METRIC DOMAINS, HOLOMORPHIC MAPPINGS AND NONLINEAR SEMIGROUPS

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ABSTRACT. We study nonlinear semigroups of holomorphic mappings on certain domains in complex Banach spaces. We examine, in particular, their differentiability and their representations by exponential and other product formulas. In addition, we also construct holomorphic retractions onto the stationary point sets of such semigroups.

0. INTRODUCTION

Let D be a topological space. A family $S = \{F_t : t \in (0,T)\},\ T > 0$, of self-mappings F_t of D is called a (one-parameter) continuous semigroup if

$$F_{s+t} \ = \ F_t \cdot F_s \ , \ 0 < s + t < T \ , \tag{0.1}$$

and for each $x \in D$,

$$\lim_{t \to 0^+} F_t(x) \ = \ x \ , \tag{0.2}$$

where the limit is taken with respect to the topology of D.

A subset $\mathcal{W}$ of D is said to be the stationary point set of S if it consists of all the points $a \in D$ such that

$$F_t(a) \ = \ a \text{ for all t} \in (0, \mathrm{T}).$$

In other words,

$$\mathcal{W} \ = \ \bigcap_{0<t<T} \mathrm{Fix}\,\mathrm{F_t} \ . \tag{0.3}$$

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It is rather important in applications to determine the structure of the set $\mathcal{W}$ in relation to the topological structure of D. Another important problem is to find constructive methods for the approximation of $\mathcal{W}$.

Now let D be a domain (open, connected subset) in a Banach space X with the topology induced by the norm of X.

A semigroup S on D is said to be generated if for each $x \in D$ there exists the strong limit

$$f(x) \;=\; \lim_{t\to 0^+} \frac{1}{t}(x - F_t(x))\ . \tag{0.4}$$

In this case the mapping $f : D \to X$ is called the (infinitesimal) generator of S.

If $f : D \to X$ is locally Lipshitzian on D, then, by using the uniqueness of the solution to the Cauchy problem, it can be shown that the stationary point set $\mathcal{W}$ is the null point set (Null f) of f in D.

In this paper we will be mainly interested in nonlinear semigroups of holomorphic self-mappings of a domain D in a complex Banach space X.

In this case, if $S = \{F_t :\; 0 < t < T\}$ is a generated semigroup and its generator $f : D \to X$ is a bounded holomorphic mapping on D, then $\mathcal{W} = \operatorname{Null} f$ is an analytic subset of D and the convergence in (0.4) (hence, in (0.2)) is uniform on each bounded subset strictly inside D (see [19]). Therefore, one of the questions in this context is whether each semigroup of holomorphic mappings which is uniformly continuous on each subset strictly inside D has a generator.

To trace an analogy with the classical linear case, we note that if f is a linear holomorphic mapping, then it is bounded by definition, and we obtain the simplest case: the semigroup S generated by f is a uniformly continuous linear semigroup $F_t = e^{-tf}$. And conversely, each semigroup of bounded linear operators which is continuous in the operator topology is differentiable at zero, and its generator is also a bounded linear operator. In addition, we have the exponential formula

$$e^{-tf} \;=\; \lim_{n\to\infty} (I + tf/n)^{-n}.$$

For the nonlinear case such facts are not trivial. For the finite dimensional case, the differentiability with respect to the parameter of nonlinear holomorphic semigroups was shown in [3], [1]. In the context of the Hille-Yosida theory the following question is also of interest. If in the infinite dimensional case we have a family of holomorphic mappings which satisfies in some sense an approximate semigroup property (see Definition 1), and converges to the identity uniformly on each subset strictly inside D, is this family differentiable with respect to the parameter and does its derivative generate a semigroup which may be represented by a product or exponential formula? We will consider these questions in Section 1.1.

Furthermore, if D is a bounded domain in a reflexive X, then it is well known that for each fixed $t \in (0, T)$, $\mathcal{W}_t = \operatorname{Fix} \mathrm{F_t}$ is an analytic submanifold in D (see [17]). Moreover, if D is convex, then $\mathcal{W}_t$ is a holomorphic retract of D and hence it is connected.

So the question arises whether these facts continue to hold for $\mathcal{W} = \bigcap\limits_{0<t<T} \mathcal{W}_t$.

For the finite dimensional case the affirmative answer to this question is an immediate consequence of [2]. However, even in this situation the problem is to find an explicit form of a retraction which will give us an approximation method for the stationary point set $\mathcal{W}$.

We will consider both questions for the general infinite dimensional case in Section 1.2.

Finally, we note that one of the main properties of a holomorphic self-mapping of a domain D in X is that each such mapping is ρ-nonexpansive with respect to each pseudometric ρ assigned to D by a Schwarz-Pick system (SPS) (see, for example, [10], [8], [9], [5]). For a bounded domain D, for example, such a pseudometric is equivalent to the original norm of X, and therefore it is actually a metric on D. For a bounded convex domain in X, all metrics in an (SPS) coincide (see [6]).

On the other hand, the class of ρ-nonexpansive self-mappings of D is much wider than the class of holomorphic self-mappings of D. For a convex domain, for instance, it contains all convex combinations of holomorphic and antiholomorphic self-mappings of D.

As a matter of fact, our approach has a more general geometric nature. Most of our arguments apply to those mappings that are nonexpansive with respect to a metric which has some of the properties enjoyed by metrics in an (SPS). Hence they are also valid in real Banach spaces (see Section 1.1).

1. Main results

1.1. Let X be a Banach space and let D be a domain in X (open, connected subset of X).

Definition 1. *We say that D is a metric domain in X if there exists a metric ρ on D such that*

(i) *for each $x \in D$ and for each $0 < d < \operatorname{dist}(x, \partial D)$ there are positive numbers $L = L(d)$, $r = r(d)$ and m=$m(d)$ such that*

$$\rho(x, y) \ \leq \ L\|x - y\|, \ \textit{whenever } \|x - y\| < d \ ,$$

and

$$\rho(x, y) \ \geq \ m\|x - y\|, \ \text{whenever } \rho(x, y) \ \leq \ r;$$

(ii) *each ρ-ball $\mathcal{B}_r(x) = \{y \in D : \rho(x, y) < r\}$ is strictly inside D, i.e., for each $x \in D$ and $r > 0$ there is $\epsilon > 0$ such that*

$$\operatorname{dist}(\mathcal{B}_r(x), \partial D) \ \geq \ \epsilon \ .$$

It is clear that (i) and (ii) imply that (D, ρ) is a complete metric space.

One of the important examples of such a domain is a bounded convex domain in a complex Banach space with a metric assigned to it by a Schwarz-Pick system (SPS).

For a bounded convex domain in a real Banach space such a metric can be induced by the complexification of X and by using $\rho \in$ (SPS) on the direct product of D by itself in the complex sense.

Other constructions of such domains can be given by using Hilbert's projective metric or Thompson's metric on a cone associated with a convex bounded domain D in X [18].

Additional examples (which use Finsler structures) can be found in [11] and [4].

Definition 2. *Let X be an arbitrary Banach space and let D be a domain in X. We say that a family $\{G_s : s \in (0,T),\ T > 0\}$ of self-mappings of D satisfies the approximate semigroup property if for each subset $\tilde{D}$ strictly inside D the following conditions hold:*

(i) *for each $\epsilon > 0$ there is a positive $\delta = \delta(\tilde{D}, \epsilon) \le T$ such that*

$$\sup_{x \in \tilde{D}} \|G_s(x) - G^p_{s/p}(x)\| < \epsilon s$$

for all positive integers p and all $s \in (0, \delta)$;

(ii) *for each pair $s, t \in (0,T),\ s + t < T$ there exists $L = L(\tilde{D})$ such that*

$$\sup_{x \in \tilde{D}} \|G_{s+t}(x) - G_s(G_t(x))\| \le L\sqrt{st}.$$

Theorem 1. *Let D be a domain in a complex Banach space X, and let $\{G_s : s \in (0,T)\}$ be a family of holomorphic self-mapping of D which satisfies the approximate semigroup property. Suppose that G_s converges to the identity, as $s \to 0^+$, uniformly on each subset $\tilde{D}$ strictly inside D, i.e.,*

$$\lim_{s \to 0^+} \sup_{x \in \tilde{D}} \|G_s(x) - x\| = 0. \tag{1.1}$$

Then

(i) *The strong limit*

$$\lim_{s \to 0^+} \frac{1}{s}(I - G_s) = f \tag{1.2}$$

exists and is a holomorphic mapping from D into X, which is bounded on each subset strictly inside D.

(ii) *Suppose that D is a metric domain in X with a metric $\rho \in$(SPS). Then, for each pair s and t, $s \in (0,T)$, $t > 0$, and each sequence of integers $\{t_n\}$ such that*

$$\frac{t_n s}{nt} \to 1, \text{ as } n \to \infty\ , \tag{1.3}$$

there exists the strong limit

$$\lim_{n \to \infty} G^{t_n}_{\frac{s}{n}} = F_t \tag{1.4}$$

uniformly on each subset strictly inside D. This limit does not depend on $\{t_n\}$ and s in (1.3), and the family $\{F_t : \ 0 < t < \infty\}$ is a one-parameter semigroup of holomorphic self-mappings of D;

(iii) *For $x \in D$ the mapping $F(t,x) = F_t(x)$ defined by (1.4) is the solution of the Cauchy problem*

$$\left\{ \begin{array}{l} \frac{\partial F(t,x)}{\partial t} + f(F(t,x)) = 0 \\ \lim\limits_{t\to 0^+} F(t,x) = x \end{array} \right. \tag{1.5}$$

where f is defined by (1.2).

