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Research Article

Modified Mann-Halpern Algorithms for Pseudocontractive Mappings

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Modified Mann-Halpern algorithms for finding the fixed points of pseudocontractive mappings are presented. Strong convergence theorems are obtained.

1. Introduction

Finding the fixed points of nonlinear operators is an important topic in fixed point theory, due to the fact that many nonlinear problems can be reformulated as fixed point equations of nonlinear mappings. The research of this area dates back to Picard's and Banach's time. Now it is well known that the Picard iterates $\{T^n x\}$ converge to the unique fixed point of T whenever T is a contraction of a complete metric space. However, if T is not a contraction (e.g., nonexpansive), then the Picard algorithm $\{T^n x\}$ does not converge. Consequently, Mann's algorithm was constructed by Mann [1] in 1953:

$$x^{n+1} = (1 - \alpha_n) x^n + \alpha_n T x^n, \quad n \geq 0. \tag{1}$$

There are a large number of papers on Mann's algorithm in the literature. See [2–5]. Now we know that if T is nonexpansive, then Mann's algorithm converges weakly to a fixed point of T. This algorithm however does not converge in the strong topology.

In order to get the strong convergence, the following Halpern's algorithm was introduced:

$$x^{n+1} = \alpha_n u + (1 - \alpha_n) T x^n, \quad n \geq 0. \tag{2}$$

The interest and importance of Halpern iterative method lie in the fact that strong convergence of the sequence $\{x_n\}$ is achieved under certain mild conditions on parameter $\{\alpha_n\}$ in a general Banach space. Please refer to [6–12].

In the present paper, we are devoted to find the fixed points of pseudocontractive mappings. For some related works, please see [13–23]. The interest of pseudocontractions lies in their connection with monotone operators. Browder and Petryshyn [24] studied weak convergence of Mann's algorithm for the class of strict pseudocontractions. But Mann's algorithm fails to converge for Lipschitzian pseudocontractions [25].

Inspired by the results in the literature, the main purpose of this paper is to construct an iterative method for finding the fixed points of pseudocontractive mappings. Under some mild conditions, strong convergence results are given.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad x, y \in C. \tag{3}$$

A mapping $T : C \to C$ is called k-Lipschitzian if there exists $k > 0$ such that

$$\|Tx - Ty\| \leq k \|x - y\|, \tag{4}$$

for all $x, y \in C$. In this case, if $k < 1$, then T is a k-contraction.

It is well known that in a real Hilbert space H the following inequality holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad (5)$$

for all $x, y \in H$.

In the present paper, we will use the following notations:

(i) we use $\mathrm{Fix}(T)$ to denote the set of fixed points of T;

(ii) $x_n \rightharpoonup x$ denotes the weak convergence of x_n to x;

(iii) $x_n \to x$ denotes the strong convergence of x_n to x.

Lemma 1 (see [26]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then $(I - T)$ is demiclosed at zero.*

Lemma 2 (see [27]). *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity; that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \leq w_{n_k+1}$ for all $k \geq 0$. For every $n \geq M$, define an integer sequence $\{W(n)\}$ as*

$$W(n) = \max\left\{i \leq n : w_{n_i} < w_{n_i+1}\right\}. \quad (6)$$

Then $W(n) \to \infty$ as $n \to \infty$ and for all $n \geq M$

$$\max\left\{w_{W(n)}, w_n\right\} \leq w_{W(n)+1}. \quad (7)$$

Lemma 3 (see [28]). *Assume that $\{A_n\}$ is a sequence of nonnegative real numbers such that*

$$A_{n+1} \leq (1-\gamma_n) A_n + \xi_n, \quad (8)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\xi_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty}(\xi_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\xi_n| < \infty$.

Then $\lim_{n\to\infty} A_n = 0$.

3. Main Results

Now we present the statement of our algorithm.

The Modified Mann-Halpern Algorithm. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudocontractive mapping and $S : C \to H$ a ρ-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real number sequences in $[0, 1]$. We have the following steps.

