ANALYTICITY OF THERMO-ELASTIC SEMIGROUPS WITH COUPLED HINGED/NEUMANN B.C.

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ABSTRACT. We consider a thermo-elastic plate system where the elastic equation does not account for rotational forces. We select the case of hinged mechanical B.C. and Neumann thermal B.C., which are *coupled* on the boundary. We show that the corresponding s.c. contraction semigroup (on a natural energy space) is *analytic* and, hence, uniformly stable. Because of the boundary (high) coupling, this case of B.C. is not contained in, and is more challenging than, recent known cases of the literature [L-R.1], [L-L.1], [L-T.1].

1. INTRODUCTION. PROBLEM STATEMENT. MAIN RESULT

Dynamics. Let Ω be a two-dimensional bounded domain with smooth boundary Γ. On Ω we consider a thermo-elastic plate problem in the displacement w and in the temperature θ, where the elastic equation does not account for rotational forces. Moreover, in this paper, we focus on the case of *coupled* B.C. which arise with hinged mechanical B.C. and Neumann (Robin) thermal B.C. (see literature below). The model, once stripped from lower-order terms and with (inessential) constants normalized to 1, is as follows

[Lag.1]:

$$\begin{cases} w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0,T] \times \Omega = Q; & (1.1a) \\[2mm] \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q; & (1.1b) \\[2mm] w(0,\,\cdot\,) = w_0;\ w_t(0,\,\cdot\,) = w_1;\ \theta(0,\,\cdot\,) = \theta_0 & \text{in } \Omega; & (1.1c) \\[2mm] w \equiv 0;\ \Delta w + (1-\mu)B_1 w + \theta = 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma; & (1.1d) \\[2mm] \frac{\partial\theta}{\partial\nu} + b\theta = 0 & \text{on } \Sigma; & (1.1e) \\[2mm] B_1 w = -c(x)\,\frac{\partial w}{\partial\nu} & \text{on } \Sigma. & (1.1f) \end{cases}$$

The assumption that the boundary Γ be smooth means that, in particular, the mean curvature $c(\,\cdot\,) \in L_\infty(\Gamma)$.

Abstract setting. First, we let $\mathcal{A}$ be the following positive, self-adjoint operator on $L_2(\Omega)$:

(1.2) $\qquad \begin{aligned} &\mathcal{A}h = \Delta^2 h; \\ &\mathcal{D}(\mathcal{A}) = \{h \in H^4(\Omega) \cap H_0^1(\Omega) : \ \Delta h + (1-\mu)B_1 h = 0 \text{ on } \Gamma\}. \end{aligned}$

Next, we introduce the positive self-adjoint operators $\mathcal{A}_D$ and $\mathcal{A}_N$, respectively:

(1.3) $\mathcal{A}_D h = -\Delta h;\ \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega),$

(1.4) $\mathcal{A}_N h = -\Delta h;\ \mathcal{D}(\mathcal{A}_N) = \left\{h \in H^2(\Omega) : \left[\frac{\partial h}{\partial\nu} + bh\right]_\Gamma = 0\right\},\ b > 0.$

We have [Gr.1],

(1.5) $\qquad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega) = \mathcal{D}(\mathcal{A}_D)$ (equivalent norms).

Accordingly, we introduce the following space (equivalent norms),

(1.6) $\qquad \begin{aligned} Y &= [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega) \times L_2(\Omega) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega) \\ &= \mathcal{D}(\mathcal{A}_D) \times L_2(\Omega) \times L_2(\Omega), \end{aligned}$

and the following Green map G, defined by

(1.7) $\qquad h = Gg \iff \begin{cases} \Delta^2 h &= \ 0 \text{ in } \Omega; \\ h|_\Gamma &= \ 0;\ (\Delta + (1-\mu)B_1)h|_\Gamma = g. \end{cases}$

Elliptic regularity [L-M.1; p. 188–189] gives:

(1.8a) $\qquad \begin{aligned} G: \ &\text{continuous } L_2(\Gamma) \to H^{\frac{5}{2}}(\Omega) \cap H_0^1(\Omega) \subset H^{\frac{5}{2}-4\epsilon}(\Omega) \cap H_0^1(\Omega) \\ &\qquad\qquad\qquad\qquad\qquad = \mathcal{D}\left(\mathcal{A}^{\frac{5}{8}-\epsilon}\right),\quad \epsilon > 0 \end{aligned}$

and

(1.8b) $\qquad\qquad \mathcal{A}^{\frac{5}{8}-\epsilon}G: \ \text{continuous } L_2(\Gamma) \to L_2(\Omega),$

where the identification in (1.8a) follows from [G.1]. By Green's second theorem, one readily obtains (see e.g., [L-T.4, Chapter 3, Section 12] for details)

$$(1.9) \qquad G^* \mathcal{A} f = \frac{\partial f}{\partial \nu}, \quad f \in \mathcal{D}(\mathcal{A}),$$

where $(Gg, y)_{L_2(\Omega)} = (g, G^* y)_{L_2(\Gamma)}$, $g \in L_2(\Gamma)$, $y \in L_2(\Omega)$. Using the definitions of G, $\mathcal{A}$, $\mathcal{A}_D$, $\mathcal{A}_N$ in (1.8), (1.2)–(1.4), we rewrite problem (1.1), as usual, as the following second-order abstract system ([L-T.4, Chapter 3, Section 12]):

$$\begin{cases} w_{tt} + \mathcal{A}w + \mathcal{A}G(\theta|_\Gamma) - \mathcal{A}_N\theta &= 0 \text{ in } [\mathcal{D}(\mathcal{A})]'; \qquad (1.10) \\ \theta_t + \mathcal{A}_N\theta + \mathcal{A}_D w_t &= 0. \qquad (1.11) \end{cases}$$

