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*Research Article*

# A Note on Double Laplace Transform and Telegraphic Equations

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Double Laplace transform is applied to solve general linear telegraph and partial integrodifferential equations. The scheme is tested through some examples, and the results demonstrate reliability and efficiency of the proposed method.

## 1. Introduction

The wave equation is known as one of three fundamental equations in mathematical physics and occurs in many branches of physics, applied mathematics, and engineering. It is also known that there are two types of these equation: the homogenous equation that has constant coefficient with many classical solutions such as separation of variables [1], the methods of characteristics [2, 3], single Laplace transform, and Fourier transform [4] and nonhomogenous equations with constant coefficient solved by double Laplace transform [5] and operation calculus [6].

In this study, we use double Laplace transform to solve telegraph equation and partial integrodifferential equation. We follow the method that was proposed by Kılıçman and Eltayeb [7] where they extended one-dimensional convolution theorem to two-dimensional case [8].

First of all, we recall the following definitions given by Kılıçman and Gadain [9]. The double Laplace transform is defined by

$$L_x L_t \left[ f(x,s) \right] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)\, dt\, dx, \quad \text{where } x,t > 0 \text{ and } p, s \text{ are complex numbers,} \tag{1}$$

and the first-order partial derivative is defined as follows:

$$L_x L_t \left[ \frac{\partial f(x,t)}{\partial x} \right] = pF(p,s) - F(0,s). \tag{2}$$

Double Laplace transform for second partial derivative with respect to $x$ is given by

$$L_{xx} \left[ \frac{\partial^2 f(x,t)}{\partial^2 x} \right] = p^2 F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x}, \tag{3}$$

and double Laplace transform for second partial derivative with respect to $t$ similarly as the previous is given by

$$L_{tt} \left[ \frac{\partial^2 f(x,t)}{\partial^2 t} \right] = s^2 F(p,s) - sF(p,0) - \frac{\partial F(p,0)}{\partial t}. \tag{4}$$

In a similar manner, the double Laplace transform of a mixed partial derivative can be deduced from a single Laplace transform as

$$L_x L_t \left[ \frac{\partial^2 f(x,t)}{\partial x \partial t} \right] = psF(p,s) - pF(p,0) - sF(0,s) - F(0,0). \tag{5}$$

**Theorem 1.** *If at the point $(p,q)$ the integral*

$$F_1(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} f_1(x,y)\, dx\, dy \tag{6}$$

*is convergent and the integral*

$$F_2(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} f_2(x,y)\, dx\, dy \tag{7}$$

*is absolutely convergent, then*

$$F(p,q) = F_1(p,q)\,F_2(p,q) \tag{8}$$

*is the Laplace transform of the function*

$$f(x,y) = \int_0^x \int_0^y f_1(x-\zeta, y-\eta)\, f_2(\zeta,\eta)\, d\zeta\, d\eta \tag{9}$$

*and the integral*

$$F(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy} f(x,y)\, dx\, dy \tag{10}$$

*is convergent at the point* $(p,q)$.

*Proof.* See [4]. □

Next, we study the uniqueness and existences of double Laplace transform. First of all, let $f(x,t)$ be a continuous function on the interval $[0,\infty)$ which is of exponential order, that is, for some $a, b \in \mathbb{R}$. Consider

$$\sup_{\substack{t>0\\x>0}} \frac{|f(x,t)|}{e^{ax+bt}} < \infty. \tag{11}$$

In this case, the double Laplace transform of

$$F(p,s) = \int_0^\infty \int_0^\infty e^{-st-px} f(x,t)\, dx\, dt \tag{12}$$

exists for all $p > a$ and $s > b$ and is in fact infinitely differentiable with respect to $p > a$ and $s > b$. All functions in this study are assumed to be of exponential order. The following theorem shows that $f(x,t)$ can be uniquely recovered from $F(p,s)$.

