_i}\}_{i\geq 1}, x_{n_i} \rightharpoonup y} r_a\left(y, \{x_{n_i}\}_{i\geq 1}\right) \leq diam_a\left(\{x_n\}\right)\right.$$

$$\text{for each sequence } \{x_n\}_{n\geq 1} \text{ such that}$$

$$\left.\overline{conv}\left(\{x_n\}_{n\geq 1}\right) \text{ is weakly compact and } x_n - x_{n+1} \to 0\right\}.$$

If $AN\,(X) > 1$, then we say that $(X, \|\cdot\|)$ has uniform asymptotic normal structure, UAN for short. If w-$AN\,(X) > 1$, then we say that $(X, \|\cdot\|)$ has uniform asymptotic normal structure with respect to the weak topology (w-UAN). Similarly, if w-$SOC\,(X) > 1$, then $(X, \|\cdot\|)$ has the uniform semi-Opial property with respect to the weak topology (w-USO).

Directly from the above definitions we get

$$1 \leq AN\,(X) \leq w\text{-}AN\,(X)\,,$$

$$(1) \qquad 1 \leq WCS\,(X) \leq w\text{-}SOC\,(X) \leq w\text{-}AN\,(X)\,.$$

We do not know if w-$AN\,(X)$ is different from w-$SOC\,(X)$, but we will present an example of a Banach space with $1 < WCS\,(X) < w\text{-}SOC\,(X)$ (Example 6.2). There are Banach spaces which have asymptotic normal structure but lack UAN, and there are also Banach spaces with $1 = AN\,(X) < w\text{-}AN\,(X)$ (Example 6.1).

Proposition 2.1. *In the definitions of w-$AN\,(X)$ and w-$SOC\,(X)$ we can replace $diam_a\left(\{x_n\}\right)$ by $diam\left(\{x_n\}\right)$.*

Proof. Let us observe that in the above definition every asymptotically regular sequence $\{x_n\}$ can be replaced by $\{x_n\}_{n\geq m}$ with arbitrary m. This yields the claimed statement. ∎

Theorem 2.1. *If a Banach space $(X, \|\cdot\|)$ has $AN\,(X) > 1$, then it is reflexive.*

Proof. It is sufficient to recall the following result of D.P. Milman and V.D. Milman [31]: If a Banach space $(X, \|\cdot\|)$ is not reflexive, then for each $\epsilon > 0$ there exists a sequence $\{y_n\}$ with the following properties:

1. $\|y_n\| = 1$ for $n = 1, 2, ...$;
2. $1 + \epsilon \geq \|z_{1j} - z_{j\omega}\| \geq 1 - \epsilon$ for each $j = 1, 2, ...$ and for each $z_{1j} \in \overline{conv}\left(\{y_n\}_{n=1}^{j}\right)$ and $z_{j\omega} \in \overline{conv}\left(\{y_n\}_{n=j+1}^{\infty}\right)$;
3. $1 - \epsilon \leq \|z_{1j}\| \leq 1$ and $1 - \epsilon \leq \|z_{j\omega}\| \leq 1$ for each $z_{1j} \in \overline{conv}\left(\{y_n\}_{n=1}^{j}\right)$ and $z_{j\omega} \in \overline{conv}\left(\{y\}_{n=j+1}^{\infty}\right)$.

Hence, if a Banach space $(X, \|\cdot\|)$ is not reflexive and $\epsilon > 0$, then we can choose elements y_n which satisfy the above conditions 1.-3. and next we construct an asymptotically regular sequence $\{x_n\}$ by dividing every segment $[y_n, y_{n+1}]$ into 2^n equal subsegments and taking their endpoints as subsequent elements of $\{x_n\}$. For this bounded sequence $\{x_n\}_{n\geq 1}$ we have $x_n - x_{n+1} \to 0$ and $\frac{1+\epsilon}{1-\epsilon} \cdot r_a \left(y, \{x_{n_i}\}_{i\geq 1}\right) \geq diam_a (\{x_n\})$ for each subsequence $\{x_{n_i}\}_{i\geq 1}$ and $y \in \overline{conv} \{x_n\}$. ∎

Remark 2.1. *The condition $w\text{-}AN(X) > 1$ implies the weak fixed point property for nonexpansive mappings as a consequence of the Baillon-Schöneberg theorem (see also* [7]*).*

Theorem 2.2. *i) If a Banach space $(X, \|\cdot\|)$ has $N(X) > 1$, then $w\text{-}SOC(X) > 1$.*

ii) If a Banach space $(X, \|\cdot\|)$ has the nonstrict Opial property, then $w\text{-}SOC(X) = w\text{-}AN(X)$.

Proof. i) If $N(X) > 1$, then X is reflexive [29], and

$$1 < N(X) \leq WCS(X) \leq w\text{-}SOC(X)$$

(see (1) and [33]).

ii) We get this equality directly from the definition of the nonstrict Opial property. ∎

Remark 2.2. *There exist w-USO spaces without the nonstrict Opial property. For example, $L^p([0, 2\pi])$ with $1 < p < \infty$ and $p \neq 2$ is such a space. It has uniform normal structure, and thus (see point i) in the above theorem) it is w-USO, but it does not satisfy the nonstrict Opial condition* [32].

We finish this section by showing the stability of the uniform asymptotic normal structure and the uniform semi-Opial properties.

