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Research Article

Subharmonics with Minimal Periods for Convex Discrete Hamiltonian Systems

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We consider the subharmonics with minimal periods for convex discrete Hamiltonian systems. By using variational methods and dual functional, we obtain that the system has a pT-periodic solution for each positive integer p, and solution of system has minimal period pT as H subquadratic growth both at 0 and infinity.

1. Introduction

Consider Hamiltonian systems

$$J\dot{u}(t) + \nabla H(t, u(t)) = 0, \qquad u(0) = u(pT), \tag{1}$$

where $u(t) \in \mathbb{R}^{2N}$, $t \in \mathbb{R}$, ∇H stands for the gradient of H with respect to the second variable, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the symplectic matrix with I_N the identity in $\mathbb{R}^N$. Moreover, H is T-periodic in the variable t, $p \in \mathbb{N} \setminus \{0\}$.

Classically, solutions for systems (1) are called subharmonics. The first result concerning the subharmonics problem traced back to Birkhoff and Lewis in 1933 (refer to [1]), in which there exists a sequence of subharmonics with arbitrarily large minimal period, using perturbation techniques. More results can be found in [1–5], where H is convex with subquadratic growth both at 0 and infinity. Using Z_p index theory and Clarke duality, Xu and Guo [1, 5] proved that the number of solutions for systems (1) with minimal period pT tends towards infinity as $p \to \infty$.

For periodic and subharmonic solutions for discrete Hamiltonian systems, Guo and Yu [6, 7] obtained some existence results by means of critical point theory, where they introduced the action functional

$$F(u) = -\frac{1}{2}\sum_{n=1}^{pT}(J\Delta Lu(n-1), u(n)) - \sum_{n=1}^{pT} H(n, Lu(n)). \tag{2}$$

Using Clarke duality, periodic solution of convex discrete Hamiltonian systems with forcing terms has been investigated in [8]. Clarke duality was introduced in 1978 by Clarke [9], and developed by Clarke, Ekeland, and others, see [10–12]. This approach is different from the direct method of variations; some scholars applied it to consider the periodic solutions, subharmonic solutions with prescribed minimal period of Hamiltonian systems; one can refer to [3, 5, 12–14] and references therein. The dynamical behavior of differential and difference equations was studied by using various methods; see [15–19]. We refer the reader to Agarwal [20] for a broad introduction to difference equations.

Motivated by the works of Mawhin and Willem [12] and Xu and Guo [21], we use variational methods and Clarke duality to consider the subharmonics with minimal periods for discrete Hamiltonian systems

$$J\Delta u(n) + \nabla H(n, Lu(n)) = 0, \quad u(n) = u(n + pT), \tag{3}$$

where $u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix}$, $Lu(n) = \begin{pmatrix} u_1(n+1) \\ u_2(n) \end{pmatrix}$, $u_i(n) \in \mathbb{R}^N$ $(i = 1, 2)$ with N a given positive integer, and $\Delta u(n) = u(n+1) - u(n)$ is the forward difference operator. $p, T \in \mathbb{N} \setminus \{0\}$. Moreover, hamiltonian function H satisfies the following assumption:

(A1) $H : \mathbb{Z} \times \mathbb{R}^{2N} \to \mathbb{R}$ is continuous differentiable on $\mathbb{R}^{2N}$, $H(n, \cdot)$ convex for each $n \in \mathbb{Z}$ and $H(n + T, u) = H(n, u)$ for each $u \in \mathbb{R}^{2N}$;

(A2) there exist constants $a_1 > 0$, $a_2 > 0$, $1 < \theta < 2$, such that

$$\frac{a_1}{\theta}|u|^{\theta} \le H(n,u) \le \frac{a_2}{\theta}|u|^{\theta}, \quad u \in \mathbb{R}^{2N}, \tag{4}$$

which implies H subquadratic growth both at 0 and infinity.

Theorem 1. *Assume (A1) holds. $H(n,u) \to +\infty$, $H(n,u)/|u|^2 \to 0$, as $|u| \to \infty$ uniformly in $n \in \mathbb{Z}$. Then there exists a pT-periodic solution u_p of (3), such that $\|u_p\|_{\infty} \triangleq \max_{n\in Z[1,pT]}\{|u_p(n)|\} \to \infty$, and the minimal period T_p of u_p tends to $+\infty$ as $p \to \infty$.*

Theorem 2. *Under the assumptions (A1) and (A2), if*

$$\frac{a_2}{a_1} \le \begin{cases} \left(\frac{1}{4}\sin\frac{\pi}{pT}\right)^{\theta/2}, & \text{when } pT \text{ is even},\\ \left(\frac{1}{2}\sin\frac{\pi}{2pT}\right)^{\theta/2}, & \text{when } pT \text{ is odd} \end{cases} \tag{5}$$

for given integer $p > 1$, then the solution of (3) has minimal period pT.