(1) Initialization:

$$\forall x_0 \in C. \quad (9)$$

(2) Mann step: for a given x_n, define a sequence y_n by

$$y_n = (1-\gamma_n) x_n + \gamma_n T x_n, \quad (10)$$

for all $n \geq 0$.

(3) Halpern step: for a given x_n and y_n, define

$$x_{n+1} = \alpha_n S(x_n) + (1-\alpha_n-\beta_n) x_n + \beta_n T y_n, \quad (11)$$

for all $n \geq 0$.

In the following, we assume that

(i) the mapping $T : C \to C$ is k-Lipschitzian;

(ii) the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions (C1)–(C5):

(C1): $\lim_{n\to\infty}\alpha_n = 0$;
(C2): $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3): $\alpha_n + \beta_n \leq \gamma_n$;
(C4): $0 < \liminf_{n\to\infty}\beta_n$;
(C5): $0 < \limsup_{n\to\infty}\gamma_n < 1/(\sqrt{1+k^2}+1)$.

Now, we prove our main result as follows.

Theorem 4. *Suppose* $\mathrm{Fix}(T) \neq \emptyset$. *Then the sequence $\{x_n\}$ defined by* (11) *converges strongly to a fixed point of T.*

Proof. Since S is a ρ-condition, then $\mathrm{Proj}_{\mathrm{Fix}(T)} S$ is a contractive mapping (where Proj is the metric projection). Hence, there exists a unique u such that $u = \mathrm{Proj}_{\mathrm{Fix}(T)} S(u)$. In the sequel, we will show that the sequence $\{x_n\}$ defined by (11) converges strongly to u.

From (11), we get

$$\begin{aligned}
&\|x_{n+1}-u\| \\
&\quad = \|\alpha_n S(x_n) + (1-\alpha_n-\beta_n) x_n + \beta_n T y_n - u\| \\
&\quad \leq \|\alpha_n (S(x_n)-u) + (1-\alpha_n-\beta_n)(x_n-u) + \beta_n (T y_n - u)\| \\
&\quad = \Bigg\|\alpha_n (S(x_n)-u) + (1-\alpha_n) \\
&\qquad \times \left(\frac{1-\alpha_n-\beta_n}{1-\alpha_n}(x_n-u) + \frac{\beta_n}{1-\alpha_n}(T y_n - u)\right)\Bigg\| \\
&\quad \leq \alpha_n \|S(x_n)-u\| + (1-\alpha_n) \\
&\qquad \times \left\|\frac{(1-\alpha_n-\beta_n)(x_n-u)}{1-\alpha_n} + \frac{\beta_n (T y_n - u)}{1-\alpha_n}\right\|.
\end{aligned} \quad (12)$$

It is well known that there holds the following inequality in Hilbert spaces:

$$\begin{aligned}
\|tx + (1-t) y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 \\
- t(1-t)\|x-y\|^2
\end{aligned} \quad (13)$$

for all $x, y \in H$ and $t \in [0, 1]$. Hence, we have

$$\begin{aligned}
&\left\|\frac{(1-\alpha_n-\beta_n)(x_n-u)}{1-\alpha_n} + \frac{\beta_n(T y_n-u)}{1-\alpha_n}\right\|^2 \\
&\quad \leq \frac{1-\alpha_n-\beta_n}{1-\alpha_n}\|x_n-u\|^2 + \frac{\beta_n}{1-\alpha_n}\|T y_n - u\|^2 \\
&\qquad - \frac{\beta_n(1-\alpha_n-\beta_n)}{(1-\alpha_n)^2}\|x_n - T y_n\|^2.
\end{aligned} \quad (14)$$

We know that T is pseudocontractive if and only if T satisfies the condition

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \tag{15}$$

for all $x, y \in C$. Since $u \in \mathrm{Fix}(T)$, we have from (15) that

$$\|Tx - u\|^2 \leq \|x - u\|^2 + \|x - Tx\|^2, \tag{16}$$

for all $x \in C$.