Corollary 1. *Let D be an arbitrary domain in a complex Banach space X, and let $\{F_t : t \in (0,T),\ T > 0\}$ be a one-parameter semigroup of holomorphic self-mappings of D, such that*

$$\lim_{t\to 0^+} F_t \ = \ I$$

uniformly on each subset strictly inside D. Then this semigroup is differentiable at $t = 0$, i.e., there exists the infinitesimal generator

$$f \ = \ \lim_{t\to 0^+} \frac{1}{t}(I - F_t)$$

which is a holomorphic mapping from D into X. This mapping is bounded on each subset strictly inside D.

We will say that a mapping $f : D \to X$ satisfies the range condition if there exists a positive $T > 0$ such that for each $s \in (0,T)$,

$$(I + sf)(D) \supseteq D \tag{1.6}$$

and $(I + sf)^{-1}$ is a well-defined self-mapping of D.

Corollary 2. *Let D be a metric domain in a complex Banach space X, with a metric $\rho \in$(SPS), and let $f \in \mathrm{Hol}(D,X)$ be bounded on each subset strictly inside D and satisfy the range condition. Then f is the infinitesimal generator of the one-parameter semigroup of holomorphic self-mappings of D, which can be defined by the following analogs of the exponential formula:*

$$F_t \ = \ \lim_{n\to\infty} (I + \frac{t}{n} f)^{-n} \tag{1.7}$$

or

$$F_t \ = \ \lim_{n\to\infty} (I + \frac{1}{n} f)^{[-tn]} \tag{1.8}$$

where the convergence in (1.7) and (1.8) is uniform on each subset strictly inside D.

This semigroup is the solution of the Cauchy problem (1.5).

In this context it is natural to look for the geometrical conditions which will ensure that any semigroup of holomorphic mappings can be represented by exponential formulas or, in other words, to find out when the range condition holds for each holomorphic generator. To answer this query we need the following definition.

Definition 3. *Let D be a convex metric domain in a Banach space X (real or complex) with a corresponding metric ρ. We say that the metric ρ is compatible with the convex structure of D if the following conditions hold:*

(a) *For each four elements x, y, z, w in D and each $0 \le \alpha \le 1$,*

$$\rho(\alpha x + (1-\alpha)y,\ \alpha z + (1-\alpha)w) \ \le\ \max\{\rho(x,z),\ \rho(y,w)\};$$

(b) *There is a real function $\varphi : [0,1] \to [0,1]$ such that*

$$\limsup_{\alpha \to 1^-} \frac{1-\alpha}{1-\varphi(\alpha)} < \infty\ ,$$

and for each three elements x, y, z in D and each $0 \le \alpha \le 1$,

$$\rho(\alpha x + (1-\alpha)y,\ \alpha z + (1-\alpha)y) \ \le\ \varphi(\alpha)\rho(x,z).$$

Such a convex metric domain D with the metric ρ, which is compatible with the convex structure of D, will be called a compatible metric domain.

Once again it can be shown by using the Earle-Hamilton theorem [7] that each convex bounded domain in a complex Banach space is a compatible metric domain with the hyperbolic metric ρ on D. (Note that in this case all hyperbolic metrics on D coincide [6].)

Theorem 2. *Let D be a compatible metric domain in a Banach space X and let $\{F_t :\ t \in (0,T)\}$ be a family of ρ-nonexpansive self-mappings of D, i.e., for each pair $x, y \in D$, and $t \in (0,T)$,*

$$\rho(F_t(x),\ F_t(y)) \ \le\ \rho(x,y)\ . \tag{1.9}$$

Suppose that

$$f(x) \ =\ \lim_{t \to 0^+} (x - F_t(x))/t$$

exists uniformly on each ρ-ball in D and that $f : D \to X$ is continuous. Then f satisfies the range condition for all $s > 0$.

Corollary 3. *Let D be a bounded convex domain in X, and let f be a bounded holomorphic mapping from D into X. Then f generates a one-parameter semigroup of holomorphic self-mappings of D on R^+ if and only if it satisfies the range condition, i.e., for each $s > 0$ the resolvent*

$$\mathcal{J}_s \ =\ (I + sf)^{-1}$$

is a well-defined holomorphic self-mapping of D.

In this case the semigroup $\{F_t\}$ can be obtained by the exponential formulas (1.7) and (1.8) or, more generally,

$$F_t \ =\ \lim_{n \to \infty} \mathcal{J}_{\frac{s}{n}}^{t_n},$$

where $\{t_n\}$ is a sequence of integers which satisfies (1.3).

Remark 1. Note also that in this case the range condition on some interval $(0,T)$ of R^+ implies the same condition globally on all of R^+.

The sufficiency part of Corollary 3 follows from Corollary 2.

The crucial point in the proof of Corollary 2 and in establishing the exponential formulas is to show that the family of resolvents $\{\mathcal{J}_s\}_{s>0}$ satisfies the approximate semigroup property (see Section 2), and has a right-hand derivative at $s = 0$ which is equal to f.

The question is what happens when we have an arbitrary continuous family $\{G_s\}_{s>0} \subset \mathrm{Hol}(D, D)$ which is differentiable at $s = 0^+$. Actually, Theorem 2 and Corollary 3 imply a somewhat more general assertion than the exponential formula, namely, the product formula.

Theorem 3. *Let D be a bounded convex domain in a complex Banach space X, and let $\{G_s\}_{s\in(0,T)}$ be an arbitrary family of holomorphic self-mappings of D such that*

$$\lim_{s\to 0^+} \frac{x - G_s(x)}{s} = f(x)$$

exists uniformly on each subset strictly inside D and is bounded on such subsets. Then

(1) *the Cauchy problem (1.5) has a global solution $F(\cdot,\cdot)$ defined on $R^+ \times D$;*
(2) *this solution can be obtained by the following product formula*

$$F(t,\cdot) = \lim_{n\to\infty} G^n_{\frac{t}{n}} \ ,$$

where the limit is uniform on each subset strictly inside D.

Corollary 4. *Let D be as in Theorem 3, and let f and g be two holomorphic generators of one-parameter semigroups on D, i.e., $\{F_t\}_{t>0}$ and $\{G_t\}_{t>0}$, respectively. Then the mapping $h = f + g$ is also a generator and the semigroup H_t generated by it can be obtained by the formula*

$$H_t = \lim_{n\to\infty} \left[F_{\frac{t}{n}} \cdot G_{\frac{t}{n}}\right]^n \ ,$$

where the limit is uniform on each subset strictly inside D. This implies that the family of holomorphic generators on a bounded convex domain is a real cone.

Theorem 3 and Corollary 4 provide affirmative answers to two questions raised in Section 9 of [19].

1.2. Now we turn to the question of approximation processes for the stationary points of a one-parameter semigroup of holomorphic mappings. Let D be a bounded convex domain in a complex Banach space X, and let $\{F_t : \ t \in (0,T)\}$ be a semigroup of holomorphic mappings such that F_t converges to the identity uniformly on each subset strictly inside D.

Let $\mathcal{W}$ be the stationary point set of $\{F_t\}$, i.e.,

$$\mathcal{W} \ = \bigcap_{t\in(0,T)} \mathrm{Fix}\, F_t.$$

Since $\{F_t\}$ is differentiable at $t = 0$ (see Corollary 1) and it solves the Cauchy problem (1.5) with $f = \frac{dF_t}{dt}|_{t=0}$, it follows by the uniqueness of the solution to the Cauchy problem that

$$\mathcal{W} \;=\; \mathrm{Null_D f}.$$

Thus $\mathcal{W}$ is an analytic subset of D. Furthemore, if X is reflexive, it is well-known that for each $t \in (0,T)$ the set $\mathcal{W}_t = Fix_D F_t$ is a holomorphic retract of D [17]. Hence $\mathcal{W}$ is an intersection of analytic submanifolds of D.

For a finite dimensional X it was shown in [2] that this set is also a holomorphic retract of D, and therefore it is also an analytic submanifold of D. But even in this case we only know that a retraction exists, but we have no constructive approximative process for the points in $\mathcal{W}$.

On the other hand, it follows from Corollary 3 and the definition of the resolvent that for each $s > 0$,

$$\mathcal{W} \;=\; \mathrm{Null_D f} \;=\; \mathrm{Fix_D}\mathcal{J}_\mathrm{s}\ , \tag{1.10}$$

and therefore it is a holomorphic retract of D, for each bounded convex domain in a reflexive Banach space. So, the question is how to construct a retraction onto this set.

A possible way is to extend our semigroup $\{F_t\}$ to all of R^+ and to investigate its asymptotic behavior as $t \to \infty$. It will become clear that this may be done only if we know a priori at least one point $a \in \mathcal{W}$ and the spectrum of the linear operator $f'(a)$ satisfies certain conditions (see [12]) (i.e., does not intersect the imaginary axis with, perhaps, the exception of zero).

Another way would be to apply the fact (1.10), and for a fixed $s > 0$ to construct the sequence of the discrete Cesaro averages

$$G_n \;=\; \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{J}_s^j\ ,$$

so that a subsequence $\{G_{n_k}\}$ weakly converges to a mapping $G : D \to \mathcal{W}$ which is a holomorphic retraction of D onto $\mathcal{W}$.

As a matter of fact, this method is superfluous because as we will see below, the iterates of the resolvents strongly converge to a holomorphic retraction of D onto $\mathcal{W}$.