**Theorem 2.** *Let $f(x,t)$ and $g(x,t)$ be continuous functions defined for $x,t \geq 0$ and having Laplace transforms, $F(p,s)$, and $G(p,s)$, respectively. If $F(p,s) = G(p,s)$, then $f(x,t) = g(x,t)$.*

*Proof.* If $\alpha$ and $\beta$ are sufficiently large, then the integral representation, by

$$f(x,t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{px} \left( \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{st} F(p,s)\, ds \right) dp \tag{13}$$

for the double inverse Laplace transform, can be used to obtain

$$\begin{aligned} f(x,t) &= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{px} \left( \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{st} F(p,s)\, ds \right) dp \\ &= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{px} \left( \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{st} G(p,s)\, ds \right) dp \\ &= g(x,t), \end{aligned} \tag{14}$$

and the theorem is proven. □

**Theorem 3.** *A function $f(x,t)$ which is continuous on $[0,\infty)$ and satisfies the growth condition (11) can be recovered from $F(p,s)$ as*

$$f(x,t) = \lim_{\substack{m\to\infty\\n\to\infty}} \frac{(-1)^{m+n}}{m!n!} \left(\frac{m}{x}\right)^{m+1} \left(\frac{n}{t}\right)^{n+1} \Psi^{m+n}\left(\frac{m}{x}, \frac{n}{t}\right), \tag{15}$$

*where $\Psi^{m+n}$ is denoted by $(m+n)$th mixed partial derivatives of $F(p,s)$, defined by $\Psi^{m+n} = \partial^{m+n}F(p,s)/\partial p^m \partial s^n$ for $x,t \geq 0$ since the previous theorem obtains $f(x,t)$ in terms of $F(p,s)$.*

Of course, the main difficulty in using Theorem 3 for computing the inverse Laplace transform is the repeated symbolic differentiation of $F(p,s)$. However, one can apply Theorem 3 in the next type of examples.

*Example 4.* Let $f(x,t) = e^{-ax-bt}$. The Laplace transform is easily found to be as follows:

$$F(p,s) = \frac{1}{(p+a)(s+b)}. \tag{16}$$

It is also simple to verify that

$$\frac{\partial^{m+n} F(p,s)}{\partial p^m \partial s^n} = m!n!(-1)^{m+n}(p+a)^{-m-1}(s+b)^{-n-1}. \tag{17}$$

Putting this expression for $\partial^{m+n}F(p,s)/\partial p^m \partial s^n$ into Theorem 3 gives the following:

$$\begin{aligned} f(x,t) &= \lim_{\substack{m\to\infty\\n\to\infty}} \frac{m^{m+1}n^{n+1}}{x^{m+1}t^{n+1}} \left(a+\frac{m}{x}\right)^{-m-1} \left(b+\frac{n}{t}\right)^{-n-1} \\ &= \lim_{\substack{m\to\infty\\n\to\infty}} \left(1+\frac{ax}{m}\right)^{-m-1} \left(1+\frac{bt}{n}\right)^{-n-1}. \end{aligned} \tag{18}$$

The last limit is easy to evaluate. Take the natural log of both sides, and write the result in the form of $-(\ln(1 + ax/m)/(1/(m+1))) - (\ln(1+bt/n)/(1/(n+1)))$. L'Hopital's rule reveals that the indeterminate from approaches $-ax-bt$. The continuity of the natural logarithm shows that $\ln(f(x,t)) = -ax - bt$; then, $f(x,t) = e^{-ax-bt}$.

## 2. Properties of Double Laplace Transform

In this part, we consider some of the properties of the double Laplace Transform that will enable us to find further transform pairs $\{f(x,t), F(p,s)\}$ without having to compute consider the following.

(I) $F(p+d, s+c) = L_x L_t \left[e^{-dx-ct} f(x,t)\right](p,s)$,

(II) $\frac{1}{k} F\left(\frac{p}{\alpha}, \frac{s}{\beta}\right) = L_x L_t \left[f(\alpha x, \beta t)\right](p,s)$,

where $k = \alpha\beta$,

(III) $\frac{\partial^{m+n} F(p,s)}{\partial p^m \partial^n} = L_x L_t \left[(-1)^{m+n} x^m t^n f(x,t)\right](p,s)$. (19)