Setting $y = [w, w_t, \theta]$, we then rewrite the above second-order system as the first-order equation

$$(1.12) \quad \dot{y} = Ay, \ A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{A}_N - \mathcal{A}G(\,\cdot\,|_\Gamma) \\ 0 & -\mathcal{A}_D & -\mathcal{A}_N \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y,$$

to be interpreted in the sense that

$$(1.13) \qquad A = \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = \begin{bmatrix} w_2 \\ -\mathcal{A}[G(\theta|_\Gamma) + w_1] + \mathcal{A}_N\theta \\ -\mathcal{A}_D w_2 - \mathcal{A}_N\theta \end{bmatrix}, \ \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \in \mathcal{D}(A),$$

where, recalling Y in (1.6), we obtain from (1.13),

$$(1.14) \qquad \begin{aligned} \mathcal{D}(A) = \Big\{ &w_1 \in \mathcal{D}(\mathcal{A}_D) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \ w_2 \in \mathcal{D}(\mathcal{A}_D); \ \theta \in \mathcal{D}(\mathcal{A}_N); \\ &w_1 + G(\theta|_\Gamma) \in \mathcal{D}(\mathcal{A}) \Big\}. \end{aligned}$$

Semigroup generation. The following result can be proved by standard methods: part (i) via the Lumer-Phillips Theorem; part (ii) by direct computation; see e.g. [L-T.4, Chapter 3, Section 12] for details.

Proposition 1.1. (i) The operator A in (1.13), (1.14) is densely defined, dissipative, in fact maximal dissipative, and thus generates a s.c. contraction semigroup: $[w_1, w_2, \theta_0] \in Y \to e^{At}[w_1, w_2, \theta_0] = [w(t), w_t(t), \theta(t)]$ on Y.

(ii) The operator A has compact (inverse A^{-1}, explicitly given in [L-T.4, Chapter 3, Section 12], hence) resolvent on Y, and there is no spectrum (= point spectrum) of A on the closed right half-plane $\{\lambda : \text{Re } \lambda \geq 0\}$.

Analyticity of e^{At}. The goal of this paper is to prove the following.

Theorem 1.2. The s.c. contraction semigroup e^{At} of Proposition 1.1 is, moreover, analytic on Y, $t > 0$.

Uniform stability of e^{At}. Once Theorem 1.2 is established, it then follows via Proposition 1.1(ii) that the s.c. contraction, analytic semigroup e^{At} is also uniformly stable in $\mathcal{L}(Y)$.

Corollary 1.3. There exist constants $M \geq 1$ and $\sigma > 0$, such that $\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-\sigma t}, \ t \geq 0$.

Literature. The elastic equation (1.1a) is of Euler-Bernoulli type and thus it does not account for rotational terms. At first, thermo-elastic plate equations with (Kirchoff-type) or without (Euler-Bernoulli-type) rotational terms were the object of several studies showing asymptotic exponential stability of their solutions. See a detailed literature overview with a comprehensive list of references in [Lag.1], [Las.1], [L-R.1], [L-L.1], [L-T.1]. This means that heat dissipation alone is sufficiently strong to induce exponential energy decay.

Elastic model with no rotational term. Here, however, it was only recently that a much stronger and more desirable result was established in [L-R.1], at least for one demanding set of B.C., via a technical ad hoc proof: that in the case of clamped B.C. for w/Dirichlet B.C. for θ, the associated s.c. contraction semigroup is, in fact, *analytic.* [Once analyticity is established, it is not difficult to infer that the semigroup is also *uniformly stable*, by excluding the possibility that the generator has spectrum on the imaginary axis]. It is plainly desirable to have an abstract setting and an abstract proof of analyticity, which covers and encompasses at least several sets of physical B.C. Such goal is achieved in both the recent papers [L-L.1] and in [L-T.1] (see also [L-T.4, Chapter 3; Appendix F] and [T.1]). The proof in [L-L.1] is by a contradiction argument: it assumes that the well-known characterization of analyticity based on the resolvent of the generator is violated and gets a contradiction. By contrast, two direct proofs of analyticity for an abstract thermo-elastic model are given in [L-T.1]. Though technically and conceptually very different, these two proofs of [L-T.1] have in common the following basic idea: to fall into the setting of [C-T.1–2], and use the analyticity of the 2×2 structurally damped matrix operator as the main 'driver,' which is responsibile for carrying the analyticity of the heat component onto the mechanical component through the coupling. Moreover, [L-T.1] presents several P.D.E. thermo-elastic plate examples which recover, in particular: (i) the case of clamped B.C. in w/Dirichlet B.C. for θ treated ad hoc in [L-R.1]; (ii) the case of physical hinged B.C./Dirichlet B.C. for θ treated ad hoc in [T.1]; (iii) the case of partially clamped/partially hinged B.C. in w and Dirichlet B.C. in θ, which is the only thermo-elastic illustration for analyticity given in [L-L.1]. In adition, [L-T.1] offers a few more examples: clamped/Neumann B.C.; and a damped free B.C. It should be noted that both the abstract operator theoretic thermo-elastic model of [L-T.1], [L-T.4, Chapter 3; Appendix F] and the variational, differently conceived, abstract model of [L-L.1] do *not* cover the case of *coupled* B.C. such as they occur in (1.1d–e). Moreover, their abstract, operator proofs cannot be adjusted

to include a problem such as (1.1), which—because of the high *coupling* in the B.C. between w and θ, represents an additional serious level of difficulty over [L-L.1], [L-T.1] when it comes to showing analyticity. By contrast, the proof of analyticity of the present paper requires P.D.E. methods and trace estimates. An even much more demanding case—the case of free B.C.—will be handled separately, again by ad hoc, more elaborate, P.D.E. methods and trace estimates [L-T.5].