2. Clarke Duality and Eigenvalue Problem

First we introduce a space E_{pT} with dimension $2NpT$ as follows:

$$E_{pT} = \{u = \{u(n)\} \in S \mid u(n+pT) = u(n),\ p \in \mathbb{N}\setminus\{0\},\ n \in \mathbb{Z}\}, \tag{6}$$

where

$$S = \left\{u = \{u(n)\} \mid u(n) = \begin{pmatrix} u_1(n) \\ u_2(n)\end{pmatrix} \in \mathbb{R}^{2N},\ u_j(n) \in \mathbb{R}^N,\ j = 1,2,\ n \in \mathbb{Z}\right\}. \tag{7}$$

Equipped with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ in E_{pT} as

$$\langle u, v\rangle = \sum_{n=1}^{pT}(u(n), v(n)),\qquad \|u\| = \left(\sum_{n=1}^{pT}|u(n)|^2\right)^{1/2}, \quad \forall u, v \in E_{pT}, \tag{8}$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote the usual scalar product and corresponding norm in $\mathbb{R}^{2N}$, respectively. It is easy to see that $(E_{pT}, \langle\cdot,\cdot\rangle)$ is a Hilbert space with $2NpT$ dimension, which can be identified with $\mathbb{R}^{2NpT}$. Then for any $u \in E_{pT}$, it can be written as $u = (u^T(1), u^T(2), \ldots, u^T(pT))^T$, where $u(j) = \binom{u_1(j)}{u_2(j)} \in \mathbb{R}^{2N}$, $j \in Z[1, pT]$, the discrete interval $\{1, 2, \ldots, pT\}$ is denoted by $Z[1, pT]$, and $\cdot^T$ denotes the transpose of a vector or a matrix.

Denote the subspace $\overline{Y} = \{u \in E_{pT} \mid u(1) = u(2) = \cdots = u(pT) \in \mathbb{R}^{2N}\}$. Let Y be the direct orthogonal complement of E_{pT} to $\overline{Y}$. Thus E_{pT} can be split as $E_{pT} = Y \oplus \overline{Y}$, and for any $u \in E_{pT}$, it can be expressed in the form $u = \tilde{u} + \overline{u}$, where $\tilde{u} \in Y$, $\overline{u} \in \overline{Y}$.

Next we recall Clarke duality and some lemmas.

The Legendre transform (see [12]) $H^*(t,\cdot)$ of $H(t,\cdot)$ with respect to the second variable is defined by

$$H^*(t, v) = \sup_{u\in R^{2N}}\{(v,u) - H(t,u)\}, \tag{9}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{2N}$. It follows from (A1) and (A2) that

(B1) $H^*(n,\cdot)$ is continuous differentiable on $\mathbb{R}^{2N}$,

(B2) for $\tau = \theta/(\theta-1)$, $v \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}$, one has

$$\frac{1}{\tau}\left(\frac{1}{a_2}\right)^{\tau-1}|v|^{\tau} \le H^*(n,v) \le \frac{1}{\tau}\left(\frac{1}{a_1}\right)^{\tau-1}|v|^{\tau}. \tag{10}$$

Associated with variational functional (2), the dual action functional is defined by

$$\chi(v) = \sum_{n=1}^{pT}\frac{1}{2}(L(J\Delta v(n-1)), v(n)) + \sum_{n=1}^{pT}H^*(n, \Delta v(n)), \quad v \in E_{pT}. \tag{11}$$

Indeed, by (11), we have $\chi(v + \overline{u}) = \chi(v)$ for any $\overline{u} \in \overline{Y}$ and $v \in Y$. Therefore, χ can be restricted in subspace Y of E_{pT}. Moreover, in terms of Lemma 2.6 in [8] and the following lemma, the critical points of (11) correspond to the subharmonic solutions of (3).

Lemma 3 (see [8, Theorem 1]). *Assume that*

(H1) *$H(n,\cdot) \in C^1(\mathbb{R}^{2N}, \mathbb{R})$, $H(n,\cdot)$ is convex in the second variable for $n \in \mathbb{Z}$,*

(H2) *there exists $\beta : \mathbb{Z} \to \mathbb{R}^{2N}$ such that for all $(n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}$, $H(n,u) \ge (\beta(n), u)$, and $\beta(n+T) = \beta(n)$,*

(H3) *there exist $\alpha \in (0, 2\sin(\pi/pT))$ and $\gamma : \mathbb{Z} \to \mathbb{R}^+$, such that for any $(n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}$, $H(n,u) \le (\alpha/2)|u|^2 + \gamma(n)$, and $\gamma(n+T) = \gamma(n)$,*

(H4) *for each $u \in \mathbb{R}^{2N}$, $\sum_{n=1}^{pT} H(n,u) \to +\infty$ as $|u| \to \infty$.*

Then system (3) has at least one periodic solution u, such that $v = -J[u - (1/pT)\sum_{n=1}^{pT} u(n)]$ minimizes the dual action $\chi(v) = \sum_{n=1}^{pT}(1/2)(LJ\Delta v(n-1), v(n)) + \sum_{n=1}^{pT} H^*(n, \Delta v(n))$.

The following lemmas will be useful in our proofs, where Lemma 4 can be proved by means of Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, and Lemma 5 is a Hölder inequality.