Definition 4. *Let f be a holomorphic mapping from D into X and let $\mathcal{W} =$ $\mathrm{Null_D f} \neq \emptyset$. A point $a \in \mathcal{W}$ is said to be quasi-regular if the following condition holds:*

$$\mathrm{Ker}\, f'(a) \oplus Im\ f'(a) \;=\; X. \tag{1.11}$$

If, in addition, $\mathrm{Ker} f'(a) = \{0\}$, *then we say that a is a regular null point of f.*

Theorem 4. *Let D be a bounded convex domain in X, and let $f \in \mathrm{Hol}(D,X)$ be a generator of a one-parameter semigroup of holomorphic self-mappings of D. Suppose that $\mathcal{W} = \mathrm{Null_D f} \neq \emptyset$. Then*

(i) *If $\mathcal{W}$ contains a quasi-regular point $a \in D$, then for each $s > 0$ the sequence*

$$\{\mathcal{J}_s^n \;=\; (I + sf)^{-n}\}_1^\infty\ ,$$

converges to a holomorphic retraction of D onto $\mathcal{W}$ as $n \to \infty$, uniformly on each ball strictly inside D.

(ii) *If $\mathcal{W}$ contains a regular point $a \in D$, then $\mathcal{W} = \{a\}$ and the net*

$$\{\mathcal{J}_s \ = \ (I + sf)^{-1}\}_{s>0}$$

converges to a as $s \to \infty$, uniformly on each ball strictly inside D.

2. Proofs of the results

2.1. To prove our theorems we need several lemmas. Some of them may be interesting in themselves.

Lemma 1. *Let (D, ρ) be a complete metric space, and let $\{G_s : \ 0 < s < T\}$ be a family of ρ-nonexpansive mappings on (D, ρ) with the following properties:*

(i) *For each ρ-ball $\mathcal{B} \subset (D, \rho)$, and each $\epsilon > 0$, there is a positive $\delta = \delta(\mathcal{B}, \epsilon) \ \leq \ T$, such that*

$$\rho(G_s(x), G^p_{\frac{s}{p}}(x)) < \epsilon \cdot s$$

for all $x \in \mathcal{B}$ and for all integers p, whenever $s \in (0, \delta)$;

(ii) *For each ρ-ball $\mathcal{B} \subset (D, \rho)$ there exist $\mu = \mu(\mathcal{B}) > 0$ and $\mathcal{L} = \mathcal{L}(\mathcal{B})$ such that*

$$\rho(G_s(x), x) \ \leq \ \mathcal{L} \cdot s$$

for all $x \in \mathcal{B}$, whenever $s \in (0, \mu)$.

Then for each pair $s \in (0, T)$ and $t > 0$, and each sequence of integers $\{t_n\}$ such that

$$\frac{t_n s}{nt} \to 1 \ , \quad as \ n \to \infty \ ,$$

there exists the limit

$$\lim_{n\to\infty} G^{t_n}_{\frac{s}{n}} \ = \ F_t$$

uniformly on each ρ-ball in (D, ρ). This limit does not depend on the sequence $\{t_n\}$ and it is a locally uniformly continuous one-parameter semigroup with respect to $t > 0$.

Proof. First we establish two simple inequalities. For each $\tau \in (0, T)$ and each integer ℓ we have

$$\rho(G^\ell_\tau(x), x) \ \leq \ \sum_{j=0}^{\ell-1} \rho(G^{j+1}_\tau(x), G^j_\tau(x)) \ \leq \ \ell\rho(G_\tau(x), x). \tag{2.1}$$

Now if ℓ_1 and ℓ_2 are two arbitrary integers, (2.1) implies

$$\begin{aligned} \rho(G^{\ell_1}_\tau(x), G^{\ell_2}_\tau(x)) \ &= \ \rho\left(G^{\min(\ell_1,\ell_2)}_\tau(G^{|\ell_1-\ell_2|}_\tau(x)), G^{\min(\ell_1,\ell_2)}_\tau(x)\right) \\ &\leq \ \rho(G^{|\ell_1-\ell_2|}_\tau(x), x) \ \leq \ |\ell_1 - \ell_2|\rho(G_\tau(x), x) \end{aligned} \tag{2.2}$$

for each $\tau \in (0, T)$.

Take a ρ-ball $\mathcal{B} \subset (D, \rho)$ and choose $\mu > 0$ so that condition (ii) holds. Then, for each $\tau \in (0, \mu)$ we have by (2.1) and (2.2)

$$\rho(G_\tau^\ell(x), x) \ \leq \ \ell \cdot \mathcal{L}\tau \tag{2.3}$$

and

$$\rho(G_\tau^{\ell_1}(x), G_\tau^{\ell_2}(x)) \ \leq \ |\ell_1 - \ell_2|\mathcal{L}\tau \tag{2.4}$$

for all $x \in \mathcal{B}$ and for all integers ℓ, ℓ_1, ℓ_2.

For a given $s \in (0, T)$ and $t > 0$ consider the sequence of mappings $\{G_{\frac{s}{n}}^{t_n}\}_1^\infty$ on $\mathcal{B}$ where $\{t_n\}$ is a sequence of integers which satisfies (1.3). Taking an integer N so that $s/N < \mu$ we get by (2.3)

$$\rho(G_{\frac{s}{n}}^{t_n}(x), x) < \frac{t_n \cdot s}{n}\mathcal{L} < \infty$$

for all $n \geq N$ and all $x \in \mathcal{B}$. In addition, for each $j = 1, 2, \ldots, t_n$, and $m = 1, 2, \ldots,$

$$\rho\left(G_{\frac{s}{nm}}^{mj}(x), x\right) \ \leq \ mj \cdot \frac{s}{nm}\mathcal{L} < \infty$$

whenever $n \geq N$. This means that there exists a ρ-ball $\mathcal{B}_1 \subset (D, \rho)$ such that the sequences $\left\{G_{\frac{s}{n}}^{t_n}(x)\right\}_N^\infty$ and $\left\{G_{\frac{s}{nm}}^{mj}(x)\right\}_N^\infty$ are in $\mathcal{B}_1$ for all $x \in \mathcal{B}$, $j = 1, 2, \ldots, t_n$, $m = 1, 2, \ldots$. Now for a given $\epsilon > 0$ we can choose by (ii) $\delta = \delta(\epsilon, \mathcal{B}_1) \ \leq \ T$ such that

$$\rho(G_\tau(z), G_{\frac{\tau}{m}}^m(z)) < \epsilon\tau \tag{2.5}$$

for all $z \in \mathcal{B}_1$ and all $p = 1, 2, \ldots,$ whenever $0 < \tau < \delta$.

Taking N so large that $s/N < \min\{\mu, \delta\}$ and setting $z = G_{\frac{s}{nm}}^{jm}(x)$, $x \in \mathcal{B}$, $m = 1, 2, \ldots,$ $j = 1, 2, \ldots, t_n$, $n \geq N$ and $\tau = \frac{s}{n}$, we obtain by the triangle inequality, the nonexpansiveness of G_s and (2.5),

$$\begin{aligned}
&\rho\left(G_{\frac{s}{n}}^{t_n}(x), G_{\frac{s}{nm}}^{t_n \cdot m}(x)\right) \\
&\quad \leq \sum_{j=0}^{t_n-1} \rho\left(G_{\frac{s}{n}}^{t_n-j}(G_{\frac{s}{nm}}^{j\cdot m}(x)), G_{\frac{s}{n}}^{t_n-j-1}(G_{\frac{s}{nm}}^{(j+1)m}(x))\right) \\
&\quad = \sum_{j=0}^{t_n-1} \rho\left(G_{\frac{s}{n}}^{t_n-j-1}(G_{\frac{s}{n}}(G_{\frac{s}{nm}}^{j\cdot m}(x))), G_{\frac{s}{n}}^{t_n-j-1}(G_{\frac{s}{nm}}^{(j+1)m}(x))\right) \\
&\quad \leq \sum_{j=0}^{t_n-1} \rho\left(G_{\frac{s}{n}}(G_{\frac{s}{nm}}^{jm}(x)), (G_{\frac{s}{nm}}^{(j+1)m}(x)\right) \\
&\quad = \sum_{j=0}^{t_n-1} \rho\left(G_{\frac{s}{n}}(G_{\frac{s}{nm}}^{jm}(x)), G_{\frac{s}{nm}}^{m}(G_{\frac{s}{nm}}^{jm}(x))\right) \\
&\quad \leq t_n \cdot \epsilon \cdot \frac{s}{n} < a \cdot \epsilon,
\end{aligned} \tag{2.6}$$

where $a = \sup\{\frac{t_n \cdot s}{n}\} < \infty$ because of (1.3). In the same way and for the same $\epsilon > 0$ we obtain the inequality

$$\rho\left(G^{t_m}_{\frac{s}{m}}(x), G^{t_m \cdot n}_{\frac{s}{nm}}(x)\right) < \epsilon \cdot a \tag{2.7}$$

for all $x \in \mathcal{B}$, whenever $m \geq N,\ n = 1, 2, \dots$. In addition, it follows by (2.4) that

$$\rho\left(G^{t_n \cdot m}_{\frac{s}{nm}}(x), G^{t_{nm}}_{\frac{s}{nm}}(x)\right) \ \leq\ |\frac{st_n}{n} - \frac{st_{nm}}{nm}|\mathcal{L} < \epsilon \cdot a \tag{2.8}$$

and

$$\rho\left(G^{t_m \cdot n}_{\frac{s}{nm}}(x), G^{t_{nm}}_{\frac{s}{nm}}(x)\right) \ \leq\ |\frac{st_m}{m} - \frac{st_{nm}}{nm}|\mathcal{L} < \epsilon \cdot a \tag{2.9}$$

for $n, m \geq N$, where N is large enough.

Thus, by the triangle inequality, (2.6)-(2.9) imply that for a given ϵ, there is $N > 0$ such that

$$\rho\left(G^{t_n}_{\frac{s}{n}}(x), G^{t_m}_{\frac{s}{m}}(x)\right) < 4a\epsilon$$

whenever $n, m > N$. This means that $\left\{G^{t_n}_{\frac{s}{n}}\right\}$ is a Cauchy sequence uniformly on each ρ-ball $\mathcal{B} \subset (D, \rho)$, and since (D, ρ) is complete, its limit exists and is a ρ-nonexpansive mapping on (D, ρ).