Theorem 2.3. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be isomorphic Banach spaces and let $d(X_1, X_2)$ be the Banach-Mazur distance between them. Then we have*

$$AN(X_1) \leq d(X_1, X_2) \cdot AN(X_2),$$
$$w\text{-}AN(X_1) \leq d(X_1, X_2) \cdot w\text{-}AN(X_2),$$

and

$$w\text{-}SOC(X_1) \leq d(X_1, X_2) \cdot w\text{-}SOC(X_2).$$

Proof. All the inequalities have similar proofs. For example, we prove the third one:

$$w\text{-}SOC(X_1) \leq d(X_1, X_2) \cdot w\text{-}SOC(X_2).$$

Let $\{x_n\}$ be asymptotically regular in X_2, and let $\overline{conv}\{x_n\}$ be weakly compact. Let $T : X_2 \to X_1$ be an isomorphism and assume $0 < k < w\text{-}SOC(X_1)$. Then there exists a weakly convergent to y subsequence $\{Tx_{n_i}\}$ such that

$$kr_a \left(T^{-1}y, \{x_{n_i}\}_{i\geq 1}\right) \leq k \left\|T^{-1}\right\| r_a \left(y, \{Tx_{n_i}\}_{i\geq 1}\right)$$

$$\leq \left\| T^{-1} \right\| \cdot diam_a \left(\{Tx_n\} \right) \leq \left\| T^{-1} \right\| \cdot \|T\| \cdot diam_a \left(\{x_n\} \right).$$

Hence we get

$$\frac{k}{\|T^{-1}\| \cdot \|T\|} \leq w\text{-}SOC\,(X_2)$$

which yields the claimed inequality. ∎

Remark 2.3. *Theorem 2.3 can be understood as a stability result for the weak fixed point property for nonexpansive mappings. This means that if* $w\text{-}AN\,(X_1) > 1$ *and* $d\,(X_1, X_2) < w\text{-}AN\,(X_1)$, *then* $w\text{-}AN\,(X_2) > 1$, *and by Remark 2.1 the space* X_2 *also has the weak fixed point property for nonexpansive mappings.*

3. Connections between asymptotically regular sequences and properties of Banach spaces

It is natural to ask, when either $AN\,(X)$ or $w\text{-}SOC\,(X)$ is equal to ∞. The following theorem gives the answer.

Theorem 3.1. *i)* $AN\,(X) = \infty$ *if and only if* $(X, \|\cdot\|)$ *is finite dimensional.*
ii) $w\text{-}SOC\,(X) = \infty$ *if and only if* $(X, \|\cdot\|)$ *is a Schur space.*
iii) $w\text{-}AN\,(X) = \infty$ *if and only if* $(X, \|\cdot\|)$ *is a Schur space.*

Proof. i) The equality $AN\,(X) = \infty$ is equivalent to the following

$$\sup \left\{ \inf_{\{x_{n_i}\}_{i\geq 1}} r_a \left(\{x_{n_i}\}_{i\geq 1} \right) : \{x_n\}_{n\geq 1} \text{ is bounded and } x_n - x_{n+1} \to 0 \right\} = 0$$

If X is finite dimensional, then the above equality is obvious.

When X is infinite dimensional, then by the Riesz Lemma [12] there exists a sequence $\{y_n\}$ such that

$$\|y_n\| = 1 \quad \text{for n=1,2,...}$$

and for $n = 1, 2, ...$

$$\|y - y_{n+1}\| \geq 1 - \frac{1}{n+1} \qquad \text{for every } y \in lin\,\{y_1, y_2, ..., y_n\}.$$

Let us observe that

$$\liminf_n \|x - y_n\| \geq 1$$

for each $x \in lin\,\{y_n\}$. Now we construct a new sequence $\{x_n\}$ in the following way. We divide each segment $[y_n, y_{n+1}]$ into 2^n equal parts and take the endpoints as subsequent elements of $\{x_n\}$. This sequence satisfies $x_n - x_{n+1} \to 0$. We will show that for each $x \in lin\,\{x_n\}$ we have

$$\inf_{\{x_{n_i}\}_{i\geq 1}} r_a \left(x, \{x_{n_i}\}_{i\geq 1} \right) \geq \frac{1}{4}. \tag{2}$$

Indeed, every x_n can be written in the following way:

$$x_n = \alpha_n y_{k(n)} + (1 - \alpha_n)\, y_{k(n)+1},$$

where $0 \leq \alpha_n \leq 1$. If we choose any subsequence $\{x_{n_i}\}_{i\geq 1}$, then without loss of generality we may assume that $\alpha_{n_i} \to \alpha$. It is obvious that $k(n) \to \infty$ and therefore $k(n_i) \to \infty$ too.

First we claim that for $0 \leq \alpha \leq \frac{3}{4}$ and for each $x \in lin\{x_n\} = lin\{y_n\}$ we have

$$\liminf_i \|x - x_{n_i}\| \geq \frac{1}{4}. \tag{3}$$

Indeed, for such an α we get

$$\begin{aligned}\liminf_i \|x - x_{n_i}\| &= \liminf_i \left\|x - \alpha_{n_i} y_{k(n_i)} - (1-\alpha_{n_i})\, y_{k(n_i)+1}\right\| \\ &\geq \lim_i (1-\alpha_{n_i})\left(1 - \frac{1}{k(n_i)+1}\right) = 1 - \alpha \geq \frac{1}{4}.\end{aligned}$$

Next we obtain

$$\begin{aligned}\liminf_i \|x - x_{n_i}\| &= \liminf_i \left\|x - \alpha_{n_i} y_{k(n_i)} - (1-\alpha_{n_i})\, y_{k(n_i)+1}\right\| \\ &\geq \liminf_i \left\|x - \alpha_{n_i} y_{k(n_i)}\right\| - \lim_i \left\|(1-\alpha_{n_i})\, y_{k(n_i)+1}\right\| \\ &\geq \lim_i \alpha_{n_i}\left(1 - \frac{1}{k(n_i)}\right) - \lim_i (1-\alpha_{n_i}) = 2\alpha - 1 \geq \frac{1}{2}\end{aligned} \tag{4}$$

for $\frac{3}{4} \leq \alpha \leq 1$ and for each $x \in lin\{x_n\}$.