By using (13) and (16), we obtain

$$\begin{aligned}
&\|Ty_n - u\|^2 \\
&\quad \leq \|y_n - u\|^2 + \|y_n - Ty_n\|^2 \\
&\quad = \|(1 - \gamma_n)x_n + \gamma_n Tx_n - u\|^2 \\
&\qquad + \|(1 - \gamma_n)x_n + \gamma_n Tx_n - Ty_n\|^2 \\
&\quad = \|(1 - \gamma_n)(x_n - u) + \gamma_n(Tx_n - u)\|^2 \\
&\qquad + \|(1 - \gamma_n)(x_n - Ty_n) + \gamma_n(Tx_n - Ty_n)\|^2 \\
&\quad = (1 - \gamma_n)\|x_n - u\|^2 + \gamma_n\|Tx_n - u\|^2 \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\qquad + (1 - \gamma_n)\|x_n - Ty_n\|^2 + \gamma_n\|Tx_n - Ty_n\|^2 \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\quad \leq (1 - \gamma_n)\|x_n - u\|^2 \\
&\qquad + \gamma_n\left(\|x_n - u\|^2 + \|x_n - Tx_n\|^2\right) \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\qquad + (1 - \gamma_n)\|x_n - Ty_n\|^2 + \gamma_n\|Tx_n - Ty_n\|^2 \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2.
\end{aligned} \tag{17}$$

Note that T is k-Lipschitzian and

$$\|x_n - y_n\| = \gamma_n\|x_n - Tx_n\|. \tag{18}$$

From (17), we have

$$\begin{aligned}
&\|Ty_n - u\|^2 \\
&\quad \leq (1 - \gamma_n)\|x_n - u\|^2 \\
&\qquad + \gamma_n\left(\|x_n - u\|^2 + \|x_n - Tx_n\|^2\right) \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\qquad + (1 - \gamma_n)\|x_n - Ty_n\|^2 + \gamma_n k^2\|x_n - y_n\|^2 \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\quad = (1 - \gamma_n)\|x_n - u\|^2 \\
&\qquad + \gamma_n\left(\|x_n - u\|^2 + \|x_n - Tx_n\|^2\right) \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\qquad + (1 - \gamma_n)\|x_n - Ty_n\|^2 + \gamma_n^3 k^2\|x_n - Tx_n\|^2 \\
&\qquad - \gamma_n(1 - \gamma_n)\|x_n - Tx_n\|^2 \\
&\quad = \|x_n - u\|^2 + (1 - \gamma_n)\|x_n - Ty_n\|^2 \\
&\qquad - \gamma_n\left(1 - 2\gamma_n - \gamma_n^2 k^2\right)\|x_n - Tx_n\|^2.
\end{aligned} \tag{19}$$

By condition (C5), without loss of generality, we may assume that $\gamma_n \leq a < 1/(\sqrt{1 + k^2} + 1)$ for all n. Then, we have $1 - 2\gamma_n - \gamma_n^2 L^2 > 0$ for all $n \geq 0$. Substituting (19) to (14) and noting condition (C3), we have

$$\begin{aligned}
&\left\|\frac{(1 - \alpha_n - \beta_n)(x_n - u)}{1 - \alpha_n} + \frac{\beta_n(Ty_n - u)}{1 - \alpha_n}\right\|^2 \\
&\quad \leq \frac{1 - \alpha_n - \beta_n}{1 - \alpha_n}\|x_n - u\|^2 + \frac{\beta_n}{1 - \alpha_n} \\
&\qquad \times \left(\|x_n - u\|^2 + (1 - \gamma_n)\|x_n - Ty_n\|^2\right) \\
&\qquad - \frac{\beta_n(1 - \alpha_n - \beta_n)}{(1 - \alpha_n)^2}\|x_n - Ty_n\|^2 \\
&\quad = \|x_n - u\|^2 + \frac{\beta_n(\alpha_n + \beta_n - \gamma_n)}{(1 - \alpha_n)^2}\|x_n - Ty_n\|^2 \\
&\quad \leq \|x_n - u\|^2.
\end{aligned} \tag{20}$$