Lemma 4. *For any $k \in \mathbb{Z}$, $\sum_{n=1}^{pT}\sin((2k\pi/pT)n) = \sum_{n=1}^{pT}\cos((2k\pi/pT)n) = 0$.*

Lemma 5. *For any $u_j > 0$, $v_j > 0$, $k \in \mathbb{Z}$, one has $\sum_{j=1}^{k} u_j v_j \le (\sum_{j=1}^{k} u_j^p)^{1/p}(\sum_{j=1}^{k} v_j^q)^{1/q}$, where $p > 1$, $q > 1$ and $1/p + 1/q = 1$.*

Lemma 6 (see [12, proposition 2.2]). *Let $H : \mathbb{R}^m \to \mathbb{R}$ be of C^1 and convex functional, $-\beta \le H(u) \le \alpha q^{-1}|u|^q + \gamma$, where $u \in \mathbb{R}^m$, $\alpha > 0$, $q > 1$, $\beta \ge 0$, $\gamma \ge 0$. Then $\alpha^{-p/q} p^{-1}|\nabla H(u)|^p \le (\nabla H(u), u) + \beta + \gamma$, where $1/p + 1/q = 1$.*

In order to know the form of $u \in E_{pT}$, we consider eigenvalue problem

$$LJ\Delta u(n-1) = \lambda u(n), \qquad u(n+pT) = u(n), \tag{12}$$

where $u(n) = \binom{u_1(n)}{u_2(n)}$, $Lu(n-1) = \binom{u_1(n)}{u_2(n-1)} \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{R}$. We can rewrite (12) as the following form:

$$\begin{gathered} u_1(n+1) = (1-\lambda^2) u_1(n) + \lambda u_2(n), \\ u_2(n+1) = -\lambda u_1(n) + u_2(n), \\ u_1(n+pT) = u_1(n), \qquad u_2(n+pT) = u_2(n). \end{gathered} \tag{13}$$

Denoting

$$M(\lambda) = \begin{pmatrix} (1-\lambda^2) I_N & \lambda I_N \\ -\lambda I_N & I_N \end{pmatrix}, \tag{14}$$

then problem (12) is equivalent to

$$u(n+1) = M(\lambda) u(n), \qquad u(n+pT) = u(n). \tag{15}$$

Letting $u(n) = \mu^n c$ be the solution of (15), for some $c \in \mathbb{C}^{2N}$, we have $\mu c = M(\lambda)c$ and $\mu^{pT} = 1$. Consider the polynomial $|M(\lambda) - \mu I_{2N}| = 0$ and condition $\mu^{pT} = 1$; it follows that

$$\begin{gathered} \mu = e^{2k\pi i/pT}, \qquad \lambda = 2\sin\frac{k\pi}{pT}, \\ k \in Z[-pT+1, pT-1]. \end{gathered} \tag{16}$$

In the following we denote by $\mu_k = e^{2k\pi i/pT}$, $\lambda_k = 2\sin(k\pi/pT)$, $k \in Z[-pT+1, pT-1]$, and $\rho \in \mathbb{R}^N$. By $(M(\lambda_k) - \mu_k I_{2N})c = 0$, it follows that

$$c_k = \begin{pmatrix} \rho \\ ie^{(-k\pi i/pT)}\rho \end{pmatrix}. \tag{17}$$

Thus

$$\begin{aligned} u_k(n) &= \mu_k^n c_k = e^{2k\pi n i/pT} \begin{pmatrix} \rho \\ ie^{(-k\pi i/pT)}\rho \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\frac{2k\pi}{pT} n\right)\rho \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix} \\ &\quad + i \begin{pmatrix} \sin\left(\frac{2k\pi}{pT} n\right)\rho \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}. \end{aligned} \tag{18}$$

Let

$$\begin{gathered} \xi_k(n) = \begin{pmatrix} \cos\left(\frac{2k\pi}{pT} n\right)\rho \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}, \\ \eta_k = \begin{pmatrix} \sin\left(\frac{2k\pi}{pT} n\right)\rho \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}. \end{gathered} \tag{19}$$

Obviously, $\xi_k(n)$ and $\eta_k(n)$ satisfy (15). Moreover $LJ\Delta\xi_k(n-1) = \lambda_k \xi_k(n)$, $LJ\Delta\eta_k(n-1) = \lambda_k\eta_k(n)$, $\xi_{2pT+k}(n) = \xi_k(n)$, $\eta_{2pT+k}(n) = \eta_k(n)$, $\xi_{pT-k}(n) = \xi_k(n)$, $\eta_{pT-k}(n) = -\eta_k(n)$.

For $k \ne pT/2$, subspace Y_k is defined by

$$Y_k = \begin{cases} \operatorname{span}\{\xi_k(n), \eta_{k+(pT/2)}(n)\}, & k \in Z\left[-\frac{pT}{2}+1, \frac{pT}{2}-1\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is even}, \\ \operatorname{span}\{\xi_k(n), \eta_{k+((pT+1)/2)}(n)\}, & k \in Z\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is odd}, \end{cases} \tag{20}$$

where $[\cdot]$ denotes the greatest-integer function and

$$\begin{gathered} Y_{pT/2} = \operatorname{span}\{\xi_{pT/2}(n), n \in \mathbb{Z}\}, \\ Y_{-pT/2} = \operatorname{span}\{\xi_{-pT/2}(n), n \in \mathbb{Z}\}. \end{gathered} \tag{21}$$

Therefore,

$$\begin{gathered} Y = \oplus Y_k, \quad k \in Z\left[-\frac{pT}{2}, \frac{pT}{2}\right] \setminus \{0\}, \text{ if } pT \text{ is even}, \\ Y = \oplus Y_k, \quad k \in Z\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \text{ if } pT \text{ is odd}. \end{gathered} \tag{22}$$

Moreover, for any $u = \{u(n)\} \in E_{pT}$, we may express $u(n)$ as

$$\begin{aligned} u(n) = \sum_{k=-pT+1}^{pT-1} \Bigg[&\begin{pmatrix} \cos\left(\frac{2k\pi}{pT} n\right) a_k \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right) a_k \end{pmatrix} \\ &+ \begin{pmatrix} \sin\left(\frac{2k\pi}{pT} n\right) b_k \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right) b_k \end{pmatrix} \Bigg], \end{aligned} \tag{23}$$

where $a_k, b_k \in \mathbb{R}^N$.