Once again it follows from (2.4) that if $\{r_n\}$ is another sequence of integers such that

$$\frac{sr_n}{n} \to t$$

then, for a given $\epsilon > 0$ and $x \in \mathcal{B}$,

$$\rho\left(G^{t_n}_{\frac{s}{n}}(x), G^{r_n}_{\frac{s}{n}}(x)\right) \ \leq |\ \frac{st_n}{n} - \frac{sr_n}{n}\ |\ \mathcal{L} < \epsilon$$

whenever n is large enough. This means that

$$F_t \ =\ \lim_{n\to\infty} G^{t_n}_{\frac{s}{n}}$$

does not depend on the sequence $\{t_n\}$ satisfying (1.3).

Now let $s \in (0, T),\ t > 0$ and $r > 0$ be given numbers. Let $\{t_n\}_1^\infty$ and $\{r_n\}_1^\infty$ be two sequences of integers such that $\frac{t_n \cdot s}{n} \to t$ and $\frac{r_n \cdot s}{n} \to r$ as $n \to \infty$. Then, for a given $\epsilon > 0$ and $x \in \mathcal{B}$,

$$\begin{aligned}
\rho\left(F_t(F_r(x)), F_{t+r}(x)\right) &\leq \rho\left(F_t(F_r(x)), G^{t_n + r_n}_{\frac{s}{n}}(x)\right) \\
&\quad + \rho\left(G^{t_n + r_n}_{\frac{s}{n}}(x), F_{t+r}(x)\right) \\
&\leq \rho\left(F_t(F_r(x)), G^{t_n}_{\frac{s}{n}}(G^{r_n}_{\frac{s}{n}}(x))\right) + \epsilon \\
&\leq \rho\left(F_t(F_r(x)), G^{t_n}_{\frac{s}{n}}(F_r(x))\right) \\
&\quad + \rho\left(G^{t_n}_{\frac{s}{n}}(F_r(x)), G^{t_n}_{\frac{s}{n}}(G^{r_n}_{\frac{s}{n}}(x))\right) + \epsilon \\
&\leq \rho\left(F_r(x), G^{r_n}_{\frac{s}{n}}(x)\right) + 2\epsilon \ \leq\ 3\epsilon
\end{aligned}$$

whenever n is big enough. Since $\epsilon > 0$ is arbitrary, we have

$$F_{t+r} \;=\; F_t \cdot F_r \;.$$

Thus $F_t : R^+ \to (D, \rho)$ is a one-parameter semigroup which is uniformly continuous on each ρ-ball in (D, ρ) with respect to $t > 0$. The lemma is proved. ∎

Lemma 2. *Let D be a convex domain in a Banach space X, and let $f : D \to X$ be a mapping which satisfies the range condition, i.e., for a positive $T > 0$ and each $t \in [0, T)$, the resolvent mapping $\mathcal{J}_t = (I + tf)^{-1}$ is a well-defined self-mapping of D. Then for $0 \le s \le t < T$ the following resolvent identity holds:*

$$\mathcal{J}_t = \mathcal{J}_s\left(\frac{s}{t}I + (1 - \frac{s}{t})\mathcal{J}_t\right). \tag{RI}$$

Proof. For each $x \in D$ the element $y = \frac{s}{t}x + (1 - \frac{s}{t})\mathcal{J}_t(x)$ belongs to D, by the convexity of D. It follows by the definition of the resolvent that $I - \mathcal{J}_t = tf(\mathcal{J}_t)$ for $t \in [0, T)$. Thus

$$y = \mathcal{J}_t(x) + \frac{s}{t}(x - \mathcal{J}_t(x)) = \mathcal{J}_t(x) + sf(\mathcal{J}_t(x)) = (I + sf)\mathcal{J}_t(x).$$

Hence $\mathcal{J}_s(y) = (I + sf)^{-1}(y) = \mathcal{J}_t(x)$, and we are done. ∎

Lemma 3. *Let D be a domain in a complex Banach space and let $\phi \in$ $\mathrm{Hol}(D, D)$. Suppose that for some subset $D_1 \subset D$ with $\mathrm{dist}(D_1, \partial D) > 0$ there are two numbers μ and d, $0 < \mu < d$, an integer $p \ge 1$, and a domain D_2, $D_1 \subset D_2 \subset D$, with $\mathrm{dist}(D_1, \partial D_2) \ge d$, such that*

$$\sup_{x \in D_2} \|x - \phi^k(x)\| < \mu \tag{2.10}$$

for all $k = 0, 1, 2, \ldots, p-1$. Then, for $x \in D_1$ the following inequality holds:

$$\|x - \phi^p(x) - p(x - \phi(x))\| \;\le\; \frac{\mu}{d - \mu}(p-1)\|x - \phi(x)\|. \tag{2.11}$$

Proof. Let $x \in D_1$ and $z \in D_2$ be such that $\|z - x\| \le \mu$. Then the ball $B_{d-\mu}(z)$ with its center at z and radius $d - \mu$ lies in D_2. Hence it follows from (2.10) and the Cauchy inequality that

$$\|(I - \phi^k)'(z)\| \;\le\; \frac{\mu}{d - \mu}. \tag{2.12}$$

Therefore, for $x \in D_1$ and $y \in D$ such that $\|x - y\| < \mu$ we have by (2.12),

$$\|x - \phi^k(x) - (y - \phi^k(y)\| \;\le\; \frac{\mu}{d - \mu}\|x - y\|. \tag{2.13}$$

Now setting $y = \phi(x)$ and using (2.10) and (2.13) we obtain by the triangle inequality

$$\begin{aligned}\|x - \phi^p(x) - p(x - \phi(x))\| &= \|\sum_{k=0}^{p-1}[\phi^k(x) - \phi^{(k+1)}(x) - x + \phi(x)]\| \\ &\leq \sum_{k=1}^{p-1} \|\phi^k(x) - x - [\phi^k(\phi(x)) - \phi(x)]\| \\ &\leq \frac{\mu}{d-\mu}(p-1)\|x - \phi(x)\|,\end{aligned}$$

and we are done. ■

Lemma 4. *Let D be a domain in a complex Banach space and let a family $\{G_s: \ 0 < s < T\}$, $G_s \in \mathrm{Hol}(D, D)$, satisfy the approximate semigroup property. Suppose that G_s converges to the identity as $s \to 0^+$, uniformly on each subset strictly inside D. Then for $s > 0$ small enough the net*

$$f_s = \frac{1}{s}(I - G_s) \tag{2.14}$$

is uniformly bounded on each subset strictly inside D.

Proof. Let D_1 be a subset strictly inside D and let $0 < d < \mathrm{dist}(\partial D, D_1)$. Take any domain $D_2 \subset\subset D$ such that $D_1 \subset\subset D_2$ and $\mathrm{dist}(D_1, \partial D_2) > d$. Choose μ, $0 < \mu < d$, such that $\mu(d-\mu)^{-1} < \frac{1}{2}$ and choose σ, $0 < \sigma \leq T$, such that for all $\tau \in (0, \sigma]$,

$$\sup_{x \in D_2} \|x - G_\tau(x)\| < \frac{\mu}{2}. \tag{2.15}$$

In addition, it follows by the approximate semigroup property (i) that there exists $0 < \delta < \frac{\sigma}{2}$ such that

$$\|G_s^k(x) - G_{sk}(x)\| < \frac{\mu}{2} \tag{2.16}$$

for all $x \in D_2$ and each $k = 1, 2, \ldots$. Now set $n = [\frac{\sigma}{s}]$. For $s \in (0, \delta)$ we have $n \geq 2$, $ns \geq \frac{\sigma}{2}$ and $ks \leq \sigma$ for all $k = 1, 2, \ldots, n$. Hence it follows from (2.15) and (2.16) that

$$\sup_{x \in D_2} \|x - G_s^k(x)\| < \mu, \ s \in (0, \delta).$$

Now for all $x \in D_1$ we get, by Lemma 3,

$$\begin{aligned}n\|x - G_s(x)\| - \|x - G_s^n(x)\| &\leq \|n(x - G_s(x)) - (x - G_s^n(x))\| \\ &\leq \frac{1}{2}n\|x - G_s(x)\|,\end{aligned}$$

or

$$\|x - G_s(x)\| \leq \frac{2}{n}\| x - G_s^n(x)\|. \tag{2.17}$$

Therefore, by (2.14)-(2.17), we obtain

$$\begin{aligned}\|f_s(x)\| &\leq \frac{2}{ns}(\|x - G_{ns}(x)\| + \|G_{ns}(x) - G_s^n(x)\|) \\ &\leq \frac{2}{ns}\left(\frac{\mu}{2} + \frac{\mu}{2}\right) \leq \frac{4\mu}{\sigma} = \mathcal{L} < \infty ,\end{aligned}$$

whenever $s \in (0, \delta)$. The lemma is proved. ∎

Lemma 5. (A. Markus [15]). *Let A be a bounded linear operator on a Banach space X such that*

$$\|(I - A)^n\| \leq M, \; n = 1, 2, \ldots \tag{2.18}$$

for some $M < \infty$. Then the following conditions are equivalent:

$$\mathrm{Ker} A \oplus \mathrm{Im}\, A = X \tag{$*$}$$

$$\overline{\mathrm{Im}\, A} = \mathrm{Im}\, A\ . \tag{$**$}$$

Proof. The implication $(*) \Rightarrow (**)$ is obvious. Now let $(**)$ hold, and let a functional $x^* \in X^*$ vanish on the sum $\mathrm{Ker}\, A \oplus \mathrm{Im}\, A$. Then $x^* \in \mathrm{Ker}\, A^*$. Furthermore, it follows from $(**)$ and the Banach-Hausdorff theorem that the condition $< u, x^* >= 0$ for all $u \in \mathrm{Ker}\, A$ implies that $x^* \in \mathrm{Im}\, A^*$. Thus $x^* \in \mathrm{Ker}\, A^* \cap \mathrm{Im}\, A^*$. But because of (2.18), $\mathrm{Ker}\, A^* \cap \mathrm{Im}\, A^* = \{0\}$ by the Yosida mean ergodic theorem [22]. So $x^* = 0$, and this implies $(*)$. ∎

Remark 2. More results in this direction can be found in the recent paper [14].