Hence (3) and (4) imply that the inequality (2) is valid and therefore

$$\sup\left\{\inf_{\{x_{n_i}\}_{n\geq 1}} r_a\left(\{x_{n_i}\}_{i\geq 1}\right) : \{x_n\}_{n\geq 1} \text{ is bounded and } x_n - x_{n+1} \to 0\right\} \geq \frac{1}{4}.$$

This means that $AN(X) < \infty$.

ii) If $(X, \|\cdot\|)$ is a Schur space [12], then the following equality

$$\sup\left\{\inf\left\{r_a\left(y, \{x_{n_i}\}_{i\geq 1}\right) : \{x_{n_i}\}_{i\geq 1} \text{ is weakly convergent and } y = w\text{-}\lim_i x_{n_i}\right\} : \{x_n\}_{n\geq 1} \text{ with a weakly compact } \overline{conv}\left(\{x_n\}_{n\geq 1}\right) \text{ and } x_n - x_{n+1} \to 0\right\} = 0$$

is obvious.

Let us assume that $(X, \|\cdot\|)$ is not a Schur space. We will show

$$\sup\left\{\inf\left\{r_a\left(y, \{x_{n_i}\}_{i\geq 1}\right) : \{x_{n_i}\}_{i\geq 1} \text{ is weakly convergent and } y = w\text{-}\lim_i x_{n_i}\right\} : \{x_n\}_{n\geq 1} \text{ with a weakly compact } \overline{conv}\left(\{x_n\}_{n\geq 1}\right) \text{ and } x_n - x_{n+1} \to 0\right\} \geq \frac{1}{8}.$$

In X there exists a weakly null sequence $\{y_n\}$ with $\|y_n\| = 1,\ n = 1, 2, \ldots$. Therefore, we can choose a subsequence $\{y_{n_k}\}$ such that for every $y \in [y_{n_k}, y_{n_{k+1}}]$ we have $\|y\| \geq \frac{1}{8}$. Indeed, we take $y_{n_1} = y_0$ and next if we have chosen $y_{n_1}, ..., y_{n_k}$, then we take $n_{k+1} > n_k$ so large that

$$\left\|(1-\alpha)\, y_{n_k} + \alpha y_{n_{k+1}}\right\| \geq \frac{1}{8}$$

for every $0 \leq \alpha \leq \frac{3}{4}$. This is possible because by the lower semicontinuity of $\|\cdot\|$ with respect to the weak topology we get

$$\liminf_n \|(1-\alpha)\, y_{n_k} + \alpha y_n\| \geq (1-\alpha)\, \|y_{n_k}\| \geq \frac{1}{4}.$$

Now for this $y_{n_{k+1}}$ and each $\frac{3}{4} \leq \alpha \leq 1$ we also have

$$\left\|(1-\alpha)\, y_{n_k} + \alpha y_{n_{k+1}}\right\| \geq \left\|\alpha y_{n_{k+1}}\right\| - \|(1-\alpha)\, y_{n_k}\| = 2\alpha - 1 \geq \frac{1}{2}.$$

Now we construct an asymptotically regular sequence $\{x_n\}$ by dividing every segment $[y_{n_k}, y_{n_{k+1}}]$ into 2^n equal parts and then taking the endpoints as subsequent elements of $\{x_n\}$. It is obvious that

$$\inf_{\{x_{n_i}\}_{i\geq 1}} r_a\left(0, \{x_{n_i}\}_{i\geq 1}\right) \geq \frac{1}{8}$$

and the proof is complete.

iii) Assume that X is not Schur. We will show that

$$\sup\left\{ \inf_{\{x_{n_i}\}_{i\geq 1}} r_a\left(\{x_{n_i}\}_{i\geq 1}\right) : \{x_n\}_{n\geq 1} \text{ with a weakly compact } \overline{conv}\left(\{x_n\}_{n\geq 1}\right) \text{ and } x_n - x_{n+1} \to 0 \right\} > 0.$$

We use the asymptotically regular sequence $\{x_n\}$ constructed in the proof of ii). We know that $\|x_n\| \geq \frac{1}{8}$ for each n and that $w\text{-}\lim x_n = 0$. Now we prove that

$$\liminf_n \|x - x_n\| \geq \frac{1}{16}$$

for every $x \in X$. Indeed, if $\|x\| \geq \frac{1}{16}$, then by the lower semicontinuity of $\|\cdot\|$ with respect to the weak topology we get

$$\liminf_n \|x - x_n\| \geq \|x\| \geq \frac{1}{16}.$$

On the other hand for $\|x\| \leq \frac{1}{16}$ we obtain

$$\liminf_n \|x - x_n\| \geq \liminf_n \left(\|x_n\| - \|x\|\right) \geq \frac{1}{16}$$

and this completes the proof. ∎

We end this section with a characterization of reflexive spaces by asymptotically regular sequences.