Elastic model with rotational terms. Here the structure of the corresponding s.c. semigroup is quite different; and is more akin to the stucture of a s.c. uniformly stable *group* (hence not analytic, not differentiable, not even continuous in the uniform topology for $t > 0$, not compact); see [Cg-T.1], [L-T.2]; see also [H-P-L.1] for a different thermo-elastic *wave* (rather than *plate*) model, and [T-Z.1] for the specific case of a thermo-elastic plate with clamped/Dirichlet B.C.

The focus of this paper is Theorem 1.2 on analyticity. Results in Section 1 leading to this are standard and hence simply stated.

2. Proof of Theorem 1.2

2.1. **General strategy and preliminaries. General strategy**. With reference to the space Y in (1.6), let $f_0 \in Y$ be arbitrary

$$(2.1.1) \quad \begin{cases} f_0 = [u_0, v_0, \theta_0] \in Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega), \\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega) \text{ (equivalent norms)}. \end{cases}$$

With reference to the operator A in (1.12)–(1.14), let ω be real, $\omega \in \mathbb{R}$, and define

$$(2.1.2) \quad y(\omega) = [u(\omega), v(\omega), \theta(\omega)] = [i\omega I - A]^{-1} f_0 = R(i\omega, A) f_0 \in \mathcal{D}(A),$$

where the resolvent of A is well-defined on the imaginary axis, see Proposition 1.1(ii).

Our *goal* is to show that the following uniform estimate holds true: there exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$, with say $|\omega| \geq 1$,

$$(2.1.3) \quad \left\| \begin{bmatrix} u(\omega) \\ v(\omega) \\ \theta(\omega) \end{bmatrix} \right\|_Y = \|y(\omega)\|_Y = \|R(i\omega, A) f_0\|_Y \leq \frac{C}{|\omega|} \|f_0\|_Y.$$

Once estimate (2.1.3) has been established for the generator A of the s.c. contraction semigroup e^{At} asserted by Proposition 1.1, we can invoke a known result e.g. [L-T.4, Chapter 3, Theorem E.3 of Appendix E] and obtain that the s.c. semigroup e^{At} is, in fact, analytic on Y, $t > 0$. In order to establish estimate (2.1.3)—and hence prove Theorem 1.2—we shall pursue the following strategy which consists in proving the following three simultaneous estimates for the components of $y(\omega)$ in (2.1.2): for all $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$, with $|\omega| > 1$, the vector

$y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$ in (2.1.2) satisfies

$$
\left\{
\begin{aligned}
\|u(\omega)\|_{H^2(\Omega)}^2 &\leq \epsilon\|y(\omega)\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2; &\qquad (2.1.4)\\[2ex]
\|v(\omega)\|_{L_2(\Omega)}^2 &\leq \epsilon\|y(\omega)\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2; &\qquad (2.1.5)\\[2ex]
\|\theta(\omega)\|_{L_2(\Omega)}^2 &\leq \epsilon\|y(\omega)\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2. &\qquad (2.1.6)
\end{aligned}
\right.
$$

Hereafter, we drop from $y = [u, v, \theta]$ the explicit dependence on ω. Estimates (2.1.4)–(2.1.6) are proved below, in Proposition 2.4.1, Eqn. (2.4.1); Proposition 2.6.1, Eqn. (2.6.1); and Proposition 2.7.1, Eqn. (2.7.1), respectively. Clearly, summing up estimates (2.1.4) through (2.1.6) (once established) yields the final desired estimate (2.1.3) with constant $C = (3C_\epsilon/1 - 3\epsilon)^{\frac{1}{2}}$.

Preliminaries. By (1.13), we obtain explicitly from (2.1.2),

(2.1.7)
$$
(i\omega I - A)
\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}
=
\begin{bmatrix}
i\omega u - v \\
i\omega v + \mathcal{A}[u + G(\theta|_\Gamma)] - \mathcal{A}_N\theta \\
i\omega\theta + \mathcal{A}_D v + \mathcal{A}_N\theta
\end{bmatrix}
=
\begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix}
= f_0 \in Y,
$$

or, upon dividing by ω, $|\omega| \geq 1$:

$$
\left\{
\begin{aligned}
I : &\qquad\qquad\qquad\qquad iu - \frac{v}{\omega} &= \frac{u_0}{\omega}; &\qquad (2.1.8)\\[2ex]
II : &\quad iv + \frac{1}{\omega}\mathcal{A}[u + G(\theta|_\Gamma)] - \frac{1}{\omega}\mathcal{A}_N\theta &= \frac{v_0}{\omega}; &\qquad (2.1.9)\\[2ex]
III : &\qquad\qquad i\theta + \frac{1}{\omega}\mathcal{A}_D v + \frac{1}{\omega}\mathcal{A}_N\theta &= \frac{\theta_0}{\omega}, &\qquad (2.1.10)
\end{aligned}
\right.
$$

where, recalling (1.14) we have *a-fortiori* the following regularity properties

(2.1.11) $y = [u, v, \theta] \in \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_N)$,

Orientation. The basic "driving" term in the present proof is the thermal estimate (2.2.3) below for θ, which follows at once from the basic *a-priori* dissipativity condition (2.2.2). To achieve the desired estimates (2.1.4) through (2.1.6), we shall employ the "driving" estimate (2.2.3) repeatedly, along with *a-priori* bounds in the right norms, *to dominate* each norm-quantity $\|q\|$ of interest, as follows:

(2.1.12) $\|q\| \leq [a + b][\epsilon a + k_\epsilon b] \leq 2\epsilon a^2 + C_\epsilon b^2, \ a, b \geq 0$,

to be specialized with $a = \|y\|_Y$ and $b = \left\|\frac{f_0}{\omega}\right\|_Y$ [inequality (2.1.12) is obtained with $C_\epsilon = \left(\frac{k_\epsilon^2}{2\epsilon} + k_\epsilon + \frac{\epsilon}{2}\right)$ by using $\epsilon ab \leq \frac{\epsilon}{2}(a^2 + b^2)$ and $k_\epsilon ab \leq \frac{k_\epsilon}{2}\left(\frac{\epsilon}{k_\epsilon}a^2 + \frac{k_\epsilon}{\epsilon}b^2\right)$].