We recall that a linear operator $A : X \to X$ is said to be m-accretive if for each $r > 0$ the operator $\mathcal{I}_r = (I + rA)^{-1}$ is well defined on X and $\|(I + rA)^{-1}\| \leq 1$.

Lemma 6. *Let A be a bounded linear operator in X which is m-accretive with respect to some norm equivalent to the norm of X. If A satisfies the condition $(*)$, then for each $r > 0$ the linear operator $\mathcal{I}_r = (I + rA)^{-1}$ satisfies the condition*

$$\mathrm{Ker}(I - \mathcal{I}_r) \oplus \mathrm{Im}(I - \mathcal{I}_r) = X. \tag{2.19}$$

Proof. Returning to the original norm of X we have by the definition

$$\|\mathcal{I}_r^n\| \leq M < \infty, \; n = 1, 2, \ldots, \; r > 0. \tag{2.20}$$

By Lemma 5 and (2.20) it is sufficient to show that $\mathrm{Im}(I - \mathcal{I}_r)$ is closed in X. Indeed, if $y_n \in \mathrm{Im}(I - \mathcal{I}_r)$ converge to $y \in X$, we get a sequence $\{x_n\} \subset X$ such that

$$(I - \mathcal{I}_r)x_n = rA\mathcal{I}_r x_n \to y \in \mathrm{Im}\, A. \tag{2.21}$$

Note that it follows by the definition of $\mathcal{I}_r$ that

$$\mathrm{Ker}(I - \mathcal{I}_r) = \mathrm{Ker}\, A. \tag{2.22}$$

Therefore, if we represent $x_n \in X$ in the form $x_n = u_n + v_n$, where $u_n \in \operatorname{Im} A$ and $v_n \in \operatorname{Ker} A$ (see $(*)$), we get from (2.21) and (2.22),

$$\begin{aligned} (I - \mathcal{I}_r)u_n &= r(A\mathcal{I}_r u_n + A\mathcal{I}_r v_n) = r(A\mathcal{I}_r u_n + Av_n) \\ &= rA\mathcal{I}_r u_n \to y \in \operatorname{Im} A. \end{aligned} \tag{2.23}$$

Denote $z_n = \mathcal{I}_r u_n \in X$. We have by (2.23)

$$z_n = u_n - rAz_n$$

and hence $z_n \in \operatorname{Im} A$. Since $\operatorname{Im} A$ is closed and invariant under A, and $Az_n \to \frac{1}{r}y \in \operatorname{Im} A$, the sequence $\{z_n\}$ converges to some element $z \in X$ and hence $u_n = z_n + rAz_n \to z + y = x$. Once again, it follows by (2.23) that $(I - \mathcal{I}_r)x = y$. ∎

Lemma 7. *Let the conditions of Lemma 6 hold. Then $\sigma(\mathcal{I}_r)$, the spectrum of the operator $\mathcal{I}_r$, is contained in the open unit disk Δ, except perhaps for 1, i.e.,*

$$\sigma(\mathcal{I}_r) \subset \Delta \bigcup \{1\}.$$

Proof. Fix $r > 0$. It follows by (2.20) that $\sigma(\mathcal{I}_r) \subseteq \overline{\Delta}$. It is known that $\sigma(A)$, the spectrum of the accretive operator A, lies in the right half-plane. Therefore there is an open domain $\Omega \subset \mathbf{C}$ such that $\sigma(A) \subset\subset \Omega$, and $\Gamma = \partial\Omega$ separates $\sigma(A)$ and the real number $\lambda = -r^{-1}$. Thus the function $f(\lambda) = (1 + r\lambda)^{-1}$ is holomorphic in Ω and $\mathcal{I}_r$ can be represented in the form

$$(I + rA)^{-1} = \frac{1}{2\pi i}\int_\Gamma (1 + r\lambda)^{-1}(\lambda I - A)^{-1} d\lambda = \tilde{f}(A).$$

It follows by the spectral mapping theorem (see, for example, [20]) that $\sigma(\tilde{f}(A)) = f(\sigma(A))$. Thus, if we assume that $e^{i\varphi} \in \sigma(\mathcal{I}_r)$ we get $\lambda = r^{-1}(e^{-i\varphi} - 1) \in \sigma(A)$ and hence $\operatorname{Re}\lambda \geq 0$. This implies that $\varphi = 0$ and the lemma is proved. ∎

Lemma 8. *(*cf. E. Vesentini [21]*). Let D be a bounded domain in a complex Banach space X, and let $F \in \operatorname{Hol}(D, D)$ have a fixed point $a \in D$ such that*

$$\sigma(F'(a)) \subset \Delta \bigcup \{1\}$$

and

$$\operatorname{Ker}(I - F'(a)) \oplus \operatorname{Im}(I - F'(a)) = X.$$

Then the sequence of iterates $\{F^n\}$ converges in the topology of local uniform convergence over D.

Proof. First we note that by P. Mazet's Theorem (see [16]) and the Vitali property of holomorphic mappings in the topology of local uniform convergence over D, we can assume that D is a convex domain in X. Then, by the Mazet-Vigué Theorem [17], there is a retraction $\psi : D \to \operatorname{Fix} F$ which satisfies the condition

$$\psi \circ F = \psi.$$

In addition, by the H. Cartan Theorem [4], in a neighbourhood U of the fixed point a of F we can find a local chart $g : U \to V$ such that $g(a) = 0$ and such that

$$g \circ \psi \circ g^{-1} = P$$

is a linear projection.

Now consider the mapping

$$G = g \circ F \circ g^{-1},$$

defined on some neighborhood W of zero, together with its iterates $G^n = g \circ F^n \circ g^{-1}$. (Indeed, by the boundedness of $\{F^n\}$ this sequence is uniformly Lipshitzian in some neighborhood of a. Hence, since $F^n(a) = a$, we can find a neighborhood W such that $F^n(g^{-1}(W)) \subseteq U$.) We now have

$$PG = g \circ \psi \circ g^{-1} g \circ F \circ g^{-1} = g \circ \psi \circ g^{-1} = P.$$

In addition, $G'(0) = g'(a) \circ F'(a) \circ [g'(a)]^{-1}$ and therefore $\sigma(G'(0)) = \sigma(F'(a))$, and P is a projection on $\mathrm{Ker}(I - G'(0))$. Thus, if $u = Px$ and $v = (I-P)x$, $x \in X$, we have $G(u,v) = (u, G_2(u,v))$, where $\sigma\left(\frac{\partial G_2}{\partial v}(0,0)\right) \subset\subset \Delta$, and $G_2(0,0) = 0$. Hence, for u small enough, the iterates $G^n(u,v) = (u, G_2(u, G_2^{(n-1)}(u,v)))$ converge locally uniformly to the mapping $h \in \mathrm{Hol}(g(W), W_1)$, where W_1 is a neighborhood of zero in X (see, for example, [17] and [13]). But then it follows that the iterates $F^n = g^{-1} \circ G^n \circ g$ also converge locally uniformly to the mapping $\varphi = g^{-1} \circ h \circ g$ in W. Using the Vitali property once again we obtain our assertion. ∎

2.2. *Proof of Theorem 1.* (1) Let D be a domain in a complex Banach space X, and let $\{G_s : s \in (0,T)\}$ be a family of holomorphic self-mappings of D which satisfies the approximate semigroup property (see Definition 2). Suppose that G_s converges to the identity uniformly on each subset strictly inside D. To show that the net

$$f_s = \frac{1}{s}(I - G_s) \tag{2.24}$$

is a Cauchy net, as $s \to 0^+$, on each subset D_1 strictly inside D, assume that $\epsilon > 0$ has been given. Choose $0 < d < \mathrm{dist}(D_1, \partial D)$, $0 < \mu < d$, such that

$$\frac{\mu}{d-\mu} < \epsilon, \tag{2.25}$$

and choose $0 < \omega \leq T$ such that

$$\sup_{x \in D_1} \|x - G_\tau(x)\| < \frac{\mu}{2} \tag{2.26}$$

for all $\tau \in (0, \omega)$.