Theorem 3.2. *A Banach space $(X, \|\cdot\|)$ is reflexive if and only if every asymptotically regular sequence has a weakly convergent subsequence.*

Proof. It is known [11] that in reflexive spaces each bounded sequence has a weakly convergent subsequence.
Let us now assume that in the Banach space $(X, \|\cdot\|)$ every asymptotically regular sequence has a weakly convergent subsequence. To get the reflexivity of $(X, \|\cdot\|)$ it is sufficient to prove ([11]) that each decreasing sequence $\{C_n\}$ of nonempty, bounded, closed and convex sets has a nonempty intersection. Without loss of generality we can assume that $diam\,(C_n) > 0$ for each n. Now we choose y_n from each set C_n and next we construct an asymptotically regular sequence $\{x_n\}$ by dividing every segment $[y_n, y_{n+1}]$ into 2^n equal parts and the taking the endpoints as subsequent elements of $\{x_n\}$. This sequence contains a weakly convergent subsequence $\{x_{n_i}\}$. Its weak limit is a common element of C_n for $n = 1, 2, \ldots$. ■

Remark 3.1. *A proof similar to the above one was used in* [8] *to prove that every Banach space with the SO property is reflexive.*

4. On the 3-space problem

In this section we consider the following problem: When can the uniform asymptotic normal structure property or the uniform semi-Opial property be extended from a subspace to the whole space? Two slightly different approaches to the solution of this problem will be demonstrated in the following theorems.

Theorem 4.1. *Suppose that $X = W \oplus Z$, where W is a closed subspace of X, Z is a Schur space, and the projection onto W has norm 1. Then we have $w\text{-}SOC\,(X) = w\text{-}SOC\,(W)$.*

Proof. Suppose $\{x_n\} = \{w_n + z_n\}$ is an asymptotically regular sequence, $w_n \in W$, $z_n \in Z$ for $n = 1, 2, ...$ and $\overline{conv}\,\{x_n\}$ is weakly compact. For each $k < w\text{-}SOC\,(W)$ we find a subsequence $\{x_{n_i}\}$ such that

$$x_{n_i} = w_{n_i} + z_{n_i} \rightharpoonup w + z, \qquad w \in W, z \in Z$$

and

$$k \lim_i \|w_{n_i} - w\| \leq diam_a\,\{w_n\}\,.$$

Then we have $w + z \in \overline{conv}\,\{x_n\}$, $z_{n_i} \to z$ and

$$k \lim_i \|w_{n_i} + z_{n_i} - w - z\| = k \lim_i \|w_{n_i} - w\| \leq diam_a\,\{w_n\}\,.$$

Now, since the projection on W is of norm 1, we obtain

$$k \lim_i \|w_{n_i} + z_{n_i} - w - z\| \leq diam_a\,\{w_n\} \leq diam_a\,\{x_n\}$$

and therefore $w\text{-}SOC\,(X) = w\text{-}SOC\,(W)$. ■

Now we consider the Cartesian product of two spaces.

Theorem 4.2. *Let* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *be Banach spaces. If* $(X_1, \|\cdot\|_1)$ *is w-USO and* $(X_2, \|\cdot\|_2)$ *has* $WCS(X_2) > 1$*, then* $X_1 \times X_2$ *equipped with the* l_p*-norm* $\|\cdot\| = (\|\cdot\|_1^p + \|\cdot\|_2^p)^{\frac{1}{p}}$ $(1 \leq p < \infty)$ *is also w-USO.*

Proof. Let $0 < \theta < 1$ be such that $\frac{1}{\theta} < \min(w\text{-}SOC(X_1), WCS(X_2))$. Let us take an arbitrary asymptotically regular sequence $\{x_n\} = \{(x_{1n}, x_{2n})\}$ in $(X_1 \times X_2, \|\cdot\|)$ with a weakly compact $\overline{conv}\{x_n\}$. Then $\{x_{1n}\}$ is also asymptotically regular in $(X_1, \|\cdot\|_1)$ and we can choose a subsequence $\{x_{n_i}\}$ such that $\{x_{n_i}\}$ tends weakly to (x_1, x_2) (see [13, 17, 33, 40]) and

$$\begin{aligned}
d &= diam_a\{x_n\} \\
&\geq diam_a\{x_{n_i}\} = \lim_{\substack{i,k\to\infty \\ i\neq k}} \|x_{n_i} - x_{n_k}\| = \lim_{i\to\infty}\lim_{k\to\infty} \|x_{n_i} - x_{n_k}\|, \\
r &= \lim_{i\to\infty} \|x_{n_i} - x\|, \\
d_1 &= diam_a\{x_{1n}\} \\
\geq \overline{d}_1 &= diam_a\{x_{1n_i}\} = \lim_{\substack{i,k\to\infty \\ i\neq k}} \|x_{1n_i} - x_{1n_k}\|_1 = \lim_{i\to\infty}\lim_{k\to\infty} \|x_{1n_i} - x_{1n_k}\|_1, \\
r_1 &= \lim_{i\to\infty} \|x_{1n_i} - x_1\|_1 \leq \theta d_1, \\
\overline{d}_2 &= diam_a\{x_{2n_i}\} = \lim_{\substack{i,k\to\infty \\ i\neq k}} \|x_{2n_i} - x_{2n_k}\|_2 = \lim_{i\to\infty}\lim_{k\to\infty} \|x_{2n_i} - x_{2n_k}\|_2, \\
r_2 &= \lim_{i\to\infty} \|x_{2n_i} - x_2\|_2.
\end{aligned}$$

Let us observe that

$$d_1 \leq d,$$

$$r^p = r_1^p + r_2^p \leq \overline{d}_1^p + \overline{d}_2^p \leq d^p,$$

$$r_1 \leq \theta d_1 \leq \theta d \tag{5}$$

and

$$r_2 \leq \theta \overline{d}_2.$$

Now we have to consider two possibilities: either

$$r_1^p + \overline{d}_2^p \leq \frac{1 + 3\theta^p}{4} d^p$$

or

$$r_1^p + \overline{d}_2^p \geq \frac{1 + 3\theta^p}{4} d^p.$$

For the first possibility we obtain

$$r^p = r_1^p + r_2^p \leq r_1^p + \overline{d}_2^p \leq \frac{1 + 3\theta^p}{4} d^p. \tag{6}$$