Therefore,

$$\left\|\frac{(1 - \alpha_n - \beta_n)(x_n - u)}{1 - \alpha_n} + \frac{\beta_n(Ty_n - u)}{1 - \alpha_n}\right\| \leq \|x_n - u\|. \tag{21}$$

It follows from (12) and (21) that

$$\begin{aligned}
&\|x_{n+1}-u\| \\
&\quad\le \alpha_n\|S(x_n)-u\|+(1-\alpha_n)\|x_n-u\| \\
&\quad\le \alpha_n\|S(x_n)-S(u)\|+\alpha_n\|S(u)-u\| \\
&\qquad+(1-\alpha_n)\|x_n-u\| \\
&\quad\le \alpha_n\rho\|x_n-u\|+\alpha_n\|S(u)-u\| \\
&\qquad+(1-\alpha_n)\|x_n-u\| \\
&\quad= \alpha_n\|S(u)-u\|+[1-(1-\rho)\alpha_n]\|x_n-u\| \\
&\quad\le \max\left\{\|x_n-u\|,\frac{\|S(u)-u\|}{1-\rho}\right\} \\
&\quad\le \max\left\{\|x_0-u\|,\frac{\|S(u)-u\|}{1-\rho}\right\}.
\end{aligned} \tag{22}$$

This implies that the sequence $\{x_n\}$ is bounded.

From (5) and (11), we have

$$\begin{aligned}
&\|x_{n+1}-u\|^2 \\
&\quad=\|(1-\alpha_n)(x_n-u)-\beta_n(x_n-Ty_n) \\
&\qquad+\alpha_n(S(x_n)-u)\|^2 \\
&\quad\le\|(1-\alpha_n)(x_n-u)-\beta_n(x_n-Ty_n)\|^2 \\
&\qquad+2\alpha_n\langle S(x_n)-u,x_{n+1}-u\rangle \\
&\quad=\|(1-\alpha_n)(x_n-u)\|^2 \\
&\qquad-2\beta_n(1-\alpha_n)\langle x_n-Ty_n,x_n-u\rangle \\
&\qquad+\beta_n^2\|x_n-Ty_n\|^2 \\
&\qquad+2\alpha_n\langle S(x_n)-u,x_{n+1}-u\rangle.
\end{aligned} \tag{23}$$

Note that (19) is equivalent to

$$\begin{aligned}
&2\langle x_n-Ty_n,x_n-u\rangle \\
&\quad\ge\gamma_n\|x_n-Ty_n\|^2 \\
&\qquad+\gamma_n\left(1-2\gamma_n-\gamma_n^2k^2\right)\|x_n-Tx_n\|^2.
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
&\|x_{n+1}-u\|^2 \\
&\quad\le(1-\alpha_n)\|x_n-u\|^2 \\
&\qquad-\beta_n(1-\alpha_n)\gamma_n\|x_n-Ty_n\|^2 \\
&\qquad+\beta_n^2\|x_n-Ty_n\|^2 \\
&\qquad-\beta_n(1-\alpha_n)\gamma_n\left(1-2\gamma_n-\gamma_n^2k^2\right)\|x_n-Tx_n\|^2 \\
&\qquad+2\alpha_n\langle S(x_n)-u,x_{n+1}-u\rangle \\
&\quad\le(1-\alpha_n)\|x_n-u\|^2 \\
&\qquad-\beta_n(1-\alpha_n)\gamma_n\left(1-2\gamma_n-\gamma_n^2k^2\right)\|x_n-Tx_n\|^2 \\
&\qquad+2\alpha_n\langle S(x_n)-u,x_{n+1}-u\rangle.
\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}
&\|x_{n+1}-u\|^2-\|x_n-u\|^2 \\
&\quad+\beta_n(1-\alpha_n)\gamma_n\left(1-2\gamma_n-\gamma_n^2k^2\right)\|x_n-Tx_n\|^2 \\
&\qquad\le\alpha_n\left(\langle 2S(x_n)-u,x_{n+1}-u\rangle-\|x_n-u\|^2\right).
\end{aligned} \tag{26}$$