Was first verify (I) as

$$\begin{aligned} &L_x L_t\left[e^{-dx-ct} f(x,t)\right](p,s) \\ &\quad = \int_0^\infty \int_0^\infty e^{-st-px} e^{-dx-ct} f(x,t)\, dt\, dx \\ &\quad = \int_0^\infty e^{-dx-px}\left(\int_0^\infty e^{-st-ct} f(x,t)\, dt\right) dx. \end{aligned} \tag{20}$$

We calculate the integral inside bracket as

$$\int_0^\infty e^{-st-ct} f(x,t)\, dt = F(x, s+c). \tag{21}$$

By substituting, we obtain

$$\begin{aligned} L_x L_t\left[e^{-dx-ct} f(x,t)\right](p,s) &= \int_0^\infty e^{-dx-px} F(x, s+c)\, dx \\ &= F(p+d, s+c). \end{aligned} \tag{22}$$

Second, the right hand side of (II) can be written in the form of

$$\begin{aligned} &L_x L_t\left[f(\alpha x, \beta t)\right](p,s) \\ &\quad = \int_0^\infty \int_0^\infty e^{-st-px} f(\alpha x, \beta t)\, dx\, dt \\ &\quad = \int_0^\infty e^{-st}\left(\int_0^\infty e^{-px} f(\alpha x, \beta t)\, dx\right) dt \\ &\quad = \int_0^\infty e^{-st} \frac{1}{\alpha} F\left(\frac{p}{\alpha}, \beta t\right) dt = \frac{1}{\alpha\beta} F\left(\frac{p}{\alpha}, \frac{s}{\beta}\right). \end{aligned} \tag{23}$$

The last property, from definition of double Laplace transform

$$F(p,s) = \int_0^\infty \int_0^\infty e^{-st-px} f(x,t)\, dx\, dt, \tag{24}$$

so that

$$\frac{\partial^{m+n} F(p,s)}{\partial p^m \partial s^n} = \frac{\partial^{m+n}}{\partial p^m \partial s^n} \int_0^\infty \int_0^\infty e^{-st-px} f(x,t)\, dx\, dt. \tag{25}$$

Owing to the convergence properties of the improper integral involved, we can interchange the operation of differentiation and integration and differentiate with respect to $p$, $s$ under the integral sign. Thus,

$$\frac{\partial^{m+n} F(p,s)}{\partial p^m \partial s^n} = \int_0^\infty \frac{\partial^n}{\partial s^n} e^{-st}\left(\int_0^\infty \frac{\partial^m}{\partial p^m} e^{-px} f(x,t)\, dx\right) dt, \tag{26}$$

which, on carrying out the repeated differentiation with respect to $p$, $s$, gives the following:

$$\begin{aligned} \frac{\partial^{m+n} F(p,s)}{\partial p^m \partial s^n} &= (-1)^{m+n} \int_0^\infty \int_0^\infty x^m t^n e^{-st-px} f(x,t)\, dx\, dt \\ &= (-1)^{m+n} L_x L_t\left[x^m t^n f(x,t)\right](p,s). \end{aligned} \tag{27}$$

The previous three properties are very useful at the proof of Theorem 3.

*Proof of Theorem 3.* Let us define the set of functions depending on parameters $m$, $n$ as

$$\begin{aligned} g_{m,n}(x,t) &= \frac{m^{m+1} n^{n+1}}{m!n!} x^m t^n e^{-mx-nt} \\ \text{so } \int_0^\infty \int_0^\infty g_{m,n}(x,t)\, dx\, dt &= 1, \\ \lim_{m\to\infty} \lim_{n\to\infty} \int_0^\infty \int_0^\infty g_{m,n}(x,t)\, \varphi(x,t)\, dx\, dt &= \varphi(1,1), \end{aligned} \tag{28}$$

where $\varphi(x,t)$ is any continuous function. Let us denote its Laplace transform as a function of $p$, $s$ by $L_x L_t[\varphi(x,t)](p,s)$. Now, we define the function $\Psi(x,t) = f(xx_0, tt_0)$, and using property (II), we have

$$\begin{aligned} &L_x L_t\left[\Psi(x,t)\right](p,s) \\ &\quad = L_x L_t\left[f(xx_0, tt_0)\right](p,s) = \frac{1}{x_0 t_0} F\left(\frac{p}{x_0}, \frac{s}{t_0}\right). \end{aligned} \tag{29}$$