For the second possibility we have

$$\overline{d}_2^p \geq \frac{1 + 3\theta^p}{4} d^p - r_1^p \geq \frac{1 + 3\theta^p}{4} d^p - \theta^p d^p = \frac{1 - \theta^p}{4} d^p$$

by (5) and therefore we get

$$\begin{aligned} r^p &= r_1^p + r_2^p \le r_1^p + \theta^p \overline{d}_2^p \\ &\le \overline{d}_1^p + \overline{d}_2^p - (1-\theta^p)\,\overline{d}_2^p \le d^p - \frac{(1-\theta^p)^2}{4} d^p \qquad (7) \\ &= \left[1 - \frac{(1-\theta^p)^2}{4}\right] d^p. \end{aligned}$$

Finally, inequalities (6) and (7) imply

$$\begin{aligned} r &\le \max\left\{\left(\frac{1+3\theta^p}{4}\right)^{\frac{1}{p}}, \left[1-\frac{(1-\theta^p)^2}{4}\right]^{\frac{1}{p}}\right\} d \\ &= \left[1-\frac{(1-\theta^p)^2}{4}\right]^{\frac{1}{p}} d. \end{aligned}$$

This completes the proof. ■

5. The space X_β^p and its w-*SOC*

In this section we give an example of a space with $N(X) < AN(X) < w\text{-}SOC(X)$. To this end, let us consider l^p with the norm

$$\|x\| = \max\left\{\|x\|_\infty, \frac{\|x\|_p}{\beta}\right\},$$

where $p > 1$, $1 < \beta < +\infty$, $\|x\|_\infty = \max\{|x(j)| : j = 1,2,...\}$ and $\|x\|_p = \left(\sum_{j=1}^\infty |x(j)|^p\right)^{\frac{1}{p}}$. We denote this space by X_β^p. The space X_β^2 was introduced by R.C. James [5]. This is essentially the space which has been discussed in various places in the literature, e.g., [1, 2, 4, 5, 7, 8, 10, 15, 16, 19, 20, 21, 22, 23, 25, 26, 28, 39].

For the convenience of the reader we recall the notations from Section 2. If $\{x_n\}_{n\ge1}$ is a bounded sequence in $(l^p, \|\cdot\|)$ and $\{x_{n_i}\}_{i\ge1}$ is a subsequence, then $r_a\left(\{x_{n_i}\}_{n\ge1}\right)$ denote the asymptotic radius of this sequence with respect to the set $\overline{conv}\left(\{x_n\}_{n\ge1}\right)$ in the norm $\|\cdot\|$. We also have

$$\alpha_k = diam_{\|\cdot\|}\left(\{x_n\}_{n\ge k}\right) \quad \text{and} \quad diam_a(\{x_n\}) = \lim_k \alpha_k = \alpha.$$

Let us observe that for each $n \in \mathbb{N}$ and for each $y \in C$ there exists an index $j_{n,y}$
(we fix it here for every pair n,y) such that

$$\|x_n - y\|_\infty = |x_n(j_{n,y}) - y(j_{n,y})| \qquad (8)$$

$(x_n = (x_n(j))_{j\ge1}$ and $y = (y(j))_{j\ge1})$.

The space X_β^p has the nonstrict Opial property [22].

Theorem 5.1. *If a sequence* $\{x_n\}_{n\geq 1}$ *is bounded and* $x_n - x_{n+1} \to 0$, *then*

$$\begin{aligned}
&\inf_{x\in\overline{conv}\left(\{x_n\}_{n\geq 1}\right)} \left(\liminf_n \|x_n - x\|\right) \\
(9) \qquad &= \inf_{\{x_{n_i}\}} \left[r_a\left(\{x_{n_i}\}_{i\geq 1}\right)\right] \\
&\leq \min\left[1, \max\left(2^{-\frac{1}{p}}, \frac{\beta}{4^{\frac{1}{p}}}\right)\right] \cdot diam_a\left(\{x_n\}\right)
\end{aligned}$$

and this constant is the best possible. Therefore

$$w\text{-}SOC\left(X_\beta^p\right) = \max\left[1, \min\left(2^{\frac{1}{p}}, \frac{4^{\frac{1}{p}}}{\beta}\right)\right].$$

Proof. We begin our proof in the case $1 < \beta < 4^{\frac{1}{p}}$.

Let

$$C = \overline{conv}\left(\{x_n\}_{n\geq 1}\right) \quad \text{and} \quad C_k = \overline{conv}\left(\{x_n\}_{n\geq k}\right), \quad k = 1, 2, ...$$

Clearly $diam C_k = \alpha_k$. Let us observe the following fact. For every subsequence $\{x_{n_i}\}$ which is weakly convergent to y we have ([2, 39])

$$\begin{aligned}
(10) \qquad \liminf_i \|y - x_{n_i}\|_p &\leq \limsup_i \|y - x_{n_i}\|_p \leq \frac{1}{2^{\frac{1}{p}}} diam_{a,\|\cdot\|_p}\left(\{x_{n_i}\}\right) \\
&\leq \lim_k \frac{1}{2^{\frac{1}{p}}} diam_{\|\cdot\|_p} C_k \leq \lim_k \frac{\beta}{2^{\frac{1}{p}}} diam_{\|\cdot\|} C_k = \frac{\beta}{2^{\frac{1}{p}}}\alpha.
\end{aligned}$$