2.2. **A-priori bounds for** θ, v, **and** u. Part (i) of the following lemma is obtained by standard integration by parts, and is in fact behind the property of dissipativity of A noted in Proposition 1.1(i). See [L-T.4, Chapter 3, Section 12] for details.

Lemma 2.2.1. (Preliminary *a-priori* bounds for θ) With reference to (2.1.1), (2.1.2), we have with $\omega \in \mathbb{R}$:
(i)

$$(2.2.1) \qquad (\mathcal{A}_N \theta, \theta)_{L_2(\Omega)} = \mathrm{Re}\left([i\omega I - A]\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right)_Y = \mathrm{Re}(f_0, y)_Y;$$

(ii)

$$(2.2.2) \qquad \|\theta\|_{H^1(\Omega)}^2 \doteq \|\mathcal{A}_N^{\frac{1}{2}}\theta\|_{L_2(\Omega)}^2 \leq \|f_0\|_Y \|y\|_Y,$$

where here and henceforth, $\doteq$ denotes equivalence of norms;
(iii) for any $\epsilon > 0$ and $\omega \in \mathbb{R}$, $\omega \neq 0$,

$$(2.2.3) \qquad \frac{1}{|\omega|}\|\theta\|_{H^1(\Omega)}^2 \doteq \frac{1}{|\omega|}\|\mathcal{A}_N^{\frac{1}{2}}\theta\|_{L_2(\Omega)}^2 \leq \frac{\epsilon}{2}\|y\|_Y^2 + \frac{1}{2\epsilon}\left\|\frac{f_0}{\omega}\right\|_Y^2,$$

Lemma 2.2.2 (*A-priori* bounds for v). With reference to (2.1.1) and (2.1.2), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:
(i)

$$(2.2.4) \qquad \frac{1}{|\omega|}\|\mathcal{A}_D v\|_{L_2(\Omega)} \doteq \frac{1}{|\omega|}\|v\|_{H^2(\Omega)} \leq \|u\|_{H^2(\Omega)} + \left\|\frac{u_0}{\omega}\right\|_{H^2(\Omega)}$$

$$(2.2.5) \qquad \qquad \qquad \qquad \leq \|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y;$$

(ii)

$$(2.2.6) \qquad \frac{1}{\sqrt{|\omega|}}\|\mathcal{A}_D^{\frac{1}{2}}v\|_{L_2(\Omega)} \doteq \frac{1}{\sqrt{|\omega|}}\|v\|_{H^1(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right].$$

Proof. (i) The validity of estimate (2.2.4) stems at once from Eqn. I = (2.1.8), and the norm equivalence in (2.1.1). Then, (2.2.4) implies at once (2.2.5) by majorizing u and u_0/ω in $H^2(\Omega)$ by y and f_0/ω in Y, via (2.1.1) and (2.1.2).

(ii) By interpolation (moment inequality [L-M.1]), we compute [henceforth we use freely $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$ throughout], via (2.2.5) and majorizing v by y, by (2.1.1), (2.1.2):

$$(2.2.7) \qquad \|v\|_{H^1(\Omega)} \leq C\|v\|_{H^2(\Omega)}^{\frac{1}{2}}\|v\|_{L_2(\Omega)}^{\frac{1}{2}}$$

$$(2.2.8) \qquad \text{(by (2.2.5))} \qquad \leq C|\omega|^{\frac{1}{2}} \left[\|y\|_Y^{\frac{1}{2}} + \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}} \right] \|y\|_Y^{\frac{1}{2}}$$

$$(2.2.9) \qquad\qquad\qquad \leq C|\omega|^{\frac{1}{2}} \left[\|y\|_Y + \frac{1}{2} \left\| \frac{f_0}{\omega} \right\| + \frac{1}{2} \|y\|_Y \right],$$

and (2.2.9) proves estimate (2.2.6), as desired. ∎

Lemma 2.2.3 (Further *a-priori* bound for θ). With reference to (2.1.1) and (2.1.2), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:

$$(2.2.10) \qquad \frac{1}{|\omega|} \|\theta\|_{H^2(\Omega)} \doteq \frac{1}{|\omega|} \|\mathcal{A}_N \theta\|_{L_2(\Omega)} \leq 2 \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right].$$

Proof. We return to Eqn. III = (2.1.10), where we use estimate (2.2.5) for v,

$$(2.2.11) \quad \left\| \frac{1}{\omega} \mathcal{A}_N \theta \right\|_{L_2(\Omega)} = \left\| \frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega} \mathcal{A}_D v \right\|_{L_2(\Omega)}$$

$$(2.2.12) \qquad\qquad \leq \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)} + \|\theta\|_{L_2(\Omega)} + \frac{1}{|\omega|} \|\mathcal{A}_D v\|_{L_2(\Omega)}$$

$$(2.2.13) \quad \text{(by (2.2.5))} \quad \leq \left\| \frac{f_0}{\omega} \right\|_Y + \|y\|_Y + \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right],$$

majorizing, in the last step, θ_0 and θ by f_0 and y via (2.1.1), (2.1.2). Then, (2.2.13) proves (2.2.10). ∎

Lemma 2.2.4 (*a-priori* bounds for u). With reference to (2.1.1) and (2.1.2), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:

(i)

$$(2.2.14) \qquad\qquad \frac{1}{|\omega|} \|u\|_{H^4(\Omega)} \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right];$$