Let $0 < \delta = \delta(D_1, \epsilon) \leq \omega$ be such that condition (i) (see Definition 2) is satisfied, i.e.,

$$(i)\ \|G_\tau(x) - G^p_{\frac{\tau}{p}}(x)\| \leq \epsilon\tau$$

whenever $x \in D_1$, and $\tau \in (0,\delta)$, $p = 1,2,\dots$. Now choose an integer $N > 0$ such that $N^{-1} < \delta$ and $\epsilon \cdot N^{-1} < \frac{1}{2}\mu$. Then, for all integers $m,n \geq N$ and all $k = 0,1,2,\dots,p = \max\{m,n\}$, we have by (2.2.6) and (i),

$$\sup_{x\in D_1} \|x - G^k_{\frac{1}{m\cdot n}}(x)\| \;\leq\; \sup_{x\in D_1} \|G^k_{\frac{1}{mn}}(x) - G_{\frac{k}{mn}}(x)\|$$

$$+ \sup_{x\in D_1} \|x - G_{\frac{k}{mn}}(x)\| \;<\; \epsilon\frac{k}{mn} + \frac{\mu}{2} < \mu.$$

Therefore, by (i) and Lemma 3, setting in this lemma $\phi = G_{\frac{1}{nm}}$ and $p = m$ we get for all $x \in D_1$,

$$\|x - G_{\frac{1}{n}}(x) - m\left(x - G_{\frac{1}{nm}}(x)\right)\| \;\leq\; \|x - G^m_{\frac{1}{nm}}(x) - m\left(x - G_{\frac{1}{nm}}(x)\right)\|$$

$$+\|G^m_{\frac{1}{nm}}(x) - G_{\frac{1}{n}}(x)\| \;\leq\; \frac{\mu}{d-\mu}m\|x - G_{\frac{1}{nm}}(x)\| + \epsilon\cdot\frac{1}{n}.$$

Multiplying this inequality by n and using (2.24) and (2.25) we obtain for $x \in D_1$,

$$\|f_{\frac{1}{n}}(x) - f_{\frac{1}{nm}}(x)\| \;\leq\; \epsilon\left(\|f_{\frac{1}{nm}}(x)\| + 1\right). \tag{2.27}$$

Now it follows by Lemma 4 that there is $\mathcal{L} = \mathcal{L}(D_1)$ such that

$$\|f_{\frac{1}{nm}}(x)\| < \mathcal{L}$$

for all $x \in D_1$, whenever N (and therefore $n\cdot m$) is big enough. So, by (2.26) we have

$$\|f_{\frac{1}{n}}(x) - f_{\frac{1}{nm}}(x)\| \;\leq\; \epsilon(\mathcal{L} + 1)\ .$$

In a similar way we can get

$$\|f_{\frac{1}{n}}(x) - f_{\frac{1}{nm}}(x)\| \;\leq\; \epsilon(\mathcal{L} + 1)$$

for all x in D_1 and $n,m \geq N$, and hence

$$\|f_{\frac{1}{n}}(x) - f_{\frac{1}{m}}(x)\| \;\leq\; 2\epsilon(\mathcal{L} + 1)$$

for all $x \in D_1$ whenever $n,m \geq N$. This inequality means that the sequence $\{f_{\frac{1}{n}}\}_{n=N}^{\infty}$ converges as $n \to \infty$ uniformly on each subset D_1 strictly inside D. In particular, it converges uniformly on each ball strictly inside D and is uniformly bounded on such a ball. Therefore, its limit

$$f \;=\; \lim_{n\to\infty} f_{\frac{1}{n}}$$

is a holomorphic mapping from D into X. Now we show that the net $\{f_s\}_{s\in(0,T)}$ converges to f uniformly on each subset D_1 strictly inside D. This will conclude the proof of the first assertion of our theorem.

For a given $\epsilon > 0$, and $x \in D_1$, setting $n = \left[\frac{1}{s^2}\right]$, we can choose s so small that

$$\|f_{\frac{1}{n}}(x) - f(x)\| < \epsilon. \tag{2.28}$$

In addition, for such s and n we have

$$f_s - f_{\frac{1}{n}} = \frac{1}{s}(I - G_s) - n\left(I - G_{\frac{1}{n}}\right) = \frac{1}{s}\left(G_{\frac{1}{n}}^{[sn]} - G_{\frac{[sn]}{n}}\right) + \frac{1}{s}\left(G_{\frac{[sn]}{n}} - G_s\right) + \frac{1}{s}\left[(I - G_{\frac{1}{n}}^{[sn]}) - ns(I - G_{\frac{1}{n}})\right] . \tag{2.29}$$

Observe that in our setting $n = \left[\frac{1}{s^2}\right]$, so that we have $ns \to \infty$ and $\frac{[ns]}{ns} \to 1$ as $s \to 0$. Thus we can find $\delta > 0$ such that $1 - \frac{[ns]}{ns} < \epsilon$ and $G_{\frac{[sn]}{n}}(x) \in D_2 \subset\subset D$ whenever $s \in (0, \delta)$ and $x \in D_1$. Using the approximate semigroup property (ii) (Definition 2) we get for such s and all $x \in D_1$,

$$\begin{aligned} &\frac{1}{s}\|G_{\frac{[sn]}{n}}(x) - G_s(x)\| \\ &\le \frac{1}{s}\|G_{\frac{[sn]}{n}}(x) - G_{\frac{[sn]}{n}} G_{s-\frac{[sn]}{n}}(x)\| \\ &+ \frac{1}{s}\|G_{\frac{[sn]}{n}} \circ G_{s-\frac{[sn]}{n}}(x) - G_s(x)\| \\ &\le \frac{1}{s}\left[M\|x - G_{s-\frac{[ns]}{n}}(x)\| + L\sqrt{\left(s - \frac{[sn]}{n}\right)\frac{[sn]}{n}}\,\right] \end{aligned} \tag{2.30}$$

where $M = \sup\limits_{x \in D_2, s \in (0,\delta)} \|(G_s)'(x)\|$. Once again, using Lemma 4, we have

$$\|x - G_{s-\frac{[ns]}{n}}\| \le \mathcal{L}\left(s - \frac{[ns]}{n}\right),$$

and therefore (2.30) implies

$$\frac{1}{s}\|G_{\frac{[sn]}{n}}(x) - G_s(x)\| \le \epsilon(M\mathcal{L} + L) . \tag{2.31}$$

Now condition (i) (Definition 2) implies

$$\frac{1}{s}\|G_{\frac{1}{n}}^{[sn]}(x) - G_{\frac{[sn]}{n}}(x)\| \le \epsilon\frac{[sn]}{sn} < \epsilon. \tag{2.32}$$

Finally, by Lemmas 3 and 4 we obtain for $x \in D_1$ and $s \in (0, \delta)$,

$$\begin{aligned} &\frac{1}{s}\|x - G_{\frac{1}{n}}^{[sn]}(x) - ns(x - G_{\frac{1}{n}}(x))\| \\ &\le \frac{1}{s}\|x - G_{\frac{1}{n}}^{[sn]}(x) - [ns](x - G_{\frac{1}{n}}(x))\| \\ &\quad + \frac{1}{s} \mid [ns] - ns \mid \|x - G_{\frac{1}{n}}(x)\| \\ &\le \left(\epsilon\frac{1}{s}[ns] + \frac{1}{s} \mid [ns] - ns \mid\right)\|x - G_{\frac{1}{n}}(x)\| \\ &\le \left(\epsilon\frac{[ns]}{ns} + \mid \frac{[ns]}{ns} - 1 \mid\right)\mathcal{L} \le 2\mathcal{L}\epsilon. \end{aligned} \tag{2.33}$$

Thus for $x \in D_1$ and $s \in (0, \delta)$ we get from (2.28)-(2.33),

$$\begin{aligned} \|f_s(x) - f(x)\| &\leq \|f_{\frac{1}{n}}(x) - f(x)\| + \|f_s(x) - f_{\frac{1}{n}}(x)\| \\ &\leq \epsilon(2 + M\mathcal{L} + L + 2\mathcal{L}), \end{aligned}$$

and we are done.

(2) Now we assume that D is a metric domain in X with some metric $\rho \in (SPS)$. Then it follows by Lemma 4 and Definitions 1 and 2 that conditions (i) and (ii) of Lemma 1 are satisfied. Therefore, by this lemma, for each pair s and t, $s \in (0, T)$, $t > 0$, and each sequence of integers $\{t_n\}$ such that $\frac{t_n s}{nt} \to 1$, there exists the strong limit $F_t = \lim_{n\to\infty} G^{t_n}_{\frac{s}{n}}$ uniformly on each subset strictly inside D. This limit, $F_t : R^+ \to \mathrm{Hol}(D, D)$, is a one-parameter semigroup of holomorphic self-mappings of D.

(3) Now we want to show that the mapping $f : D \to X$ defined in Assertion (1) generates the semigroup $\{F_t\}$, i.e., that

$$f = \lim_{t\to 0^+} \frac{I - G_t}{t}$$

also satisfies the condition

$$f = \lim_{t\to 0^+} \frac{I - F_t}{t}, \tag{2.34}$$

where the convergence in (2.34) is uniform on each $D_1 \subset\subset D$. Indeed, for a given $\epsilon > 0$ and $D_1 \subset\subset D$, we can find a small enough $\delta > 0$ such that

$$\|f(x) - \frac{x - G_t(x)}{t}\| < \epsilon$$

for all $x \in D_1$ and all $t \in (0, \delta)$. In addition, setting $s = t$ and $t_n = n$ we can find $\delta_1 < \delta$ such that

$$\frac{1}{t}\|G^n_{\frac{t}{n}}(x) - F_t(x)\| \leq \frac{1}{t}\epsilon t = \epsilon$$

(see (2.6)) for all $t \in (0, \delta)$. Once again, using (i), we take $\delta_2 < \delta$ such that

$$\frac{1}{t}\|G_t(x) - G^n_{\frac{t}{n}}(x)\| < \epsilon \text{ for } 0 < t < \delta_2.$$

Thus we get for $t \in (0, \delta_2)$,

$$\begin{aligned} \left\|f(x) - \frac{x - F_t(x)}{t}\right\| &\leq \left\|f(x) - \frac{x - G_t(x)}{t}\right\| \\ &\quad + \frac{1}{t}\left\|G_t(x) - G^n_{\frac{t}{n}}(x)\right\| + \frac{1}{t}\left\|G^n_{\frac{t}{n}}(x) - F_t(x)\right\| \\ &\leq 3\epsilon, \end{aligned}$$

and we are done.