Next choosing in an arbitrary way a subsequence $\left\{x_{n_{i_l}}\right\}$ such that

$$\liminf_i \|y - x_{n_i}\|_\infty = \lim_l \left\|y - x_{n_{i_l}}\right\|_\infty,$$

we get

$$\begin{aligned}
\liminf_i \|y - x_{n_i}\| &\leq \liminf_l \max\left\{\left\|y - x_{n_{i_l}}\right\|_\infty, \frac{\left\|y - x_{n_{i_l}}\right\|_p}{\beta}\right\} \\
(11) \qquad &\leq \max\left\{\lim_l \left\|y - x_{n_{i_l}}\right\|_\infty, \limsup_l \frac{\left\|y - x_{n_{i_l}}\right\|_p}{\beta}\right\} \\
&\leq \max\left\{\lim_l \left\|y - x_{n_{i_l}}\right\|_\infty, \frac{\alpha}{2^{\frac{1}{p}}}\right\} \\
&= \max\left\{\liminf_i \|y - x_{n_i}\|_\infty, \frac{\alpha}{2^{\frac{1}{p}}}\right\}.
\end{aligned}$$

Now we can begin the proof of our inequality (9). For $1 < \beta < 4^{\frac{1}{p}}$ it reduces to

$$\inf_{x \in \overline{conv}(\{x_n\}_{n\geq 1})} \left(\liminf_n \|x_n - x\|\right) \leq \max\left(2^{-\frac{1}{p}}, \frac{\beta}{4^{\frac{1}{p}}}\right) \cdot diam_a(\{x_n\}).$$

Without loss of generality we can assume that $\alpha > 0$, and this implies that for each k we have $\alpha_k \geq \alpha > 0$. Suppose that for some asymptotically regular sequence $\{x_n\}$ and for some t with $\max\left(\frac{1}{2^{\frac{1}{p}}}, \frac{\beta}{4^{\frac{1}{p}}}\right) < t < 1$ the following inequality is valid:

$$\inf_{x \in C} \left(\liminf_n \|x_n - x\|\right) > t\alpha. \tag{12}$$

We will try to reach a contradiction. For our t there exists $\epsilon > 0$ such that the inequalities

$$\epsilon < t\alpha \quad \text{and} \quad \frac{\beta^p}{2}\alpha^p + \epsilon < 2(t\alpha - \epsilon)^p \tag{13}$$

are valid.

Let us take an arbitrary subsequence $\{x_{n_i}\}$ which converges weakly to some y. Directly from the definition of the norm $\|\cdot\|$, by (11) and by $t > \frac{1}{2^{\frac{1}{p}}}$, we have

$$\begin{aligned} t\alpha < \ &\liminf_n \|x_n - y\| \leq \liminf_i \|x_{n_i} - y\| \\ &\leq \max\left\{\liminf_i \|x_{n_i} - y\|_\infty, \frac{\alpha}{2^{\frac{1}{p}}}\right\} = \liminf_i \|x_{n_i} - y\|_\infty \\ &\leq \liminf_i \max\left\{\|x_{n_i} - y\|_\infty, \frac{\|x_{n_i} - y\|_p}{\beta}\right\} = \liminf_i \|x_{n_i} - y\|, \end{aligned}$$

which implies

$$\liminf_i \|x_{n_i} - y\|_\infty = \liminf_i \|x_{n_i} - y\| > t\alpha. \tag{14}$$

By formulas (8) and (14) and because $\{x_{n_i}\}$ tends weakly to y we get

$$\lim_i j_{n_i,y} = +\infty. \tag{15}$$

Using (8, 14, 15) and $\lim_n \|x_n - x_{n+1}\| = 0$ we can find $\widetilde{n}$ and $\widetilde{i}$, $\widetilde{i} \geq \widetilde{n}$, such that for $n \geq \widetilde{n}$ and $i \geq \widetilde{i}$ we have

$$\|x_n - x_{n+1}\|_\infty \leq \frac{\epsilon}{3}, \tag{16}$$

$$\|x_{n_i} - y\|_\infty = |x_{n_i}(j_{n_i,y}) - y(j_{n_i,y})| > t\alpha, \tag{17}$$

and

$$|y(j)| \leq \frac{\epsilon}{3} \tag{18}$$

for each $j \geq \min_{i \geq \widetilde{i}} j_{n_i,y}$. Therefore (17) and (18) yield

$$\|x_{n_i}\| \geq \|x_{n_i}\|_\infty \geq |x_{n_i}(j_{n_i,y})| \geq t\alpha - \frac{\epsilon}{3} \tag{19}$$

for $i \geq \widetilde{i}$.

Now let us return to the sequence $\{x_n\}$ and set

$$j_n = \begin{cases} \max\{j : |x_n(j)| \geq t\alpha - \frac{\epsilon}{3}\} & \text{if there exists } j \\ & \text{such that } |x_n(j)| \geq t\alpha - \frac{\epsilon}{3}, \\ \max\{j : |x_n(j)| = \|x_n\|_\infty\} & \text{otherwise.} \end{cases} \tag{20}$$

We claim that

$$\lim_n j_n = +\infty. \tag{21}$$

If this were false, then there would exist a subsequence $\{n_i\}$ with a bounded $\{j_{n_i}\}$. We could then choose a subsequence $\left\{x_{n_{i_l}}\right\}$ which tends weakly to its weak limit y. In this case (see formulas (19) and (20)) we have

$$j_{n_{i_l},y} \leq j_{n_{i_l}}$$

for all l greater than or equal to some $\widetilde{l}$ and therefore (see (15)) $\lim_l j_{n_{i_l}} = +\infty$ which contradicts our assumption.