We apply property (III) (we must evaluate the $m+n$ mixed partial derivatives of $F(p,s)$ at the points $p = m/x$ and $s = n/t$) as follows:

$$\begin{aligned} &\frac{\partial^{m+n}}{\partial p^m \partial s^n}\left(L_x L_t\left[\Psi(x,t)\right]\right)(p,s) \\ &\quad = \frac{1}{x_0^{m+1} t_0^{n+1}} \frac{\partial^{m+n}}{\partial p^m \partial s^n} F\left(\frac{p}{x_0}, \frac{s}{t_0}\right). \end{aligned} \tag{30}$$

Let $\varphi(x,t) = e^{-px-st}\Psi(x,t)$. By using (28), we have

$$\begin{aligned} \varphi(1,1) &= e^{-p-s}\Psi(1,1) = e^{-p-s} f(x_0, t_0) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{m^{m+1} n^{n+1}}{m!n!} \\ &\quad \times \int_0^\infty \int_0^\infty x^m t^n e^{-px-st} e^{-mx-nt} \Psi(x,t)\, dx\, dt \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{m^{m+1} n^{n+1}}{m!n!} \\ &\quad \times L_x L_t\left[x^m t^n e^{-mx-nt} \Psi(x,t)\right](p,s). \end{aligned} \tag{31}$$

By using the previous properties (I) and (II) of double Laplace transform, (30), and the definition of $\Psi(x,t)$, we have

$$\begin{aligned} &L_x L_t\left[x^m t^n e^{-mx-nt} \Psi(x,t)\right](p,s) \\ &\quad = (-1)^{m+n} \frac{\partial^{m+n}}{\partial p^m \partial s^n}\left(L_x L_t\left(e^{-mx-nt}\Psi(x,t)\right)\right)(p,s) \\ &\quad = (-1)^{m+n} \frac{\partial^{m+n}}{\partial p^m \partial s^n}\left(L_x L_t\left(\Psi(x,t)\right)\right)(p+m, s+n) \\ &\quad = (-1)^{m+n} \frac{1}{z} \frac{\partial^{m+n}}{\partial p^m \partial s^n}\left(L_x L_t\left(f(xx_0, tt_0)\right)\right)\left(\frac{p+m}{x_0}, \frac{s+n}{t_0}\right) \\ &\quad = (-1)^{m+n} \frac{1}{z} \frac{\partial^{m+n}}{\partial p^m \partial s^n}\left(F\left(\frac{p+m}{x_0}, \frac{s+n}{t_0}\right)\right), \end{aligned} \tag{32}$$

where $1/z = 1/x_0^{m+1}t_0^{n+1}$. From (31) and (32), with $f(x_0,t_0) = e^{p+s}\varphi(1,1)$, we have

$$f(x_0,t_0) = e^{p+s}\lim_{m\to\infty}\lim_{n\to\infty}\frac{(-1)^{m+n}}{m!n!}\left(\frac{m}{x_0}\right)^{m+1}\left(\frac{n}{t_0}\right)^{n+1} \times \frac{\partial^{m+n}}{\partial p^m \partial s^n}\left(F\left(\frac{p+m}{x_0},\frac{s+n}{t_0}\right)\right). \tag{33}$$

For any $p$, $s$ the statement in Theorem 3 is actually just the special case for $p = 0$ and $s = 0$. □