Since x_n and $S(x_n)$ are bounded, there exists $M>0$ such that $\sup_n\{2\langle S(x_n)-u,x_{n+1}-u\rangle-\|x_n-u\|^2\}\le M$. So

$$\begin{aligned}
&\|x_{n+1}-u\|^2-\|x_n-u\|^2 \\
&\quad+\beta_n(1-\alpha_n)\gamma_n\left(1-2\gamma_n-\gamma_n^2k^2\right)\|x_n-Tx_n\|^2 \\
&\qquad\le\alpha_nM.
\end{aligned} \tag{27}$$

Next, we prove two cases.

Assume there exists an integer $m>0$ such that $\{\|x_n-u\|\}$ is decreasing for all $n\ge m$.

In this case, we know that $\lim_{n\to\infty}\|x_n-u\|$ exists. From (27), we deduce

$$\begin{aligned}
&\beta_n(1-\alpha_n)\gamma_n\left(1-2\gamma_n-\gamma_n^2k^2\right)\|x_n-Tx_n\|^2 \\
&\quad\le\|x_n-u\|^2-\|x_{n+1}-u\|^2+M\alpha_n.
\end{aligned} \tag{28}$$

By conditions (C4) and (C5), we have $\liminf_{n\to\infty}\beta_n(1-\alpha_n)\gamma_n(1-2\gamma_n-\gamma_n^2k^2)>0$. Thus, from (28), we get

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{29}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\begin{gathered}
x_{n_k}\longrightarrow\tilde{x}\in C, \\
\limsup_{n\to\infty}\langle S(u)-u,x_n-u\rangle=\lim_{k\to\infty}\langle S(u)-u,x_{n_k}-u\rangle.
\end{gathered} \tag{30}$$

Thus, we use the demiclosed principle of T (Lemma 1) to deduce

$$\tilde{x}\in\operatorname{Fix}(T). \tag{31}$$

So

$$\begin{aligned}\limsup_{n\to\infty}\langle S(u)-u, x_n-u\rangle &= \lim_{k\to\infty}\langle S(u)-u, x_{n_k}-u\rangle \\ &= \langle S(u)-u, \tilde{x}-u\rangle \\ &\le 0.\end{aligned} \tag{32}$$

Returning to (25) and using (5) we obtain

$$\begin{aligned}&\|x_{n+1}-u\|^2 \\ &\quad\le (1-\alpha_n)\|x_n-u\|^2 \\ &\qquad + 2\alpha_n\langle S(x_n)-u, x_{n+1}-u\rangle \\ &\quad= (1-\alpha_n)\|x_n-u\|^2 \\ &\qquad + 2\alpha_n\langle S(x_n)-S(u), x_{n+1}-u\rangle \\ &\qquad + 2\alpha_n\langle S(u)-u, x_{n+1}-u\rangle \\ &\quad\le (1-\alpha_n)\|x_n-u\|^2 \\ &\qquad + 2\alpha_n\rho\|x_n-u\|\|x_{n+1}-u\| \\ &\qquad + 2\alpha_n\langle S(u)-u, x_{n+1}-u\rangle \\ &\quad\le (1-\alpha_n)\|x_n-u\|^2 \\ &\qquad + \alpha_n\rho\left(\|x_n-u\|^2+\|x_{n+1}-u\|^2\right) \\ &\qquad + 2\alpha_n\langle S(u)-u, x_{n+1}-u\rangle.\end{aligned} \tag{33}$$

It follows that

$$\begin{aligned}\|x_{n+1}-u\|^2 \le\ & [1-(1-\rho)\alpha_n]\|x_n-u\|^2 \\ & + \frac{2\alpha_n}{1-\alpha_n\rho}\langle Su-u, x_{n+1}-u\rangle.\end{aligned} \tag{34}$$

In Lemma 3, we take $A_n = \|x_{n+1}-u\|^2$, $\gamma_n = (1-\rho)\alpha_n$, and $\xi_n = (2\alpha_n/(1-\alpha_n\rho))\langle Su-u, x_{n+1}-u\rangle$. We can check easily that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}(\xi_n/\gamma_n) \le 0$. Thus, we deduce that $x_n \to u$.