Since $(\Delta u(n), \Delta u(n)) = -(\Delta^2 u(n-1), u(n))$, we consider eigenvalue problem

$$-\Delta^2 u(n-1) = \lambda u(n), \quad u(n+pT) = u(n), \quad u(n) \in \mathbb{R}^N, \tag{24}$$

where $\Delta^2 u(n-1) = \Delta u(n) - \Delta u(n-1) = u(n+1) - 2u(n) + u(n-1)$. The second order difference equation (24) has complexity solution $u(n) = e^{in\theta}c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/pT$. Moreover, $\lambda = 2 - e^{-i\theta} - e^{i\theta} = 2(1-\cos\theta) = 4\sin^2(\theta/2)$; that is, $\lambda = 4\sin^2(k\pi/pT)$, $k \in Z[0, pT-1]$.

By the previous, it follows Lemma 7.

Lemma 7. *For any $u \in E_{pT}$, one has $-\lambda_{\max}\|u\|^2 \leq \sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n)) \leq \lambda_{\max}\|u\|^2$, and $0 \leq \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \lambda_{\max}^2\|u\|^2$, where*

$$\lambda_{\max} = \max_{k\in[0,pT-1]}\left\{2\sin\frac{k\pi}{pT}\right\} = \begin{cases} 2, & \text{if } pT \text{ is even}, \\ 2\cos\dfrac{\pi}{2pT}, & \text{if } pT \text{ is odd}. \end{cases} \tag{25}$$

Moreover, if $u \in Y$, then $4\sin^2(\pi/pT)\|u\|^2 \leq \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \lambda_{\max}^2\|u\|^2$.

3. Proofs of Main Results

Lemma 8. *Consider*

$$\sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n)) \geq -\left(2\sin\frac{\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta u(n)|^2, \quad \forall u \in E_{pT}. \tag{26}$$

Proof. Letting $\tilde{u}(n) = u(n) - (1/pT)\sum_{n=1}^{pT} u(n)$, then $\tilde{u} \in Y$. By Lemmas 5 and 7, we have

$$\begin{aligned}\sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n)) &= \sum_{n=1}^{pT}(LJ\Delta u(n-1), \tilde{u}(n)) \\ &\geq -\left(\sum_{n=1}^{pT}|LJ\Delta u(n-1)|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{pT}|\tilde{u}(n)|^2\right)^{1/2} \\ &\geq -\left(\sum_{n=1}^{pT}|\Delta u(n)|^2\right)^{1/2} \cdot \left(2\sin\frac{\pi}{pT}\right)^{-1}\left(\sum_{n=1}^{pT}|\Delta\tilde{u}(n)|^2\right)^{1/2} \\ &= -\left(2\sin\frac{\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta u(n)|^2.\end{aligned} \tag{27}$$

□

Lemma 9. *If there exist $\alpha \in (0, \sin(\pi/pT))$, $\beta \geq 0$ and $\delta > 0$, such that*

$$\delta|u| - \beta \leq H(n,u) \leq \frac{\alpha}{2}|u|^2 + \gamma \tag{28}$$

for all $n \in [1, pT]$ and $u \in \mathbb{R}^{2N}$, then each solution of (3) satisfies the inequalities

$$\begin{aligned}\sum_{n=1}^{pT}|\Delta u(n)|^2 &\leq \frac{2\alpha(\beta+\gamma)pT\sin(\pi/pT)}{\sin(\pi/pT)-\alpha}, \\ \sum_{n=1}^{pT}|Lu(n)| &\leq \frac{(\beta+\gamma)pT\sin(\pi/pT)}{\delta(\sin(\pi/pT)-\alpha)}.\end{aligned} \tag{29}$$

Proof. Let u be the solution of (3). By Lemma 6, we have

$$\begin{aligned}\frac{1}{2\alpha}|\nabla H(n, Lu(n))|^2 &\leq (\nabla H(n, Lu(n)), Lu(n)) + \beta + \gamma \\ &= -(J\Delta u(n), Lu(n)) + \beta + \gamma.\end{aligned} \tag{30}$$

Obviously, $|J\Delta u(n)|^2 = (-\nabla H(n, Lu(n)), J\Delta u(n)) = |\nabla H(n, Lu(n))|^2$ by (3), and it follows that $(1/2\alpha)\sum_{n=1}^{pT}|J\Delta u(n)|^2 + \sum_{n=1}^{pT}(J\Delta u(n), Lu(n)) \leq (\beta+\gamma)pT$; that is,

$$\frac{1}{2\alpha}\sum_{n=1}^{pT}|\Delta u(n)|^2 + \sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n)) \leq (\beta+\gamma)pT. \tag{31}$$

By means of Lemma 8, we have

$$\left[\frac{1}{2\alpha} - \left(2\sin\frac{\pi}{pT}\right)^{-1}\right]\sum_{n=1}^{pT}|\Delta u(n)|^2 \leq (\beta+\gamma)pT, \tag{32}$$

which gives first conclusion.