*Example 5.* Find double Laplace transform for a regular generalized function

$$f(x,t) = H(t)\otimes H(x)\ln(t)\ln(x), \tag{34}$$

$$\frac{\partial^2}{\partial x\partial t}f(x,t) = \frac{\partial^2}{\partial x\,\partial t}[H(t)\otimes H(x)\ln(t)\ln(x)] = pf\left[\frac{H(t)\otimes H(x)}{xt}\right], \tag{35}$$

where $H(x,t) = H(t)\otimes H(x)$ is a Heaviside function, and $\otimes$ is tensor product. The double Laplace transform with respect to $x$, $t$ of (1) becomes

$$L_xL_t[f(x,t)] = \int_0^\infty e^{-px}\ln(x)\int_0^\infty e^{-st}\ln(t)\,dt\,dx = -\frac{1}{s}\int_0^\infty e^{-px}\ln(x)[\gamma+\ln s]\,dx, \tag{36}$$

$$L_xL_t[f(x,t)] = \frac{1}{sp}\left[\gamma^2+\ln(p)\ln(s)\right],$$

where $\gamma$ is Euler's constant [10]. Thus,

$$L_xL_t[f(x,t)] = \frac{1}{sp}\left[\gamma^2+\ln(p)\ln(s)\right], \quad \text{where } Re>0. \tag{37}$$

Double Laplace transform of (35) with respect to $x$ and $t$ is obtained as follows:

$$L_xL_t\left[\frac{\partial^2}{\partial x\,\partial t}f(x,t)\right] = L_xL_t[H(t)H(x)\ln(t)\ln(x)] = ps\left[\frac{1}{sp}\left[\gamma^2+\ln(p)\ln(s)\right]\right] = \gamma^2+\ln(p)\ln(s). \tag{38}$$

*Definition 6.* A linear continuous function over the space $L$ of test functions is called a distribution of exponential growth. This dual space of $L$ is denoted by $L'$ [10].

*Example 7.* Let us find double laplace transform of the function $(x^\alpha+t^\beta) = H(x)\otimes H(t)x^\alpha t^\beta$, where $\alpha,\beta\neq -1,-2,\ldots$

Since $(x^\alpha+t^\beta)\in L'$, then double laplace transform of the function $(x^\alpha+t^\beta) = H(x)\otimes H(t)x^\alpha t^\beta$ is given by

$$L_xL_t\left[\left(x^\alpha+t^\beta\right)\right] = \int_0^\infty x^\alpha e^{-px}\int_0^\infty t^\beta e^{-st}dt\,dx. \tag{39}$$

Letting $u = px$ and $v = st$ for $p,t>0$, it follows that

$$L_xL_t\left[\left(x^\alpha+t^\beta\right)\right] = \frac{1}{p^{\alpha+1}s^{\beta+1}}\int_0^\infty u^\alpha e^{-u}\int_0^\infty v^\beta e^{-v}dv\,du, = \frac{1}{p^{\alpha+1}s^{\beta+1}}\Gamma(\alpha+1)\Gamma(\beta+1). \tag{40}$$

In particular, if $\alpha,\beta = 0$, (40) becomes

$$L_xL_t[H(x)\otimes H(t)] = \int_0^\infty e^{-px}\int_0^\infty e^{-st}dt\,dx = \frac{1}{ps}. \tag{41}$$

Consider the general telegraph equation in the following form:

$$u_{xx} = u_{tt}+u_t+u+f(x,t), \tag{42}$$

with boundary conditions

$$u(0,t) = f_1(t), \qquad u_x(0,t) = f_2(t), \tag{43}$$

and initial conditions

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x). \tag{44}$$

We apply double Laplace transform for (42) and single Laplace transform for (43) and (43). after taking double inverse Laplace transform, we obtain the solution of (42) in the form

$$u(x,t) = L_p^{-1}L_s^{-1}\left[\frac{F(p,s)+pF_1(s)+F_2(s)}{(p^2-s^2-s-1)} - \frac{sG_1(p)-G_2(p)-G_1(p)}{(p^2-s^2-s-1)}\right]. \tag{45}$$

Here, we assume that the double inverse Laplace transform exists for each term in the right side of (45).