Since $\lim_n j_n = +\infty$ (see (21)), there exists a subsequence $\{n_i\}$ such that $\{x_{n_i}\}$ converges weakly to some y and

$$j_{n_i} < j_{n_{i+1}} \tag{22}$$

for $i = 1, 2, \ldots$. Since $\|x_n - x_{n+1}\| \to 0$ we get $x_{n_i+1} \rightharpoonup y$ and therefore for $i \geq \widetilde{i}$ (see formulas $(16, 17, 18, 19, 20)$) we get

$$\begin{aligned} &|x_{n_i+1}(j_{n_i+1,y}) - y(j_{n_i+1,y})| = \|x_{n_i+1} - y\|_\infty \\ &\geq \|x_{n_i} - y\|_\infty - \|x_{n_i} - x_{n_i+1}\|_\infty \geq t\alpha - \frac{\epsilon}{3}, \end{aligned} \tag{23}$$

$$\begin{aligned} &|x_{n_i+1}(j_{n_i,y}) - y(j_{n_i,y})| \\ &\geq |x_{n_i}(j_{n_i,y}) - y(j_{n_i,y})| - |x_{n_i}(j_{n_i,y}) - x_{n_i+1}(j_{n_i,y})| \\ &\geq \|x_{n_i} - y\|_\infty - \|x_{n_i} - x_{n_i+1}\|_\infty \geq t\alpha - \frac{\epsilon}{3}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} &|x_{n_i+1}(j_{n_i+1}) - y(j_{n_i+1})| \geq |x_{n_i+1}(j_{n_i+1})| - |y(j_{n_i+1})| \\ &\geq \min\left\{t\alpha - \frac{\epsilon}{3},\ \|x_{n_i+1}\|_\infty\right\} - |y(j_{n_i+1})| \\ &\geq \min\left\{t\alpha - \frac{\epsilon}{3},\ \|x_{n_i}\|_\infty - \|x_{n_i} - x_{n_i+1}\|_\infty\right\} - |y(j_{n_i+1})| \\ &\geq t\alpha - \frac{2\epsilon}{3} - \frac{\epsilon}{3} = t\alpha - \epsilon\ . \end{aligned} \tag{25}$$

We will now show that there exists $\widetilde{\widetilde{i}} > \widetilde{i}$ such that for $i \geq \widetilde{\widetilde{i}}$ we have $j_{n_i+1,y} = j_{n_i,y}$.

Indeed, let us take $\tilde{\tilde{i}} > \tilde{i}$ such that

$$\|y - x_{n_i+1}\|_p^p \leq \frac{\beta^p}{2}\alpha^p + \epsilon \tag{26}$$

is satisfied for $i \geq \tilde{\tilde{i}}$ (this is possible by (10)). If $j_{n_i+1,y} \neq j_{n_i,y}$, then by (13, 23, 24) we would obtain

$$\begin{aligned} 2\,(t\alpha - \epsilon)^p &\leq |x_{n_i+1}\,(j_{n_i,y}) - y\,(j_{n_i,y})|^p + |x_{n_i+1}\,(j_{n_i+1,y}) - y\,(j_{n_i+1,y})|^p \\ &\leq \|x_{n_i+1} - y\|_p^p \leq \frac{\beta^p}{2}\alpha^p + \epsilon < \ 2\,(t\alpha - \epsilon)^p\,. \end{aligned}$$

But this is impossible.

Therefore for every $i \geq \tilde{\tilde{i}}$ we have

$$j_{n_i,y} = j_{n_i+1,y}. \tag{27}$$

Next by (16, 22, 27) for $i \geq \tilde{\tilde{i}}$ we have

$$j_{n_i+1,y} = j_{n_i,y} \leq j_{n_i} < j_{n_i+1} \text{ and } \|x_{n_i} - x_{n_i+1}\|_\infty \leq \frac{\epsilon}{3}.$$

Hence by (13, 23, 25, 26) we get the following contradiction

$$\begin{aligned} 2\,(t\alpha - \epsilon)^p &\leq |x_{n_i+1}\,(j_{n_i+1,y}) - y\,(j_{n_i+1,y})|^p + |x_{n_i+1}\,(j_{n_i+1}) - y\,(j_{n_i+1})|^p \\ &\leq \|x_{n_i+1} - y\|_p^p \leq \frac{\beta^p}{2}\alpha^p + \epsilon < \ 2\,(t\alpha - \epsilon)^p\,. \end{aligned}$$

Thus the sequence $\{j_{n_i}\}$ cannot be strictly increasing, contrary to (22).

Hence the inequality (12) is false and therefore the claimed inequality

$$\inf_{x\in C}\left(\liminf_n \|x_n - x\|\right) \leq t\alpha$$

is valid for arbitrary t satisfying $\max\left(\frac{1}{2^{\frac{1}{p}}}, \frac{\beta}{4^{\frac{1}{p}}}\right) < t < 1$. We conclude that

$$w\text{-}SOC\left(X_\beta^p\right) \geq \max\left(2^{\frac{1}{p}}, \frac{4^{\frac{1}{p}}}{\beta}\right)$$

for $1 < \beta < 4^{\frac{1}{p}}$.

To show that the constant

$$\min\left[1, \max\left(2^{\frac{1}{p}}, \frac{4^{\frac{1}{p}}}{\beta}\right)\right]$$

is the best possible in X_β^p ($1 < p < \infty$ and $1 < \beta < \infty$), let us consider two sequences. We define the sequence $\{x_n\}$ by (see [4])

$$x_n = \begin{cases} \frac{(2k+1)^2 - n}{4k+1}e_k + e_{k+1} & \text{if } (2k)^2 < n \leq (2k+1)^2\,, \\ e_{k+1} + \frac{n-(2k+1)^2}{4k+3}e_{k+2} & \text{if } (2k+1)^2 < n \leq (2k+2)^2\,, \end{cases}$$

where $\{e_k\}$ is the standard basis in l^p. Then we have $x_n \rightharpoonup 0$, $x_{n+1} - x_n \to 0$,

$$diam_{a\|\cdot\|}\{x_n\} = \max\left(\frac{4^{\frac{1}{p}}}{\beta}, 1\right)$$

and

$$\left\|x_{(2k+1)^2}\right\| = 1.$$

This yields

$$\max\left(\frac{4^{\frac{1}{p}}}{\beta}, 1\right) \cdot \inf_{x \in \overline{conv}\{x_n\}} \left(\liminf_n \|x_n - x\|\right)$$
$$= \max\left(\frac{4^{\frac{1}{p}}}{\beta}, 1\right) \cdot r_a\left(\left\{x_{(2k+1)^2}\right\}\right) = \max\left(\frac{4^{\frac{1}{p}}}{\beta}, 1\right) = diam_{a\|\cdot\|}\left(\{x_n\}\right).$$