Now, $H(n,0) \leq \gamma$ in view of (28); therefore by convex and Lemma 8, we have

$$\begin{aligned}\delta\sum_{n=1}^{pT}|Lu(n)| - \beta pT &\leq \sum_{n=1}^{pT}H(n, Lu(n)) \\ &\leq \sum_{n=1}^{pT}[H(n,0) + (\nabla H(n, Lu(n)), Lu(n))]\end{aligned}$$

$$\begin{aligned}
&\le \gamma pT - \sum_{n=1}^{pT} (J\Delta u(n), Lu(n)) \\
&= \gamma pT - \sum_{n=1}^{pT} (JL\Delta u(n-1), u(n)) \\
&\le \gamma pT + \left(2\sin\frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2 \\
&\le \gamma pT + \frac{\alpha(\beta+\gamma)pT}{\sin(\pi/pT)-\alpha},
\end{aligned} \tag{33}$$

which gives the second conclusion. The proof is completed. □

Proof of Theorem 1. Let $c_1 = \max_{n\in\mathbb{Z}} |H(n,0)|$. By assumption in Theorem 1, there exists $R > 0$, such that $H(n,u) \ge 1 + c_1$, for $n \in \mathbb{Z}$ and $|u| \ge R$. Moreover, there exist $\alpha \in (0, 2\sin(\pi/pT))$, $\gamma > 0$ such that

$$H(n,u) \le \frac{\alpha}{2}|u|^2 + \gamma, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}. \tag{34}$$

Thus, by convex of H, for all $(n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}$ with $|u| \ge R$, we have

$$\begin{aligned}
1 + c_1 &\le H\left(n, \frac{R}{|u|}u\right) \\
&\le H(n,0) + \frac{R}{|u|}(H(n,u) - H(n,0)) \\
&\le \frac{R}{|u|}H(n,u) + c_1.
\end{aligned} \tag{35}$$

Therefore there exist $\beta > 0$ and $\delta > 0$, such that

$$H(n,u) \ge \delta|u| - \beta, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}. \tag{36}$$

Combining the previous argument, by Lemma 3, the system (3) has a pT-periodic solution u_p such that $v_p = -J[u_p - (1/pT)\sum_{n=1}^{pT} u_p(n)] \in Y$ minimizes the dual action

$$\begin{aligned}
\chi_p(v_p) = &\sum_{n=1}^{pT} \frac{1}{2}(LJ\Delta v_p(n-1), v_p(n)) \\
&+ \sum_{n=1}^{pT} H^*(n, \Delta v_p(n)) \quad \text{on } E_{pT}.
\end{aligned} \tag{37}$$

It follows that $\Delta u_p(n) = J\Delta v_p(n)$ and $Jv_p(n) = u_p(n) - (1/pT)\sum_{n=1}^{pT} u_p(n)$.

We next prove that $\|u_p\|_\infty \to \infty$ as $p \to \infty$.

Suppose not, there exist $c_2 > 0$ and a subsequence $\{p_k\}$ such that

$$p_k \to \infty, \qquad \|u_{p_k}\|_\infty \le c_2 \quad \text{as } k \longrightarrow \infty. \tag{38}$$

In terms of (3), it follows that $\|\Delta u_{p_k}\|_\infty \le c_3$ for some $c_3 > 0$, and $\|v_{p_k}\|_\infty \le 2c_2$, $\|\Delta v_{p_k}\|_\infty \le c_3$. Consequently, by $H^*(n,v) \ge -H(n,0) \ge -c_1$, we have

$$\begin{aligned}
c_{p_k} &= \chi_{p_k}(v_{p_k}) \\
&= \sum_{n=1}^{p_kT} \frac{1}{2}(LJ\Delta v_{p_k}(n-1), v_{p_k}(n)) \\
&\quad + \sum_{n=1}^{p_kT} H^*(n, \Delta v_{p_k}(n)) \\
&\ge -\frac{1}{2}\sum_{n=1}^{p_kT} |LJ\Delta v_{p_k}(n-1)||v_{p_k}(n)| - c_1 p_k T \\
&\ge -(\sqrt{2}c_2c_3 + c_1)p_kT,
\end{aligned} \tag{39}$$

where $n \in Z[1, p_kT]$ and

$$\begin{aligned}
|LJ\Delta v_{p_k}(n-1)| &= \left(|\Delta v_{2,p_k}(n)|^2 + |\Delta v_{1,p_k}(n-1)|^2\right)^{1/2} \\
&\le \sqrt{2}\|\Delta v_{p_k}\|_\infty \le \sqrt{2}c_3.
\end{aligned} \tag{40}$$