Assume there exists an integer n_0 such that $\|x_{n_0}-u\| \le \|x_{n_0+1}-u\|$. In this case, we set $\omega_n = \{\|x_n-u\|\}$. Then, we have $\omega_{n_0} \le \omega_{n_0+1}$. Define an integer sequence $\{W_n\}$ for all $n \ge n_0$ as follows:

$$W(n) = \max\{l \in \mathbb{N} \mid n_0 \le l \le n,\ \omega_l \le \omega_{l+1}\}. \tag{35}$$

It is clear that $W(n)$ is a nondecreasing sequence satisfying

$$\begin{aligned}&\lim_{n\to\infty} W(n) = \infty, \\ &\omega_{\tau(n)} \le \omega_{W(n)+1},\end{aligned} \tag{36}$$

for all $n \ge n_0$. From (28), we get

$$\lim_{n\to\infty}\|x_{W(n)} - Tx_{W(n)}\| = 0. \tag{37}$$

This implies that $\omega_w(x_{W(n)}) \subset \mathrm{Fix}(T)$. Thus, we obtain

$$\limsup_{n\to\infty}\langle S(u)-u, x_{W(n)}-u\rangle \le 0. \tag{38}$$

Since $\omega_{W(n)} \le \omega_{W(n)+1}$, we have from (34) that

$$\begin{aligned}\omega_{W(n)}^2 &\le \omega_{W(n)+1}^2 \\ &\le [1-(1-\rho)\alpha_W(n)]\omega_{W(n)}^2 \\ &\quad + \frac{2\alpha_W(n)}{1-\alpha_W(n)\rho}\langle Su-u, x_{W(n)+1}-u\rangle.\end{aligned} \tag{39}$$

It follows that

$$\omega_{W(n)}^2 \le \frac{2}{(1-\alpha_W(n)\rho)(1-\rho)}\langle Su-u, x_{W(n)+1}-u\rangle. \tag{40}$$

Combining (38) and (40), we have

$$\limsup_{n\to\infty}\omega_{W(n)} \le 0, \tag{41}$$

and hence

$$\lim_{n\to\infty}\omega_{W(n)} = 0. \tag{42}$$

From (34), we obtain

$$\begin{aligned}&\|x_{W(n)+1}-u\|^2 \\ &\quad\le [1-(1-\rho)\alpha_W(n)]\|x_W(n)-u\|^2 \\ &\qquad + \frac{2\alpha_W(n)}{1-\alpha_W(n)\rho}\langle Su-u, x_{W(n)+1}-u\rangle.\end{aligned} \tag{43}$$

It follows that

$$\limsup_{n\to\infty}\omega_{W(n)+1} \le \limsup_{n\to\infty}\omega_{W(n)}. \tag{44}$$

This together with (42) implies that

$$\lim_{n\to\infty}\omega_{W(n)+1} = 0. \tag{45}$$

Applying Lemma 2 we get

$$0 \le \omega_n \le \max\{\omega_{W(n)}, \omega_{W(n)+1}\}. \tag{46}$$

Therefore, $\omega_n \to 0$. That is, $x_n \to u$. The proof is completed. □

4. Conclusions

It is now well known that Mann's algorithm fails to converge for Lipschitzian pseudocontractions. Strong convergence of Ishikawa's algorithm has not been achieved without compactness assumption. In the present paper, modified Mann-Halpern algorithms for finding the fixed points of pseudocontractive mappings are presented. Strong convergence theorems are obtained.