Now it follows by the semigroup property and (2.34) that for each $x \in D$ the mapping $F(t, x) = F_t(x)$ is a solution of the Cauchy problem (1.5). This concludes the proof of Theorem 1. ∎

Corollary 1 is a direct consequence of Theorem 1.

To prove Corollary 2 we need to show that if $f \in \mathrm{Hol}(D, X)$ is bounded on each subset strictly inside D, and satisfies the range condition, then the family $\{\mathcal{J}_s = (I + sf)^{-1},\ s \in (0, T)\}$ converges to the identity uniformly on each subset strictly inside D, and satisfies the approximate semigroup property.

Indeed, let D be a metric domain with a metric $\rho \in (SPS)$, and let D_1 be a subset of D such that $\mathrm{dist}(D_1, \partial D) > 0$.

Denote $M_1 = \sup\{\|f(y)\| : y \in D_1\}$ and $\delta = \min\{\frac{d}{M}, T\}$, where $0 < d < \mathrm{dist}(D_1, \partial D)$. Setting $y = x + sf(x)$ for $x \in D$ and $s \in (0, \delta)$, we have $\mathcal{J}_s(y) = x,\ \|x - y\| < d$, and

$$(2.35)\quad \rho(\mathcal{J}_s(x), x) \;=\; \rho(\mathcal{J}_s(x), \mathcal{J}_s(y)) \;\le\; \rho(x, y) \;\le\; L\|x - y\| \;\le\; LM_1 s.$$

In turn, by using (2.1), this implies that for $x \in D,\ n = 1, 2, \ldots,$ and $k = 1, 2, \ldots n$,

$$(2.36)\qquad \|\mathcal{J}^{k}_{\frac{s}{n}}(x) - x\| \;\le\; \frac{1}{m}\rho\left(\mathcal{J}^{k}_{\frac{s}{n}}(x), x\right) \;\le\; \mathcal{L}\cdot s < \frac{d}{2}\,,$$

whenever $0 < s < \delta_1 = \min\{\delta, \frac{d}{2\mathcal{L}}\}$.

Firstly, (2.35) and (2.36) mean that the subset $D_2 = D_1 \bigcup\limits_{x \in D_1} B_d(x)$, which lies strictly inside D, where $B_d(x)$ is a closed ball with its center at x and radius d, contains the sets $\{\mathcal{J}_s(x)\}_{s \in (0, \delta_1)}$ and $\{\mathcal{J}^{k}_{\frac{s}{n}}(x)\},\ s \in (0, \delta),\ n = 1, 2, \ldots,$ and $k = 1, \ldots n$, where $x \in D_1$.

Secondly, it follows from (2.35) that the net $\{\mathcal{J}_s\}_{s \in (0, \delta)}$ converges to the identity uniformly on D_1. Now denote $M_2 = \sup\limits_{x \in D_2}\{\|f(x)\|\}$. It follows from the Cauchy inequalities that for each $x \in D_1$ and $y \in D$ such that $\|x - y\| < \frac{d}{2},\ \|f'(y)\| < 2M_2/d$, and hence, for such x and y we have

$$(2.37)\qquad \|f(x) - f(y)\| \;\le\; \frac{2M_2}{d}\|x - y\|.$$

Now, because of the identity

$$(2.38)\qquad x - \mathcal{J}_s(x) \;=\; sf(\mathcal{J}_s(x)),\ x \in D\,,$$

we have that

$$f(x) \;=\; \lim_{s \to 0^+} \frac{x - \mathcal{J}_s(x)}{s}$$

uniformly on D_1.

In addition, (2.36)-(2.38) imply that

$$
\begin{aligned}
&\|sf(x) - x + \mathcal{J}^n_{\frac{s}{n}}(x)\| \\
&\qquad \le \sum_{k=1}^{n} \|\frac{s}{n} f(x) + \mathcal{J}_{\frac{s}{n}}\left(\mathcal{J}^{k-1}_{\frac{s}{n}}(x)\right) - \left(\mathcal{J}^{k-1}_{\frac{s}{n}}(x)\right)\| \\
&\qquad = \sum_{k=1}^{n} \frac{s}{n} \|f(x) - f\left(\mathcal{J}^{k}_{\frac{s}{n}}(x)\right)\| \\
&\qquad \le \mathcal{L} \cdot \frac{2M_2}{d} \cdot s^2, \ x \in D.
\end{aligned}
\tag{2.39}
$$

Thus we obtain by (2.35), (2.38) and (2.39),

$$
\begin{aligned}
\|\mathcal{J}_s(x) - \mathcal{J}^n_{\frac{s}{n}}(x)\| &\le \|x - \mathcal{J}^n_{\frac{s}{n}}(x) - sf(x)\| + \|sf(x) - x + \mathcal{J}_s(x)\| \\
&\le \mathcal{L}\frac{2M_2}{d} s^2 + s\|f(x) - f(\mathcal{J}_s(x))\| \\
&\le \mathcal{L}\frac{4M_2}{d} \cdot s^2, \text{ for all } x \in D_1\ ,\ s \in (0, \delta_1),\ n = 1, 2, \dots\ .
\end{aligned}
$$

This inequality shows that condition (i) of Definition 2 is satisfied.

Now take positive s, t such that $s+t < \delta_1$. Then it follows by the resolvent identity (see Lemma 2) and (2.35) that

$$
\begin{aligned}
\|\mathcal{J}_{s+t}(x) - \mathcal{J}_t(x)\| &\le \frac{1}{m}\rho(\mathcal{J}_{s+t}(x), \mathcal{J}_t(x)) \\
&\le \frac{1}{m}\rho\left(\mathcal{J}_t(\frac{t}{s+t}x + \frac{s}{s+t}\mathcal{J}_{t+s}(x)),\ \mathcal{J}_t(x)\right) \\
&\le \frac{1}{m}\rho\left(\frac{t}{s+t}x + \frac{s}{s+t}\mathcal{J}_{t+s}(x), x\right) \\
&\le \frac{L}{M}\|x - \mathcal{J}_{s+t}(x)\|\frac{s}{s+t}\ \le\ \frac{L}{m} \cdot \mathcal{L} \cdot s.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\|\mathcal{J}_{s+t}(x) - \mathcal{J}_s(\mathcal{J}_t(x))\| &\le \frac{1}{m}\rho(\mathcal{J}_{s+t}(x), \mathcal{J}_s\mathcal{J}_t(x)) \\
&\le \frac{1}{m}\rho\left(\mathcal{J}_s(\frac{s}{s+t}x + \frac{t}{s+t}\mathcal{J}_{s+t}(x)),\ \mathcal{J}_s(\mathcal{J}_t(x))\right) \\
&\le \frac{L}{m}\|\frac{s}{s+t}x + \frac{t}{s+t}\mathcal{J}_{s+t}(x) - \mathcal{J}_t(x)\| \\
&\le \frac{L}{m}\left[\frac{s}{s+t}\|x - \mathcal{J}_t(x)\| + \frac{t}{s+t}\|\mathcal{J}_{s+t}(x) - \mathcal{J}_t(x)\|\right] \\
&\le \frac{L}{m}\frac{st}{s+t}\left[\mathcal{L} + \frac{L}{m}\mathcal{L}\right]\ \le\ \frac{1}{2}L_1\sqrt{st}.
\end{aligned}
$$

This proves condition (ii) of Definition 2 and we are done.

2.3. *Proof of Theorem 2.* Suppose that the conditions of Theorem 2 are satisfied. For a fixed $z \in D$ and $s, t > 0$, we consider the equation

$$x \;=\; \frac{s}{s+t} F_t(x) + \frac{t}{s+t} z \;=\; G_{s,t}(x). \tag{2.40}$$

Since ρ is compatible with the convex structure of D, we have by Definition 3,

$$\begin{aligned} \rho(G_{s,t}(x), G_{s,t}(y)) \;&\leq\; \varphi\left(\frac{s}{s+t}\right)\rho(F_t(x), F_t(y)) \\ &\leq\; \varphi\left(\frac{s}{s+t}\right)\rho(x,y), \end{aligned}$$

where $0 < \varphi\left(\frac{s}{s+t}\right) < 1$, and

$$\lim_{t\to 0^+}\sup \frac{t}{1-\varphi\left(\frac{s}{s+t}\right)} < \infty. \tag{2.41}$$

So, by the Banach fixed point theorem, it follows that equation (2.40) has a unique solution in D, $x = x_{s,t}$, for each $s, t > 0$. Since the equation

$$x + sf_t(x) \;=\; z,\;\; z \in D\;,$$

where $f_t = \frac{1}{t}(I - F_t)$, is equivalent to (2.40), this implies that the mapping $\mathcal{J}_{s,t} = (I + sf_t)^{-1}$ is a well-defined self-mapping of D and $\mathcal{J}_{s,t}(z) = x_{s,t}$. In addition, $\mathcal{J}_{s,t}(z)$ can be obtained by the approximation method

$$x_{s,t}^{(n)}(z) \;=\; \frac{s}{s+t}F_t\left(x_{s,t}^{(n-1)}(z)\right) + \frac{t}{s+t}z,$$

where $n = 1, 2, \ldots$, and $x_{s,t}^{0}(z) = y$ is an arbitrary element of D. Thus it follows by Definition 3 and induction that

$$\rho\left(x_{s,t}^{(n)}(z), x_{s,t}^{(n)}(w)\right) \;\leq\; \max\{\rho\left(x_{s,t}^{(n-1)}(z), x_{s,t}^{(n-1)}(w)\right), \rho(z,w)\} \;\leq\; \rho(z,w)\;,$$

because F_t is a ρ-nonexpansive mapping on D. Hence $\mathcal{J}_{s,t} : D \to D$ is also ρ-nonexpansive on D. Now we want to show that the net $\{\mathcal{J}_{s,t}\}_{s>0}$ converges to a mapping $\mathcal{J}_s : D \to D$, as $t \to 0^+$. If this holds, it is clear that $\mathcal{J}_s = (I + sf)^{-1}$ is a ρ-nonexpansive mapping of D and this will conclude our proof.