By (36), if $|v| \le \delta$, we have $(v,u) - H(n,u) \le (v,u) - \delta|u| + \beta \le \beta$, and $H^*(n,v) \le \beta$. Letting $\rho \in \mathbb{R}^N$ and $|\rho| = 1$, in terms of (12), h_p associated with $\lambda_{-1} = -2\sin(\pi/pT)$ is given by

$$\begin{aligned}
h_p(n) = &\frac{\delta}{4\sin(\pi/pT)} \\
&\cdot \begin{pmatrix} \left(\cos\frac{2\pi}{pT}n - \sin\frac{2\pi}{pT}n\right)\rho \\ \left(\sin\frac{2\pi}{pT}\left(n-\frac{1}{2}\right) + \cos\frac{2\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}
\end{aligned} \tag{41}$$

which belongs to E_{pT}, and

$$\begin{aligned}
&|\Delta h_p(n)|^2 \\
&= \left(\frac{\delta}{4\sin(\pi/pT)}\right)^2 \\
&\quad \cdot \left| 2\sin\frac{\pi}{pT} \begin{pmatrix} \left(-\sin\frac{2\pi}{pT}\left(n+\frac{1}{2}\right) - \cos\frac{2\pi}{pT}\left(n+\frac{1}{2}\right)\right)\rho \\ \left(\cos\frac{2\pi}{pT}n - \sin\frac{2\pi}{pT}n\right)\rho \end{pmatrix} \right|^2 \\
&= \frac{1}{4}\left[2 + \sin\frac{2\pi}{pT}(2n+1) - \sin\frac{2\pi}{pT}(2n)\right] \cdot |\rho|^2\delta^2 \\
&\le \delta^2.
\end{aligned} \tag{42}$$

Moreover, by Lemma 4 we have

$$\begin{aligned}\sum_{n=1}^{pT}\left|h_p(n)\right|^2 &= \sum_{n=1}^{pT}\left(\frac{\delta}{4\sin(\pi/pT)}\right)^2 \\ &\quad\cdot\left(2+\sin\frac{2\pi}{pT}(2n-1)-\sin\frac{2\pi}{pT}(2n)\right)|\rho|^2 \\ &= \left(\frac{\delta}{4\sin(\pi/pT)}\right)^2 2|\rho|^2 pT = \frac{\delta^2 pT}{8\sin^2(\pi/pT)}.\end{aligned}\tag{43}$$

Thus $c_p = \chi_p(h_p) \le \sum_{n=1}^{pT}(1/2)(LJ\Delta h_p(n-1), h_p(n)) + \beta pT = \sum_{n=1}^{pT}(1/2)(-2\sin(\pi/pT))|h_p(n)|^2 + \beta pT = -\delta^2 pT/8\sin(\pi/pT) + \beta pT$. Combining (39), we have $8(\sqrt{2}c_2c_3 + c_1 + \beta_1)\sin(\pi/p_kT) \ge \delta^2$, which is impossible as k large. So the claim $\lim_{p\to\infty}\|u_p\|_\infty = \infty$ is valid.

It remains only to prove that the minimal period T_p of u_p tends to $+\infty$ as $p \to \infty$.

If not, there exists $T > 0$ and a sequence $\{p_k\}$ such that the minimal period T_{p_k} of u_{p_k} satisfies $1 \le T_{p_k} \le T$. By assumption in Theorem 1, there exists $\alpha \in (0, \sin(\pi/T))$ and $\gamma > 0$ such that

$$H(n,u) \le \frac{\alpha}{2}|u|^2 + \gamma, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}.\tag{44}$$

By (36) and Lemma 9 with pT replaced by T_{p_k}, we get

$$\begin{aligned}\sum_{n=1}^{T_{p_k}}\left|\Delta u_{p_k}(n)\right|^2 &\le \frac{2\alpha(\beta+\gamma)T_{p_k}\sin\left(\pi/T_{p_k}\right)}{\sin\left(\pi/T_{p_k}\right)-\alpha} \\ &\le \frac{2\alpha(\beta+\gamma)T\sin(\pi/T)}{\sin(\pi/T)-\alpha},\end{aligned}\tag{45}$$

$$\begin{aligned}\sum_{n=1}^{T_{p_k}}\left|Lu_{p_k}(n)\right| &\le \frac{(\beta+\gamma)T_{p_k}\sin\left(\pi/T_{p_k}\right)}{\delta\left(\sin\left(\pi/T_{p_k}\right)-\alpha\right)} \\ &\le \frac{(\beta+\gamma)T_{p_k}\sin(\pi/T)}{\delta(\sin(\pi/T)-\alpha)}.\end{aligned}\tag{46}$$

Write $u_{p_k} = \tilde{u}_{p_k} + \overline{u}_{p_k}$, where $\overline{u}_{p_k} = (1/T_{p_k})\sum_{n=1}^{T_{p_k}} u_{p_k}(n) = (1/T_{p_k})\sum_{n=1}^{T_{p_k}} Lu_{p_k}(n) \in \overline{Y}$. Inequality (46) implies that

$$\begin{aligned}\left\|\overline{u}_{p_k}\right\|_\infty &\triangleq \max_{n\in Z[1,T_{p_k}]}\left\{\left|\overline{u}_{p_k}\right|\right\} \\ &\le \frac{1}{T_{p_k}}\sum_{n=1}^{T_{p_k}}\left|Lu_{p_k}(n)\right| \le \frac{(\beta+\gamma)\sin(\pi/T)}{\delta(\sin(\pi/T)-\alpha)}.\end{aligned}\tag{47}$$