(ii)

$$(2.2.15) \qquad\qquad \frac{1}{\sqrt{|\omega|}} \|u\|_{H^3(\Omega)} \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right].$$

Proof. (i) Eqn. II = (2.1.9) rewritten abstractly as

$$(2.2.16) \qquad\qquad \frac{1}{\omega} \mathcal{A}[u + G(\theta|_\Gamma)] = -iv + \frac{1}{\omega} \mathcal{A}_N \theta + \frac{v_0}{\omega}$$

is equivalent, via the definition (1.7) of the Green operator G, to the following elliptic boundary value problem (i.e., the original elliptic problem (1.7)), of which (2.2.16) is the abstract version, in the first place):

$$\begin{cases} \Delta^2 \left(\dfrac{u}{\omega} \right) = -iv + \frac{1}{\omega} \mathcal{A}_N \theta + \frac{v_0}{\omega} & \text{in } \Omega; & (2.2.17) \\[2mm] u|_\Gamma \equiv 0; \ [\Delta u + (1-\mu)B_1 u|_\Gamma] = -\theta|_\Gamma & \text{on } \Gamma. & (2.2.18) \end{cases}$$

From the right-hand side of (2.2.17), we readily estimate, by virtue of (2.2.10) for $\mathcal{A}_N\theta/\omega$, majorizing v and θ_0 by y and f_0 via (2.1.1), (2.1.2),

$$(2.2.19) \qquad \left\|\Delta^2\left(\frac{u}{\omega}\right)\right\|_{L_2(\Omega)} \le 3\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right].$$

Moreover, from the second B.C. in (2.2.18), we readily estimate for $|\omega| \ge 1$, by virtue of trace theory, (1.1f) for B_1, and again (2.2.10),

$$(2.2.20) \qquad \begin{aligned} \left\|\Delta\left(\frac{u}{\omega}\right)\right\|_{H^{\frac{3}{2}}(\Gamma)} &\le \frac{C}{|\omega|}\|u\|_{H^2(\Omega)} + \left\|\frac{\theta}{\omega}\right\|_{H^{\frac{3}{2}}(\Gamma)} \\ &\le C\left[\|u\|_{H^2(\Omega)} + \frac{1}{|\omega|}\|\theta\|_{H^2(\Omega)}\right] \end{aligned}$$

$$(2.2.21) \qquad \text{(by (2.2.10))} \quad \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right],$$

majorizing u in $H^2(\Omega)$ by y in Y, via (2.1.1), (2.1.2). Thus, standard elliptic theory [L-M.1; p. 188–9] applied to the 'right-hand side' estimate (2.2.19), the first B.C. in (2.2.18) and the boundary estimate (2.2.21), produces a gain of $2\frac{1}{2}$ Sobolev units from boundary to interior ($\frac{3}{2} + 2\frac{1}{2} = 4$), and a gain of 4 Sobolev units from 'right-hand side' to interior ($0 + 4 = 4$), thus yielding

$$(2.2.22) \qquad \left\|\frac{u}{\omega}\right\|_{H^4(\Omega)} \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right],$$

which proves (2.2.14). [We note that 'right-hand side' and boundary estimates produce, *independently*, the *same* interior regularity of $\frac{u}{\omega}$ in $H^4(\Omega)$.]

(ii) By interpolation (moment inequality), we estimate via (2.2.14), and majorizing u by y, by (2.1.1), (2.1.2):

$$(2.2.23) \qquad \begin{aligned} \|u\|_{H^3(\Omega)} &\le C\|u\|_{H^4(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}} \\ &\le C|\omega|^{\frac{1}{2}}\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right]\|y\|_Y^{\frac{1}{2}} \end{aligned}$$

$$(2.2.24) \qquad \le C|\omega|^{\frac{1}{2}}\left[\|y\|_Y + \frac{1}{2}\left\|\frac{f_0}{\omega}\right\|_Y + \frac{1}{2}\|y\|_Y\right],$$

and (2.2.24) proves estimate (2.2.15), as desired. ∎

Remark 2.2.1. One could, alternatively, sum up Eqns. II = (2.1.9) and Eqn. III = (2.1.10) to eliminate $[\mathcal{A}_N\theta/\omega]$, and then use estimate (2.2.5) for $[\mathcal{A}_D v/w]$ (rather than estimate (2.2.10) on $[\mathcal{A}_N\theta/\omega]$, which is a consequence of (2.2.5)), to obtain likewise the interior bound (2.2.19). It is, however, at the level of obtaining the boundary estimate (2.2.21) that estimate (2.2.10) is needed.

2.3. **A fundamental estimate on $\frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)}$.** The following result—a consequence of the 'driving' term (2.2.3)—is fundamental.

Proposition 2.3.1. With reference to (2.1.1), (2.1.2), given $\epsilon > 0$ there exists $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have

$$(2.3.1) \qquad \left| \frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)} \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2.$$

Proof. Step 1. By Green's first theorem with $v \in \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\theta \in \mathcal{D}(\mathcal{A}_N)$, we have

$$(2.3.2) \qquad \begin{aligned} -\frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)} &= \frac{1}{\omega} \int_\Omega \Delta v \bar{\theta}\, d\Omega \\ &= \frac{1}{\omega} \int_\Gamma \frac{\partial v}{\partial \nu} \bar{\theta}\, d\Gamma - \frac{1}{\omega} \int_\Omega \nabla v \cdot \nabla \bar{\theta}\, d\Omega. \end{aligned}$$

Step 2. Lemma 2.3.2. In the same notation of Proposition 2.3.1, we have

(i)

$$(2.3.3) \qquad \left| \frac{1}{\omega} \int_\Omega \nabla v \cdot \nabla \bar{\theta}\, d\Omega \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2;$$

(ii)

$$(2.3.4) \qquad \left| \frac{1}{\omega} \int_\Gamma \frac{\partial v}{\partial \nu} \bar{\theta}\, d\Gamma \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2.$$

Assuming for the moment the validity of Lemma 2.3.2, we see that the desired conclusion (2.3.1) follows by use of estimates (2.3.3) and (2.3.4) in identity (2.3.2).