*Example 8.* Consider the homogeneous telegraph equation given by

$$u_{xx}-u_{tt}-u_t-u = 0 \tag{46}$$

with boundary conditions

$$u(0,t) = e^{-t}, \qquad u_x(0,t) = e^{-t}, \tag{47}$$

and initial conditions

$$u(x,0) = e^x, \qquad u_x(x,0) = -e^x. \tag{48}$$

*Solution 1.* By taking double Laplace transform for (46) and single Laplace transform for (47) and (48), we have

$$U(p,s) = \frac{\left(p^2 - s^2 - s - 1\right)}{(s+1)(p-1)\left(p^2 - s^2 - s - 1\right)} = \frac{1}{(s+1)(p-1)}. \tag{49}$$

By using double inverse Laplace transform for (49), we get the solution as follows:

$$u(x,t) = e^{x-t}. \tag{50}$$

In the next example we apply double Laplace transform for nonhomogenous telegraphic equation as follows.

*Example 9.* Consider the nonhomogenous telegraphic equation denoted by

$$u_{xx} - u_{tt} - u_t - u = -2e^{x+t} \tag{51}$$

with boundary conditions

$$u(0,t) = e^t, \qquad u_x(0,t) = e^t, \tag{52}$$

and initial conditions

$$u(x,0) = e^x, \qquad u_x(x,0) = e^x. \tag{53}$$

By taking double Laplace transform for (51) and single Laplace transform for (52) and (53), we have

$$U(p,s) = \frac{\left(p^2 - s^2 - s - 1\right)}{(s-1)(p-1)\left(p^2 - s^2 - s - 1\right)} = \frac{1}{(s-1)(p-1)}. \tag{54}$$

By applying double inverse Laplace transform for (54), we get the solution of (51) in the following form:

$$u(x,t) = e^{x+t}. \tag{55}$$

## 3. An Application to Partial Integrodifferential Equations

Consider the following partial integrodifferential equation:

$$u_{tt} - u_{xx} + u + \int_0^x \int_0^t g(x-\alpha, t-\beta)\, u(\alpha,\beta)\, d\alpha\, d\beta = f(x,t), \tag{56}$$

with boundary conditions

$$u(0,t) = f_1(t), \qquad u_x(0,t) = f_2(t), \tag{57}$$

and initial conditions

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x). \tag{58}$$

By taking double Laplace transform for (56) and single Laplace transform for (57) and (58), we get

$$U(p,s) = \frac{p/G_1(p) + 1/G_2(p)}{\left(p^2 - s^2 + 1 + G(p,s)\right)} - \frac{p/F_1(s) - 1/F_2(s) + F(p,s)}{\left(p^2 - s^2 + 1 + G(p,s)\right)}. \tag{59}$$

By applying double inverse Laplace transform for (59), we obtain the solution of (56) in the following form:

$$u(x,t) = L_p^{-1} L_s^{-1}\left[\frac{p/G_1(p) + 1/G_2(p)}{\left(p^2 - s^2 + 1 + G(p,s)\right)} - \frac{p/F_1(s) - 1/F_2(s) + F(p,s)}{\left(p^2 - s^2 + 1 + G(p,s)\right)}\right]. \tag{60}$$

We provide the double inverse Laplace transform existing for each terms in the right side of (60). In particular, consider the following example.

*Example 10.* Consider the partial integro-differential equation

$$u_{tt} - u_{xx} + u + \int_0^x \int_0^t e^{x-\alpha+t-\beta} u(\alpha,\beta)\, d\alpha\, d\beta = e^{x+t} + xte^{x+t} \tag{61}$$

with conditions

$$u(x,0) = e^x, \qquad u_t(x,0) = e^x, \qquad u(0,t) = e^t, \qquad u_x(0,t) = e^t. \tag{62}$$

By taking double Laplace transform for (61) and single Laplace transform for (62), we have

$$\left(s^2 - p^2 + 1 + \frac{1}{(p-1)(s-1)}\right) U(p,s) = \frac{s}{p-1} + \frac{1}{p-1} - \frac{p}{s-1} - \frac{1}{s-1} + \frac{1}{(p-1)(s-1)} + \frac{1}{(p-1)^2(s-1)^2}. \tag{63}$$

By simplifying (63), we obtain

$$U(p,s) = \frac{1}{(p-1)(s-1)}. \tag{64}$$

By using double inverse Laplace transform for (64), we obtain the solution of (61) as follows:

$$u(x,t) = e^{x+t}. \tag{65}$$

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