By Lemma 7 and (45), it follows that

$$\begin{aligned}\left\|\tilde{u}_{p_k}\right\|^2 &= \sum_{n=1}^{T_{p_k}}\left|\tilde{u}_{p_k}(n)\right|^2 \\ &\le \left(2\sin\frac{\pi}{T_{p_k}}\right)^{-1}\sum_{n=1}^{T_{p_k}}\left|\Delta u_{p_k}(n)\right|^2 \\ &\le (2\sin(\pi/T))^{-1}\frac{2\alpha(\beta+\gamma)T\sin(\pi/T)}{\sin(\pi/T)-\alpha} \\ &\le \frac{\alpha(\beta+\gamma)T}{\sin(\pi/T)-\alpha},\end{aligned}\tag{48}$$

which implies that $\{\|\tilde{u}_{p_k}\|_\infty\}$ is bounded, therefore $\{\|u_{p_k}\|_\infty\}$ is bounded; a contradiction with the second claim $\lim_{p\to\infty}\|u_p\|_\infty = \infty$. This completes the proof. □

Proof of Theorem 2. Under the assumptions (A1) and (A2), all conditions in Theorem 1 are satisfied. Then, for each integer $p > 1$, there exists a pT-periodic solution u of (3) such that $v = -J[u - (1/pT)\sum_{n=1}^{pT} u(n)] \in Y$ minimizes the dual action

$$\begin{aligned}\chi(v) &= \sum_{n=1}^{pT}\frac{1}{2}(LJ\Delta v(n-1), v(n)) \\ &\quad + \sum_{n=1}^{pT}H^*(n,\Delta v(n)) \quad \text{on } E_{pT}.\end{aligned}\tag{49}$$

If the critical point v of dual action functional χ has minimal period $pT/l \in \mathbb{N}\setminus\{0\}$, where $l \in \mathbb{N}\setminus\{0\}$, then by Lemma 7 with pT replaced by pT/l, we have the following estimate:

$$4\sin^2\frac{l\pi}{pT}\sum_{n=1}^{pT}|v(n)|^2 \le \sum_{n=1}^{pT}|\Delta v(n)|^2.\tag{50}$$

By Lemma 5 and the previous inequality, we have

$$\begin{aligned}\sum_{n=1}^{pT}&(LJ\Delta v(n-1), v(n)) \\ &\ge -\left(\sum_{n=1}^{pT}|LJ\Delta v(n-1)|^2\right)^{1/2} \\ &\quad\cdot\left(\sum_{n=1}^{pT}|v(n)|^2\right)^{1/2} \\ &\ge -\left(\sum_{n=1}^{pT}|\Delta v(n)|^2\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\cdot\left(2\sin\frac{l\pi}{pT}\right)^{-1}\left(\sum_{n=1}^{pT}|\Delta v(n)|^2\right)^{1/2}\\
&= -\left(2\sin\frac{l\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta v(n)|^2\\
&\geq -\left(2\sin\frac{l\pi}{pT}\right)^{-1}(pT)^{(1-2/\tau)}\left(\sum_{n=1}^{pT}|\Delta v(n)|^\tau\right)^{2/\tau},
\end{aligned}
\tag{51}
$$

where $\tau = \theta/(\theta-1) > 2$ for $1 < \theta < 2$. It follows from assumption (B2) that

$$
H^*(n, \Delta v(n)) \geq \frac{1}{\tau}\left(\frac{1}{a_2}\right)^{\tau-1}|\Delta v(n)|^\tau, \tag{52}
$$

thus

$$
\begin{aligned}
\chi(v) \geq &-\left(2\sin\frac{l\pi}{pT}\right)^{-1}(pT)^{(\tau-2)/\tau}\left(\sum_{n=1}^{pT}|\Delta v(n)|^\tau\right)^{2/\tau}\\
&+\frac{1}{\tau}\left(\frac{1}{a_2}\right)^{\tau-1}\sum_{n=1}^{pT}|\Delta v(n)|^\tau
\end{aligned}
\tag{53}
$$

$$
\geq \frac{(1/\tau - 1/2)\, pT\left(a_2^2\right)^{(\tau-1)/(\tau-2)}}{\left(\sin(l\pi/pT)\right)^{\tau/(\tau-2)}}. \tag{54}
$$

One can obtain the previous inequality by minimizing in (53) with respect to $(\sum_{n=1}^{pT}|\Delta v(n)|^\tau)^{1/\tau}$, and the minimum is attained at $(pT)^{1/\tau}(a_2)^{(\tau-1)/(\tau-2)}/(\sin(l\pi/pT))^{1/(\tau-2)}$.

On the other hand, let

$$
v(n) = \frac{1}{\sqrt{pT}}\begin{pmatrix}\cos\dfrac{2k\pi}{pT}n\cdot a_k\\ -\sin\dfrac{2k\pi}{pT}\left(n-\dfrac{1}{2}\right)\cdot a_k\end{pmatrix}, \tag{55}
$$

where $a_k \in \mathbb{R}^N$, $k \in Z[[-pT/2],[pT/2]]\setminus\{0\}$. Then $v \in Y_k$, and

$$
\Delta v(n) = -2\sin\frac{k\pi}{pT}\frac{1}{\sqrt{pT}}\begin{pmatrix}\sin\dfrac{2k\pi}{pT}\left(n+\dfrac{1}{2}\right)\cdot a_k\\ \cos\dfrac{2k\pi}{pT}n\cdot a_k\end{pmatrix}. \tag{56}
$$