To prove Lemma 2.3.2, we shall use, for each part, inequality (2.1.12) plus *a-priori* bounds.

Step 3. Proof of inequality (2.3.3). By the *a-priori* bound (2.2.6) and the 'driving' bound (2.2.3), we estimate

$$(2.3.5) \quad \left| \frac{1}{\omega} \int_\Omega \nabla v \cdot \nabla \bar{\theta}\, d\Omega \right| \leq \left(\frac{1}{\sqrt{|\omega|}} \|v\|_{H^1(\Omega)} \right) \left(\frac{1}{\sqrt{|\omega|}} \|\theta\|_{H^1(\Omega)} \right)$$

$$(2.3.6)$$

$$\text{(by (2.2.6) and (2.2.3))} \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right] \left[\sqrt{\epsilon_1} \|y\|_Y + \frac{1}{\sqrt{\epsilon_1}} \left\| \frac{f_0}{\omega} \right\|_Y \right]$$

$$(2.3.7) \qquad \text{(by (2.1.12))} \leq C \sqrt{\epsilon_1} \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2,$$

Step 4. Proof of inequality (2.3.4). We recall the *a-priori* inequalities [Th.1], [B-S.1, p. 37]

$$(2.3.8) \qquad \left\| \frac{\partial v}{\partial \nu} \right\|_{L_2(\Gamma)} \leq C \|v\|_{H^2(\Omega)}^{\frac{1}{2}} \|v\|_{H^1(\Omega)}^{\frac{1}{2}}$$

$$(2.3.9) \qquad \|\theta|_\Gamma\|_{L_2(\Gamma)} \leq C \|\theta\|_{H^1(\Omega)}^{\frac{1}{2}} \|\theta\|_{L_2(\Omega)}^{\frac{1}{2}}.$$

Then, using (2.3.8) and (2.3.9),

$$(2.3.10) \qquad \left| \frac{1}{\omega} \int_\Gamma \frac{\partial v}{\partial \nu} \bar{\theta} \, d\Gamma \right| \leq \frac{1}{|\omega|} \left\| \frac{\partial v}{\partial \nu} \right|_\Gamma \right\|_{L_2(\Gamma)} \|\theta|_\Gamma\|_{L_2(\Gamma)}$$

$$\leq C \frac{\|v\|_{H^2(\Omega)}^{\frac{1}{2}}}{\sqrt{|\omega|}} \frac{\|v\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \|\theta\|_{L_2(\Omega)}^{\frac{1}{2}}.$$

Taking the $\frac{1}{2}$-th power of the *a-priori* bounds (2.2.5) and (2.2.6), we obtain the following uniform bound for $|\omega| \geq 1$:

$$(2.3.11) \quad \frac{\|v\|_{H^2(\Omega)}^{\frac{1}{2}}}{\sqrt{|\omega|}} \frac{\|v\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \leq C \left[\|y\|_Y^{\frac{1}{2}} + \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \left[\|y\|_Y^{\frac{1}{2}} + \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right]$$

$$(2.3.12) \qquad \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right].$$

On the other hand, taking the $\frac{1}{4}$-power of the 'driving' bound (2.2.3) and majorizing θ by y by (2.1.1), (2.1.2), we obtain the following uniform bound for $|\omega| \geq 1$,

$$(2.3.13) \qquad \frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \|\theta\|_{L_2(\Omega)}^{\frac{1}{2}} \leq 2^{-\frac{1}{4}} \left[\epsilon_1 \|y\|_Y^{\frac{1}{2}} + \frac{1}{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \|y\|_Y^{\frac{1}{2}}$$

$$(2.3.14) \qquad \leq 2^{-\frac{5}{4}} \left[3\epsilon_1 \|y\|_Y + \frac{1}{\epsilon_1^3} \left\| \frac{f_0}{\omega} \right\|_Y \right].$$

Using estimates (2.3.12) and (2.3.14) on the right-hand side of (2.3.10), we obtain

$$(2.3.15) \quad \left| \frac{1}{\omega} \int_\Gamma \frac{\partial v}{\partial \nu} \bar{\theta} \, d\Gamma \right| \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right] \left[\epsilon_1 \|y\|_Y + \frac{1}{\epsilon_1^3} \left\| \frac{f_0}{\omega} \right\|_Y \right]$$

$$(2.3.16) \qquad \text{(by (2.1.12))} \leq C \left[2\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right],$$

recalling (2.1.12) in the last step, $\epsilon_1 > 0$ being arbitrary. Then, inequality (2.3.16) proves (2.3.4), as desired. The proof of Lemma 2.3.2 is complete, and so is the proof of Proposition 2.3.1. ∎

2.4. **Proof of estimate (2.1.6) for θ.** As a corollary of the 'driving' estimate (2.2.3) for θ, as well as of Proposition 2.3.1 (which also stems from (2.2.3)), we obtain the desired inequality (2.1.6) for θ.