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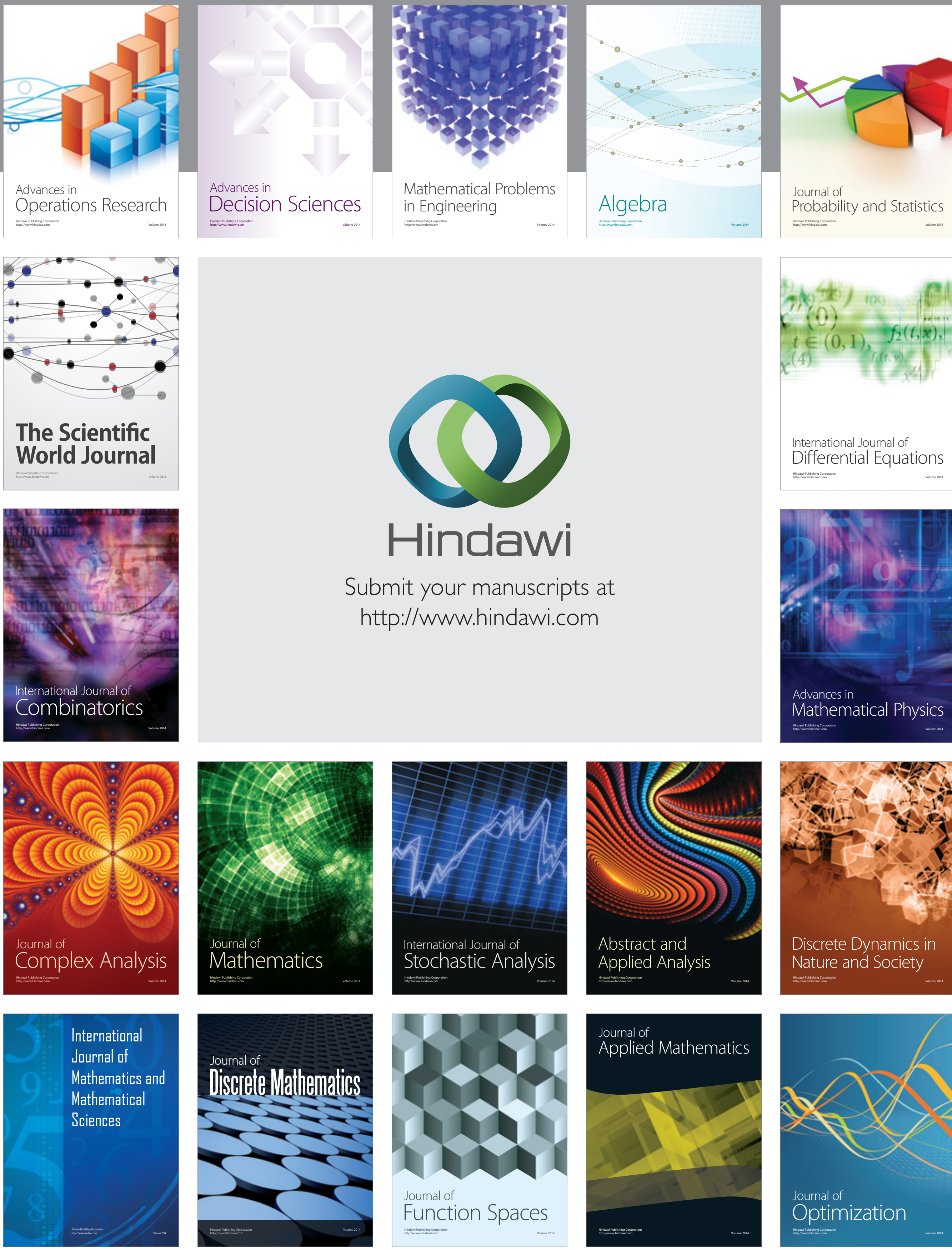
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