The second sequence is defined as follows:

$$x_n = \left[r_n \cos\left(\frac{n-(2k)^2}{8k+4} \cdot \frac{\pi}{2}\right)\right] e_k + \left[r_n \sin\left(\frac{n-(2k)^2}{8k+4} \cdot \frac{\pi}{2}\right)\right] e_{k+1},$$

where

$$r_n = \left\{\left[\cos\left(\frac{n-(2k)^2}{8k+4} \cdot \frac{\pi}{2}\right)\right]^p + \left[\sin\left(\frac{n-(2k)^2}{8k+4} \cdot \frac{\pi}{2}\right)\right]^p\right\}^{-\frac{1}{p}}$$

for $(2k)^2 < n \leq (2k+2)^2$ and $k = 1, 2, \ldots$. For this sequence we get $x_n \rightharpoonup 0$, $x_{n+1} - x_n \to 0$,

$$diam_{a\|\cdot\|}\{x_n\} = \max\left(1, \frac{2^{\frac{1}{p}}}{\beta}\right)$$

and

$$\|x_{4k^2+4k+2}\| = \max\left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{\beta}\right) = \frac{1}{2^{\frac{1}{p}}} \max\left(1, \frac{2^{\frac{1}{p}}}{\beta}\right).$$

Hence we obtain

$$2^{\frac{1}{p}} \cdot \inf_{x \in \overline{conv}\{x_n\}} \left(\liminf_n \|x_n - x\|\right) =$$
$$= 2^{\frac{1}{p}} \cdot r_a\left(\{x_{4k^2+4k+2}\}\right) = \max\left(1, \frac{2^{\frac{1}{p}}}{\beta}\right) = diam_{a\|\cdot\|}\left(\{x_n\}\right).$$

This completes the proof. ∎

Remark 5.1. *A Banach space Y which is isomorphic to X_β^p with $1 < \beta < 4^{\frac{1}{p}}$ has the weak fixed point property for nonexpansive mappings if*

$$d\left(Y, X_\beta^p\right) < \min\left(2^{\frac{1}{p}}, \frac{4^{\frac{1}{p}}}{\beta}\right)$$

(see Remark 2.3), but recently T. Domínguez Benavides and M.Á. Japón Pineda obtained a better result for X^2_β. Namely, if

$$d\left(Y, X^2_\beta\right) < M\left(X^2_\beta\right) = \begin{cases} \sqrt{3} & \text{for } 1 < \beta \le \sqrt{\frac{3}{2}}, \\ \frac{\sqrt{2}}{\beta}\left(1 + \sqrt{\frac{\beta^2-1}{2}}\right) & \text{for } \sqrt{\frac{3}{2}} < \beta < \sqrt{2}, \\ 1 + \frac{1}{\sqrt{2}} & \text{for } \sqrt{2} \le \beta, \end{cases}$$

then Y has the weak fixed point property [16]*. But we have to mention that in the second part of our paper the coefficient w-$SOC(X)$ is applied to the problem of existence of fixed points of asymptotically regular uniformly lipschitzian semigroups, where till now we have not been able to use the coefficient $M(X)$.*

6. Comparison of the basic geometric coefficients of Banach spaces

As we mentioned in Section 2 the following inequalities

$$AN(X) \le w\text{-}AN(X),$$

and

$$WCS(X) \le w\text{-}SOC(X)$$

are always valid. The following examples show that for particular spaces strict inequalities may occur.

Example 6.1. *If we take the Cartesian product $X^2_{\sqrt{2}} \times l^1$ equipped with the l^1 norm, then this space is nonreflexive and therefore by Theorem 2.1, $AN\left(X^2_{\sqrt{2}} \times l^1\right) = 1$, but after applying Theorems 2.2, 4.2 and 5.1 we obtain $w\text{-}AN\left(X^2_{\sqrt{2}} \times l^1\right) = w\text{-}SOC\left(X^2_{\sqrt{2}} \times l^1\right) = \sqrt{2}$.*

Example 6.2. *Taking X^2_β with $1 < \beta < \sqrt{2}$ and applying Theorem 5.1 we obtain*
$1 < WCS\left(X^2_\beta\right) = \frac{\sqrt{2}}{\beta} < w\text{-}SOC\left(X^2_\beta\right) = \sqrt{2}$.

Example 6.3. *Let us consider the product space*

$$X = \left\{ x = \{x_p\}_{p=1}^\infty : x_p \in l^p, \left(\sum_{p=1}^\infty \|x_p\|_p^2\right)^{\frac{1}{2}} = \|x\| \right\}.$$

Since X contains isometric copies of l^p for every p and since both the semi-Opial coefficient and the asymptotic normal structure coefficient satisfy

$$w\text{-}SOC(l^p) = AN(l^p) = 2^{\frac{1}{p}},$$

the space X has neither the uniform asymptotic normal structure nor the uniform semi-Opial properties. But it easy to observe that X is still SO. In fact, it has the Opial property [26].

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