Proposition 2.4.1. With reference to (2.1.1), (2.1.2), given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have

$$(2.4.1) \qquad \|\theta\|^2_{L_2(\Omega)} \leq \epsilon \|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y.$$

Proof. We return to Eqn. III $=$ (2.1.10), take here the $L_2(\Omega)$-inner product with θ, use estimates (2.3.1) and (2.2.3), and obtain

$$\|\theta\|^2_{L_2(\Omega)} \leq \left|\frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)}\right|$$

$$(2.4.2) \qquad + \frac{1}{|\omega|}\|\mathcal{A}_N^{\frac{1}{2}}\theta\|^2_{L_2(\Omega)} + \left|\left(\frac{\theta_0}{\omega}, \theta\right)_{L_2(\Omega)}\right|$$

$$\text{(by (2.3.1) and (2.2.3))} \leq \left[\epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y\right] + \frac{1}{2}\left[\epsilon\|y\|^2_Y + \frac{1}{\epsilon}\left\|\frac{f_0}{\omega}\right\|^2_Y\right]$$

$$(2.4.3) \qquad + \frac{\epsilon}{2}\|\theta\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon}\left\|\frac{\theta_0}{\omega}\right\|^2_Y.$$

Then, the desired inequality (2.4.1) readily follows from (2.4.3), by majorizing θ_0 by f_0 and θ by y, via (2.1.1), (2.1.2). ∎

2.5. **Improving upon** *a-priori* **bounds.** The 'driving' estimate (2.2.3) and the *a-priori* bound (2.2.6) for $[\mathcal{A}_D^{\frac{1}{2}}v/\sqrt{|\omega|}]$ yield

Lemma 2.5.1. With reference to (2.1.1), (2.1.2), given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$,

$$(2.5.1) \qquad \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| \leq \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y.$$

Proof. With $\theta \in \mathcal{D}(\mathcal{A}_N)$ and $v \in \mathcal{D}(\mathcal{A}_D) \subset H^1(\Omega) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, we estimate by (2.2.3) and (2.2.6):

$$(2.5.2) \qquad \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| = \left|\left(\frac{\mathcal{A}_N^{\frac{1}{2}}\theta}{\sqrt{|\omega|}}, \frac{\mathcal{A}_N^{\frac{1}{2}}v}{\sqrt{|\omega|}}\right)_{L_2(\Omega)}\right|$$

$$\leq \frac{\|\mathcal{A}_N^{\frac{1}{2}}\theta\|_{L_2(\Omega)}}{\sqrt{|\omega|}} \frac{\|\mathcal{A}_N^{\frac{1}{2}}v\|_{L_2(\Omega)}}{\sqrt{|\omega|}}$$

$$(2.5.3) \quad \text{(by (2.2.3), (2.2.6))} \leq C\left[\epsilon_1\|y\|_Y + \frac{1}{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]$$

$$(2.5.4) \qquad \text{(by (2.1.12))} \leq C\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2,$$

after invoking (2.1.12) in the last step, for an arbitrary $\epsilon_1 > 0$. Then, (2.5.4) proves (2.5.1), as desired. ∎

The following result—a corollary of Proposition 2.4.1, Eqn. (2.4.1), for θ—improves upon the *a-priori* bound (2.2.6) for v.

Lemma 2.5.2. With reference to (2.1.1), (2.1.2), given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have

$$(2.5.5) \qquad \frac{1}{|\omega|} \|v\|_{H^1(\Omega)}^2 \doteq \frac{1}{|\omega|} \|\mathcal{A}_D^{\frac{1}{2}} v\|_{L_2(\Omega)}^2 \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2.$$

Proof. We return to Eqn. III $=$ (2.1.10), take here the $L_2(\Omega)$-inner product with v, invoke estimates (2.4.1) and (2.5.1), thereby obtaining

$$(2.5.6) \qquad \frac{1}{|\omega|} \|\mathcal{A}_D^{\frac{1}{2}} v\|_{L_2(\Omega)}^2 = \left| \frac{1}{\omega} (\mathcal{A}_D v, v)_{L_2(\Omega)} \right|$$

$$= \left| \left(\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega} \mathcal{A}_N \theta, v \right)_{L_2(\Omega)} \right|$$

$$(2.5.7) \qquad \leq \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} + \|\theta\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}$$

$$+ \left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right|$$

$$(2.5.8) \ \text{(by (2.4.1), (2.5.1))} \leq \left[\frac{\epsilon_1}{2} \|v\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_1} \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)}^2 \right]$$

$$+ \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \|y\|_Y$$

$$+ \left[\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right]$$

$$(2.5.9) \qquad \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2,$$

majorizing v by y twice via (2.1.1), (2.1.2), once from (2.5.7) to (2.5.8), and once from (2.5.8) to (2.5.9). Eqn. (2.5.9) proves (2.5.5). ∎

2.6. **Proof of estimate (2.1.5) for v.** As a corollary of Lemma 2.5.2 and of the *a-priori* bound Eqn. (2.2.15), Lemma 2.2.4 on u, we obtain the desired estimate (2.1.5) for v.

Proposition 2.6.1. With reference to (2.1.1), (2.1.2), given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$:

$$(2.6.1) \qquad \|v\|_{L_2(\Omega)}^2 \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2.$$

Proof. Step 1. We return to Eqn. II $= (2.1.9)$ and take here the $L_2(\Omega)$-inner product with v, thereby obtaining, see definitions (1.2) and (1.7) of the operators $\mathcal{A}$ and G:

$$(2.6.2) \quad \|v\|_{L_2(\Omega)}^2 \leq \left| \frac{1}{\omega} \int_\Omega \Delta^2 u \bar{v} \, d\Omega \right| + \left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right| + \left| \left(\frac{\theta_0}{\omega}, v \right)_{L_2(\Omega)} \right|.$$

Step 2. Lemma 2.6.2. In the notation of Proposition 2.6.1, we have

$$(2.6.3) \qquad \left| \frac{1}{\omega} \int_\Omega \Delta^2 u \bar{v} \, d\Omega \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2.$$