First we show that for each $z \in D$ and each $s > 0$, the net $\{\mathcal{J}_{s,t}(z)\}$ $(= \{x_{s,t}\})$ is strictly inside D for t small enough. Indeed,

$$\begin{aligned} \rho(\mathcal{J}_{s,t}(z), z) \;&=\; \rho\left(\frac{s}{s+t}F_t(x_{s,t}) + \frac{t}{s+t}z, z\right) \\ &\leq\; \varphi\left(\frac{s}{s+t}\right)\rho(F_t(x_{s,t}), z) \\ &\leq\; \varphi\left(\frac{s}{s+t}\right)[\rho(F_t(x_{s,t}), F_t(z)) + \rho(F_t(z), z)] \\ &\leq\; \varphi\left(\frac{s}{s+t}\right)[\rho(x_{s,t}, z) + \rho(F_t(z), z)]\,. \end{aligned}$$

Thus it follows by Definition 1 and (2.41) that

$$\lim_{t\to 0^+}\sup \rho(x_{s,t}, z) \ \le \ \lim_{t\to 0^+}\sup \frac{L\cdot t}{1-\varphi\left(\frac{s}{s+t}\right)}\frac{\|z-F_t(z)\|}{t} \ \le \ C \ < \ \infty.$$

So, there is a ρ-ball in D which contains the set $\{x_{s,t} = \mathcal{J}_{s,t}(z)\}$ whenever t is small enough. Now we want to show that for a fixed $s > 0$ and a given $z \in D$ this net is a Cauchy net as $t \to 0^+$. Indeed, if we denote, as before, $x_{s,t} = \mathcal{J}_{s,t}(z)$ and $z_{t,r} = (I + sf_r)(x_{s,t})$, we have $z = (I + sf_t)(x_{s,t})$ and $\{z_{t,r}\}$ converges to z as $t, r \to 0^+$, because $\{f_t\}$ is a Cauchy net. But $\mathcal{J}_{s,r}(z_{t,r}) = x_{s,t}$ and we get

$$\begin{aligned}\rho(\mathcal{J}_{s,t}(z), \mathcal{J}_{s,r}(z)) &\le \rho(\mathcal{J}_{s,t}(z), \mathcal{J}_{s,t}(z_{t,r})) + \rho(\mathcal{J}_{s,t}(z_{t,r}), \mathcal{J}_{s,r}(z_{t,r}))\\ &\le \rho(z, z_{t,r}) \to 0\end{aligned} \tag{2.42}$$

as $t, r \to 0^+$. The proof is complete. ■

Corollary 3 is a direct consequence of Corollary 2 and Theorem 2.

2.4. *Proof of Theorem 3.* Let D be a bounded convex domain in a complex Banach space X, and let $\{G_s\}_{s\in(0,T)}$, $T > 0$, be a family of holomorphic self-mappings of D such that

$$\lim_{s\to 0^+}\frac{I-G_s}{s} = f$$

exists uniformly on each subset strictly inside D and is bounded on such subsets. Then, by Theorem 2, f satisfies the range condition. Hence Corollary 3 implies that f generates a one-parameter semigroup $F_t = F(t,\cdot)$ which is a solution of the Cauchy problem (1.5).

Now let ρ be a metric on D assigned to D by a Schwarz-Pick system. Without loss of generality suppose that $0 \in D$. Then, for an arbitrary subset K strictly inside D, one can find a ρ-ball $\mathcal{B}_R(0)$ with radius R, centered at 0, such that $K \subset \mathcal{B}_R(0)$.

Since $\{F_t\}_{t>0}$ is a semigroup which is uniformly continuous on $\mathcal{B}_R(0)$, it follows from (2.3) (see the proof of Lemma 1) that there is $\mu > 0$ such that for each $\tau \in (0, \mu)$ and each integer ℓ,

$$S(F_\tau^\ell(x), x) \le \ell\tau\mathcal{L}$$

for some $\mathcal{L} < \infty$, whenever $x \in \mathcal{B}_R(0)$.

Now fix any $t > 0$ and take n so large that $\frac{t}{n} \in (0, \mu)$. Then we have for such n and all $x \in \mathcal{B}_R(0)$,

$$\rho(F^\ell_{\frac{t}{n}}(x), 0) \le \rho(F^\ell_{\frac{t}{n}}(x), x) + \rho(x, 0) \le \frac{\ell t}{n}\mathcal{L} + R.$$

This means that for all $\ell = 0, 1, \ldots, n-1$, the set $\{F^\ell_{\frac{t}{n}}(x)\}$ lies in the ρ-ball $\mathcal{B}_{R_1}(0)$, where $R_1 = t\mathcal{L} + R$, whenever $x \in K \subset \mathcal{B}_R(0)$.

Since

$$\lim_{s\to 0^+}\frac{I-F_s}{s} = f$$

uniformly on $\mathcal{B}_{R_1}(0) := K_1 \subset\subset D$, we have that for each $\epsilon > 0$ there is $0 < \eta = \eta(K_1, \epsilon) \le \mu$ such that

$$\rho(F_s(y), G_s(y)) \le s \cdot \epsilon$$

for all $y \in K_1$. Now take N so large that $s = \frac{t}{n} \in (0, \eta)$ for all $n > N$. Then, for all $x \in K$ we have for such n,

$$\begin{aligned} \rho(F_t(x), G^n_{\frac{t}{n}}(x)) &= \rho(F^n_{\frac{t}{n}}(x), G^n_{\frac{t}{n}}(x)) \\ &\le \sum_{k=1}^{n} \rho(G_s^{k-1}(G_s(F_s^{n-k}(x))), G_s^{k-1}(F_s(F_s^{n-k}(x)))) \\ &\le \sum_{k=1}^{n} \rho(G_s(y_k), F_s(y_k)) \le t\epsilon \ , \end{aligned}$$

where $y_k = F_s^{n-k}(x) \in K_1$. The latter inequality proves our second claim.

2.5. *Proof of Theorem 4.* (i) Let $f \in \mathrm{Hol}(D, X)$ be a generator, and let $\{F_t\}_{t\ge 0}$ be the semigroup of holomorphic self-mappings of D generated by f. Suppose that $\mathcal{W} = \mathrm{Null}_D f = \bigcap_{t>0} \mathrm{Fix}_D F_t$ contains a quasi-regular point $a \in D$ (see Definition 4). Observe that the linear operator $A = f'(a)$ is the infinitesimal generator of the semigroup $\{U_t\}_{t\ge 0}$, where $U_t = (F_t)'(a)$. Since D is bounded, it follows by the Cauchy inequalities that $\{U_t = e^{-At}\}_{t\ge 0}$ is uniformly bounded and therefore A is an m-accretive operator with respect to some norm equivalent to the norm of X. In additon, for each $r > 0, (I + rA)^{-1} = [(I + rf)^{-1}]'(a)$ by the chain rule. Thus by applying Lemmas 5-8 we see that for each $r > 0$ the sequence $\{\mathcal{J}_r^n\}_{n=1}^{\infty}$, where $\mathcal{J}_r = (I + rf)^{-1} : D \to D$, converges locally uniformly to some holomorphic mapping $\varphi : D \to D$ which is a retraction onto the fixed point set of $\mathcal{J}_r$. But this set coincides with $\mathcal{W}$ and we are done.

(ii) Now let $a \in D$ be a regular null point of f. Once again, by Lemmas 5 and 6, this means that for each $r > 0$ the spectral radius of the operator $(\mathcal{J}_r)'(a)$ is less than 1. Thus there is an equivalent norm $\|\cdot\|_1$ on X such that $\|(\mathcal{J}_r')(a)\|_1 < 1$. It follows by continuity that in this norm there is a ball $B_R(a)$ centered at a with radius R $(= R(r))$ such that $B_R(a) \subset\subset D$ and $\|(\mathcal{J}_r)'(x)\|_1 \le q_r < 1$ for each $x \in B_R(a)$. Fix $r > 0$ and take any $t \ge r$. Using the resolvent identity and the equality $\mathcal{J}_t(a) = a$, $t > 0$, we have for all $x \in B_R(a)$,

$$\begin{aligned} \|\mathcal{J}_t(x) - a\|_1 &= \|\mathcal{J}_t(x) - \mathcal{J}(a)\|_1 \\ &= \|\mathcal{J}_r\left(\frac{r}{t}x + (1 - \frac{r}{t})\mathcal{J}_t(x)\right) - \mathcal{J}_r(a)\|_1 \\ &\le q_r\|\frac{r}{t}x + (1 - \frac{r}{t})\mathcal{J}_t(x) - a\|_1 \\ &\le q_r\frac{r}{t}\|x - a\|_t \ + \ q_r(1 - \frac{r}{t})\|\mathcal{J}_t(x) - a\|_1. \end{aligned} \tag{2.43}$$

Thus we obtain the inequality

$$\|\mathcal{J}_t(x) - a\|_1 \leq \frac{q_r \frac{r}{t}}{1 - q_r(1 - \frac{r}{t})} \|x - a\|_1, \tag{2.44}$$

which implies that $\mathcal{J}_t(x)$ converges to a as $t \to \infty$, uniformly on $B_R(a)$. Now it follows by the Vitali property that $\{\mathcal{J}_t\}_{t>0}$ converges to a locally uniformly on all of D. The theorem is proved. ∎

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