Taking $a_k = (d, 0, \dots, 0)^T \in \mathbb{R}^N$, where $d \in \mathbb{R}$, by Lemma 4, it follows that

$$
\begin{aligned}
&\sum_{n=1}^{pT}(LJ\Delta v(n-1), v(n))\\
&= \sum_{n=1}^{pT}\left[-\Delta v_2(n)v_1(v) + \Delta v_1(n-1)v_2(n)\right]\\
&= \sum_{n=1}^{pT}\frac{1}{pT}\cdot 2\sin\frac{k\pi}{pT}\\
&\quad\cdot\left(\cos^2\frac{2k\pi}{pT}n\cdot|d|^2 + \sin^2\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\cdot|d|^2\right)\\
&= \lambda_k\cdot|d|^2,
\end{aligned}
\tag{57}
$$

where $\lambda_k = 2\sin(k\pi/pT)$ and

$$
\begin{aligned}
&\sum_{n=1}^{pT}|\Delta v(n)|^\tau\\
&= \sum_{n=1}^{pT}|\lambda_k|^\tau(pT)^{-\tau/2}\\
&\quad\cdot\left(\sin^2\frac{2k\pi}{pT}\left(n+\frac{1}{2}\right)+\cos^2\frac{2k\pi}{pT}n\right)^{\tau/2}|d|^\tau\\
&\leq \lambda_{\max}^\tau\cdot(pT)^{1-(\tau/2)}\cdot 2^{\tau/2}|d|^\tau.
\end{aligned}
\tag{58}
$$

Therefore, taking $k = -[pT/2]$, by eigenvalue problem (24) and (B2), it follows that

$$
\begin{aligned}
\chi(v) &= \frac{1}{2}\sum_{n=1}^{pT}(LJ\Delta v(n-1), v(n))\\
&\quad+\sum_{n=1}^{pT}H^*(n,\Delta v(n))\\
&\leq -\frac{1}{2}\lambda_{\max}\cdot|d|^2\\
&\quad+\frac{1}{\tau}\left(\frac{1}{a_1}\right)^{\tau-1}\sum_{n=1}^{pT}|\Delta v(n)|^\tau\\
&\leq -\frac{1}{2}\lambda_{\max}\cdot|d|^2+\frac{1}{\tau}\left(\frac{1}{a_1}\right)^{\tau-1}\lambda_{\max}^\tau\\
&\quad\cdot(pT)^{1-(\tau/2)}\cdot 2^{\tau/2}|d|^\tau.
\end{aligned}
\tag{59}
$$

Let $f(\rho)$ equal the right-hand side of (59) where $\rho = |d|$. It is easy to see that the absolute minimum of f is attained at $\rho_{\min} = (a_1)^{(\tau-1)/(\tau-2)}(pT)^{1/2}/[\lambda_{\max}^{(\tau-1)/(\tau-2)}\cdot 2^{\tau/2(\tau-2)}]$ and given by $f_{\min} = (1/\tau - 1/2)pT(a_1^2)^{(\tau-1)/(\tau-2)}/(2\lambda_{\max})^{\tau/(\tau-2)}$. Hence, by (19), let

$$\xi(n) = \xi_{-[pT/2]}(n) = \begin{pmatrix} \cos\frac{2k\pi}{pT}n\cdot\rho \\ -\sin\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\cdot\rho \end{pmatrix}, \tag{60}$$

where $\rho \in \mathbb{R}^N$, $k = -[pT/2]$.

If pT is even, then $\xi(n) = (1,1)^T \cdot (-1)^n \rho$. Set

$$Y_{\rho_{\min}} = \left\{ \nu \in Y_{-[pT/2]} : \nu(n) = \xi(n), \; \rho = (d, 0, \ldots, 0)^T \in \mathbb{R}^N, d \in \mathbb{R} \right\}. \tag{61}$$

For $\nu \in Y_{\rho_{\min}}$, we have

$$\chi(\nu) \le f_{\min}. \tag{62}$$

Combining (54), (59), and (62), we have

$$\frac{(1/\tau - 1/2)\, pT\left(a_2^2\right)^{(\tau-1)/(\tau-2)}}{\left(\sin(l\pi/pT)\right)^{\tau/(\tau-2)}} \le \frac{(1/\tau - 1/2)\, pT\left(a_1^2\right)^{(\tau-1)/(\tau-2)}}{\left(2\lambda_{\max}\right)^{\tau/(\tau-2)}}. \tag{63}$$

By $\tau > 2$, and $\theta = \tau/(\tau-1)$, it follows that

$$\frac{\sin(l\pi/pT)}{(2\lambda_{\max}) \le (a_2/a_1)^{2/\theta}}. \tag{64}$$

For integer $p > 1$, $T \ge 1$, $l \in \mathbb{N} \setminus \{0\}$, $pT/l \in \mathbb{N} \setminus \{0\}$, we have $0 < l\pi/pT \le \pi$, $0 < \pi/pT \le \pi/2$.

If pT is even, then $\lambda_{\max} = 2$. By assumption $a_2/a_1 \le ((1/4)\sin(\pi/pT))^{\theta/2}$ we have $\sin(l\pi/pT) \le \sin(\pi/pT)$, which implies that $l = 1$ or $l = pT - 1$. If $pT > 2$, then $pT/l = pT/(pT-1) \notin \mathbb{N}$. So we have $l = 1$.

If pT is odd, then $\lambda_{\max} = 2\cos(\pi/2pT)$. By assumption $a_2/a_1 \le ((1/2)\sin(\pi/2pT))^{\theta/2}$, we have $\sin(l\pi/pT) \le \sin(\pi/pT)$, so $l = 1$. This completes the proof. □

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