Proof of Lemma 2.6.2. Since $v \in \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega)$, then $\bar{v}|_\Gamma = 0$ and Green's first theorem yields by virtue of (2.2.15) and (2.5.5),

$$(2.6.4) \qquad \left| \frac{1}{\omega} \int_\Omega \Delta^2 u \bar{v} \, d\Omega \right| = \left| \frac{1}{\omega} \int \nabla \Delta u \cdot \nabla \bar{v} \, d\Omega \right|$$

$$(2.6.5) \qquad\qquad \leq \left(\frac{1}{\sqrt{|\omega|}} \|u\|_{H^3(\Omega)} \right) \left(\frac{1}{\sqrt{|\omega|}} \|v\|_{H^1(\Omega)} \right)$$

$$(2.6.6) \text{ (by (2.2.15), (2.5.5))} \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right] \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right]$$

$$(2.6.7) \qquad \text{(by (2.1.12))} \leq C \left[\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right],$$

invoking (2.1.12) in the last step, with $\epsilon_1 > 0$ arbitrary. Then (2.6.7) proves (2.6.3). ∎

Step 3. We use estimate (2.6.3) and estimate (2.5.1) on the right-hand side of (2.6.2) to obtain

$$(2.6.8) \qquad \begin{aligned} \|v\|_{L_2(\Omega)}^2 \leq{} & 2 \left[\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] \\ & + \left[\frac{\epsilon_1}{2} \|v\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_1} \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)}^2 \right], \end{aligned}$$

from which the desired estimate (2.6.1) follows at once, majorizing v and θ_0 by y and f_0, via (2.1.1), (2.1.2). ∎

2.7. **Proof of estimate (2.1.4) for u.**

Proposition 2.7.1. With reference to (2.1.1), (2.1.2), given $\epsilon > 0$, there is $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$,

$$(2.7.1) \qquad \|u\|^2_{H^2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y.$$

Proof.   Step 1. We have already noted in Lemma 2.6.2, recalling the definitions (1.2) and (1.7) of the operators $\mathcal{A}$ and G, that

$$(2.7.2) \qquad \begin{aligned} \left|\frac{1}{\omega}(\mathcal{A}[u + G(\theta|_\Gamma)], v)_{L_2(\Omega)}\right| &= \left|\frac{1}{\omega}\int_\Omega \Delta^2 u\bar{v}\, d\Omega\right| \\ &\leq \epsilon_1\|y\|^2_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^2_Y. \end{aligned}$$

Step 2. On the other hand, by substituting $\frac{v}{w} = iu - \frac{u_0}{\omega}$ from Eqn. I $=$ (2.1.8), we obtain

$$(2.7.3)\ \frac{1}{\omega}(\mathcal{A}[u + G(\theta|_\Gamma)], v)_{L_2(\Omega)} = \left(\mathcal{A}u, iu - \frac{u_0}{\omega}\right)_{L_2(\Omega)}$$

$$+ \frac{1}{\omega}(\theta|_\Gamma, G^*\mathcal{A}v)_{L_2(\Gamma)}$$

$$(2.7.4) \qquad \text{(by (1.9))} \qquad = -i\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} - \left(\mathcal{A}^{\frac{1}{2}}u, \frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right)_{L_2(\Omega)}$$

$$+ \frac{1}{\omega}\left(\theta|_\Gamma, \left.\frac{\partial v}{\partial \nu}\right|_\Gamma\right)_{L_2(\Gamma)},$$

recalling $G^*\mathcal{A}v = \frac{\partial v}{\partial \nu}$ from (1.9) in the last step.

Step 3. Combining (2.7.3) with (2.7.2), we obtain via the norm equivalence in (1.5):

$$(2.7.5) \qquad \|u\|^2_{H^2(\Omega)} \dot{=} \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} \leq \left[\epsilon_1\|y\|^2_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^2_Y\right]$$

$$+ \left[\frac{\epsilon_1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon_1}\left\|\frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right\|^2_Y\right]$$

$$+ \left|\frac{1}{\omega}\left(\theta|_\Gamma, \left.\frac{\partial v}{\partial \nu}\right|_\Gamma\right)_{L_2(\Gamma)}\right|$$

(2.7.6)   (by (2.3.4)) $\leq \epsilon_2 \|y\|_Y^2 + C_{\epsilon_2} \left\| \dfrac{f_0}{\omega} \right\|_Y^2 + \dfrac{\epsilon_1}{2} \|\mathcal{A}^{\frac{1}{2}} u\|_{L_2(\Omega)}^2,$

where in going from (2.7.5) to (2.7.6) we have invoked the boundary estimate (2.3.4) of Lemma 2.3.2, and have majorized $\mathcal{A}^{\frac{1}{2}} u_0$ by f_0 by (2.1.1). Estimate (2.7.6) readily leads to the desired conclusion (2.7.1). ∎

The required estimates (2.1.4), (2.1.5), (2.1.6) are all proved.

Final Remark. Having shown the required estimate (2.7.1) for u, we return to Eqn. I = (2.1.8) and obtain: given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have

(2.7.7) $\dfrac{1}{|\omega|} \|\mathcal{A}_D v\|_{L_2(\Omega)} \doteq \dfrac{1}{|\omega|} \|v\|_{H^2(\Omega)} \leq \epsilon \|y\|_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|_Y.$

Moreover, armed with estimate (2.7.7), we return to Eqn. III = (2.1.10), use here also estimate (2.4.1) for θ, and obtain

(2.7.8) $\dfrac{1}{|\omega|} \|\theta\|_{H^2(\Omega)} \doteq \dfrac{1}{|\omega|} \|\mathcal{A}_N \theta\|_{L_2(\Omega)} = \left\| \dfrac{\theta_0}{\omega} - i\theta - \dfrac{1}{\omega} \mathcal{A}_D v \right\|_{L_2(\Omega)}$

(2.7.9)   (by (2.4.1), (2.7.7)) $\leq \epsilon \|y\|_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|_Y,$

majorizing also θ_0 by f_0 via (2.1.1). Inequality (2.7.9) complements the 'driving' estimate (2.2.3) as well as its consequence (2.4.1).

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